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Chebyshev-Grüss Type Inequalities for Hadamard k-Fractional Integrals

Research Article

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Abstract. Integral inequalities are taken up to be important as they are useful in the study of different classes of differential and integral equations. During the past several years, many researchers have obtained various fractional integral inequalities comprising the different fractional differential and integral operators. A considerable work is done associated with classical and variants of Grüss type inequality, which actually connects the integral of the product of two functions with the product of their integrals. In this paper, we present the Chebyshev-Grüss type inequalities for Hadamard fractional integrals in the framework of parameter $k > 0$.

Keywords. Fractional integral inequalities; Chebyshev functional; (k,s)-fractional integrals

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1. Introduction

For two measurable functions $f,g:[a,b]\to\mathbb{R}$, define the functional, which is known in the literature as Chebychev's functional

$$T(f,g;a,b)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx, \tag{1.1}$$

provided that the involved integrals exist. There are many studies involving (1.1) in the literature, see for example [2,6,20,24].

Grüss type Inequality due to Chebychev (see for example [21, p. 207]) is as follows.

If f,g are absolutely continuous on $[a,b]$ and $f',g' \in L_\infty[a,b]$ and $\|f'\|_\infty := \operatorname*{ess\,sup}_{t\in[a,b]}|f'(t)|$, then

$$|T(f,g;a,b)| \le \frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2 \tag{1.2}$$

and the constant $\frac{1}{12}$ is the best possible.

Further, a weighted version of the Chebyshev functional (see [3]) is defined as:

$$T(f,g,p) = \int_a^b p(t)dt\int_a^b f(t)g(t)p(t)dt - \int_a^b f(t)p(t)dt\int_a^b g(t)p(t)dt, \tag{1.3}$$

where f and g are integrable functions on $[a,b]$ and $p(t)$ is a positive and integrable function on $[a,b]$. In 2000, Dragomir [8] derived the following inequality, related to the weighted Chebyshev functional (1.3):

$$2|T(f,g,p)| \le \|f'\|_p\|g'\|_q\left[\int_a^b\int_a^b |x-y|\,p(x)p(y)dxdy\right], \tag{1.4}$$

where f,g are differentiable functions and $f' \in L_p(a,b)$, $g' \in L_q(a,b)$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. In mathematical analysis, the fractional calculus is a very helpful tool to perform differentiation and integration with the real number or complex number powers of the differential or integral operators. This subject has earned the attention of many researchers and mathematicians during last few decades (see [1,3–5,10,22,26,28]). There is a large number of the fractional integral operators discussed in literature but because of their applications in many fields of sciences, the Riemann-Liouville fractional integral operator and Hadamard fractional integral operator have been studied extensively.

The Hadamard fractional integral operator was introduced by Hadamard [9]. It can be defined as follows:

Let $f \in L[a,b]$, the left and right sided Hadamard fractional integrals of order $\alpha \ge 0$ and $a>0$ are defined respectively as

$$H^\alpha_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \quad 0<a<t\le b \tag{1.5}$$

and

$$H^\alpha_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b\left(\ln\frac{\tau}{t}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}, \quad 0<a\le t<b. \tag{1.6}$$

The theory of special k-functions was introduced about a decade ago when Diaz and Pariguan [7] defined the generalization of the classical gamma and beta functions in terms of a new parameter $k>0$, called gamma and beta k-functions, respectively

$$\Gamma_k(\alpha)=\int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt,\quad Re(\alpha)>0$$

and

$$B_k(\alpha,\beta)=\frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt,\quad Re(\alpha)>0,\ Re(\beta)>0. \tag{1.7}$$

This idea of generalization of special functions in terms of some new parameter fascinated many researchers and mathematicians. Several properties, identities and inequalities involving special k-functions were proved during past several years (see for instance [11–15, 28, 29]).

The functions Γ_k defined on $\mathbb{R}^+$ and $B_k(x,y)$ on $(0,1)$ hold the following four properties:

(I) $\Gamma_k(x+k)=x\Gamma_k(x)$;

(II) $\Gamma_k(k)=1$;

(III) $\Gamma_k(x)$ is logarithmically convex;

(IV) $\beta_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

For the first time, Mubeen and Habibullah [17] used this special k-functions theory in fractional calculus and introduced the k-fractional integral of the Riemann-Liouville type as

$$I^\alpha_{a,k}f(t)=\frac{1}{k\Gamma_k(\alpha)}\int_a^t (t-x)^{\frac{\alpha}{k}-1}f(x)dx,\quad t\in[a,b],$$

where Γ_k is the Euler gamma k-function.

Later, Romero *et al.* [25] introduced a new fractional operator called k-Riemann-Liouville fractional derivative by using gamma k-function. They also proved some properties of this newly defined fractional operator and found its relationship with Riemann-Liouville k-fractional integral.

In new research paper, using Γ_k and new k parameter, Mubeen *et al.* [18] have introduced left-sided and right-sided Hadamard k-fractional integrals as following:

Definition 1.1. For $k>0$, let $f\in L[a,b]$, the left and right sided k-fractional integrals of order $\alpha\geq 0$ and $a>0$ are defined respectively as

$$\mathscr{H}^\alpha_{a^+,k}\{f\}(t)=\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}f(\tau)\frac{d\tau}{\tau},\quad 0<a<t\leq b \tag{1.8}$$

and

$$\mathscr{H}^\alpha_{b^-,k}\{f\}(t)=\frac{1}{k\Gamma_k(\alpha)}\int_t^b\left(\ln\frac{\tau}{t}\right)^{\frac{\alpha}{k}-1}f(\tau)\frac{d\tau}{\tau},\quad 0<a\leq t<b. \tag{1.9}$$

Corollary 1. *Using definition of Hadamard k-fractional integral and relation* (I)*, we have*

$$\mathscr{H}^\alpha_{a^+,k}(1)=\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)},\quad \alpha,k>0 \tag{1.10}$$

and

$$\mathscr{H}^\alpha_{1^+,k}(1)=\frac{(\ln t)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)},\quad \alpha,k>0. \tag{1.11}$$

2. Hadamard k-Fractional Integral Inequalities

Throughout of this paper, we denote the Hadamard k-fractional integral of order α of a function f which have limit zero by $\mathscr{H}^{\alpha}_{0,k}[f(t)] = \mathscr{H}^{\alpha}_{k}[f(t)]$.

To use in next theorems, we want to define two functions as following:

$$\mathscr{A}(\tau,\rho) = \big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big), \quad \tau,\rho \in (0,t),\ t>0 \tag{2.1}$$

and

$$F_{\alpha}(t,\tau) = \frac{(\ln \frac{t}{\tau})^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)\tau}, \quad t,k,\alpha>0,\ \tau\in(0,t). \tag{2.2}$$

Theorem 1. *Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f'\in L_n([0,\infty)), g'\in L_m([0,\infty))$ such that $\frac{1}{n}+\frac{1}{m}=1$ with $n>1$. Then for all $t>0$, $\alpha>0$, $k>0$ and*

$$\begin{aligned}
&2\left|\mathscr{H}^{\alpha}_{k}[p(t)]\ \mathscr{H}^{\alpha}_{k}[p(t)f(t)g(t)] - \mathscr{H}^{\alpha}_{k}[p(t)f(t)]\ \mathscr{H}^{\alpha}_{k}[p(t)g(t)]\right| \\
&\quad \le \frac{\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|d\tau d\rho \\
&\quad \le \|f'\|_n\|g'\|_m t\left(\mathscr{H}^{\alpha}_{k}[p(t)]\right)^2.
\end{aligned} \tag{2.3}$$

Proof. Provided that the conditions of the theorem and for all $\tau\in(0,t)$, we can easily see $F^{\alpha}_{s}(t,\tau)>0$. Multiplying with $p(\tau)$ both side of product $\mathscr{A}(\tau,\rho)\times F_{\alpha}(t,\tau)$ and taking the integral with respect to τ on $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\mathscr{A}(\tau,\rho)d\tau \\
&\quad = \mathscr{H}^{\alpha}_{k}[p(t)f(t)g(t)] - f(\rho)\mathscr{H}^{\alpha}_{k}[p(t)g(t)] - g(\rho)\mathscr{H}^{\alpha}_{k}[p(t)f(t)] + f(\rho)g(\rho)\mathscr{H}^{\alpha}_{k}[p(t)].
\end{aligned} \tag{2.4}$$

Now, multiplying above identity (2.4) by $F_{\alpha}(t,\rho)p(\rho)$ and then integrating with respect to ρ on $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}\mathscr{A}(\tau,\rho)d\tau d\rho \\
&\quad = 2\left(\mathscr{H}^{\alpha}_{k}[p(t)]\ \mathscr{H}^{\alpha}_{k}[p(t)f(t)g(t)] - \mathscr{H}^{\alpha}_{k}[p(t)f(t)]\ \mathscr{H}^{\alpha}_{k}[p(t)g(t)]\right).
\end{aligned} \tag{2.5}$$

With the help of fundamental theorem of calculus, identity (2.1) can be written as

$$\mathscr{A}(\tau,\rho) = \int_{\tau}^{\rho}\int_{\tau}^{\rho} f'(y)g'(z)dydz.$$

By using the Hölder's inequality for double integrals, we get

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho} f(y)g(z)dydz\right| \le \left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|f(y)|^n dydz\right|^{\frac{1}{n}}\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|g(z)|^m dydz\right|^{\frac{1}{m}},$$

$$\left(\frac{1}{n}+\frac{1}{m}=1, n>1\right),$$

then, we obtain

$$|\mathscr{A}(\tau,\rho)| \le \left|\int_\tau^\rho\int_\tau^\rho |f'(y)|^n dydz\right|^{\frac{1}{n}} \left|\int_\tau^\rho\int_\tau^\rho |g'(z)|^m dydz\right|^{\frac{1}{m}}. \tag{2.6}$$

Since

$$\left|\int_\tau^\rho\int_\tau^\rho |f'(y)|^n dydz\right|^{\frac{1}{n}} = |\tau-\rho|^{\frac{1}{n}} \left|\int_\tau^\rho |f'(y)|^n dy\right|^{\frac{1}{n}}$$

and

$$\left|\int_\tau^\rho\int_\tau^\rho |g'(z)|^m dydz\right|^{\frac{1}{m}} = |\tau-\rho|^{\frac{1}{m}} \left|\int_\tau^\rho |g'(z)|^m dz\right|^{\frac{1}{m}},$$

thus, from inequality (2.6) we get

$$|\mathscr{A}(\tau,\rho)| \le |\tau-\rho| \left|\int_\tau^\rho |f'(y)|^n dy\right|^{\frac{1}{n}} \left|\int_\tau^\rho |g'(z)|^m dz\right|^{\frac{1}{m}}. \tag{2.7}$$

As a result of (2.7) and from equality (2.5) can be written following inequality:

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho \\
&\le \frac{1}{k\Gamma_k(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho} \\
&\quad \times |\tau-\rho|\left|\int_\tau^\rho |f'(y)|^n dy\right|^{\frac{1}{n}}\left|\int_\tau^\rho |g'(z)|^m dz\right|^{\frac{1}{m}} d\tau d\rho.
\end{aligned} \tag{2.8}$$

Now again using weighted Hölder's integral inequality, on the right-hand side of (2.8), we have

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho \\
&\le \left[\frac{1}{k^n\Gamma_k^n(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|\left|\int_\tau^\rho \left|f'(y)\right|^n dy\right| d\tau d\rho\right]^{\frac{1}{n}} \\
&\quad \times \left[\frac{1}{k^m\Gamma_k^m(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|\left|\int_\tau^\rho \left|g'(z)\right|^m dz\right| d\tau d\rho\right]^{\frac{1}{m}}.
\end{aligned} \tag{2.9}$$

Taking into account the fact that

$$\left|\int_\tau^\rho |f(y)|^n dy\right| \le \|f\|_n^n \quad \text{and} \quad \left|\int_\tau^\rho |g(z)|^m dz\right| \le \|g\|_m^m,$$

we obtain

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho \\
&\le \left[\frac{\|f\|_n^n}{k^n\Gamma_k^n(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho| d\tau d\rho\right]^{\frac{1}{n}} \\
&\quad \times \left[\frac{\|g\|_m^m}{k^m\Gamma_k^m(\alpha)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho| d\tau d\rho\right]^{\frac{1}{m}}.
\end{aligned} \tag{2.10}$$

Inequality (2.10) gives us following inequality:

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho\\
&\le\frac{\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\left[\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|d\tau d\rho\right]^{\frac{1}{n}}\\
&\quad\times\left[\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|d\tau d\rho\right]^{\frac{1}{m}}.
\end{aligned}\tag{2.11}$$

With the fact that $\frac{1}{n}+\frac{1}{m}=1$, from (2.11) we get

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho\\
&\le\frac{\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|d\tau d\rho.
\end{aligned}\tag{2.12}$$

On the other hand using equality (2.5) we can easily seen that

$$\begin{aligned}
&2\left|\mathscr{H}_k^\alpha[p(t)]\,\mathscr{H}_k^\alpha[p(t)f(t)g(t)]-\mathscr{H}_k^\alpha[p(t)f(t)]\,\mathscr{H}_k^\alpha[p(t)g(t)]\right|\\
&\le\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}\left|\mathscr{A}(\tau,\rho)\right|d\tau d\rho.
\end{aligned}\tag{2.13}$$

Taking into account the inequalities (2.12) and (2.13), we conclude the left-hand side of the inequality (2.3).

Now, to obtain the right-hand side of the inequality (2.3), since $0\le\tau\le t$ and $0\le\rho\le t$, we will use the fact that

$$0\le|\tau-\rho|\le t.$$

Clearly, from (2.12), we obtain

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho\\
&\le\frac{\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|d\tau d\rho\\
&\le\frac{t\|f\|_n\|g\|_m}{k^2\Gamma_k^2(\alpha)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\alpha}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}d\tau d\rho\\
&=\|f'\|_n\|g'\|_m t\left(\mathscr{H}_k^\alpha[p(t)]\right)^2
\end{aligned}$$

which completes of the proof of Theorem 1. □

The following theorem put forward a further generalization of Theorem 1 with α and β positive parameters.

Theorem 2. *Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f' \in L_n([0,\infty))$, $g' \in L_m([0,\infty))$ such that $\frac{1}{n}+\frac{1}{m}=1$ with $n>1$. Then for $t>0$ following inequality holds:*

$$\begin{aligned}
&\Big|\mathscr{H}_k^{\beta}[p(t)]\ \mathscr{H}_k^{\alpha}[p(t)f(t)g(t)] - \mathscr{H}_k^{\beta}[p(t)f(t)]\ \mathscr{H}_k^{\alpha}[p(t)g(t)] \\
&\quad - \mathscr{H}_k^{\beta}[p(t)g(t)]\ \mathscr{H}_k^{\alpha}[p(t)f(t)] + \mathscr{H}_k^{\beta}[p(t)f(t)g(t)]\ \mathscr{H}_k^{\alpha}[p(t)]\Big| \\
&\quad \le \frac{\|f'\|_n\|g'\|_m}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|\,d\tau d\rho \\
&\quad \le\ t\|f'\|_n\|g'\|_m\ \mathscr{H}_k^{\alpha}[p(t)]\ \mathscr{H}_k^{\beta}[p(t)]
\end{aligned} \tag{2.14}$$

where $\alpha,\beta,k>0$.

Proof. To prove this theorem, we multiply (2.4) by $F_\beta(t,\rho)$ ($\rho\in(0,t)$, $t>0$) and take the integral on $(0,t)$ (with respect to ρ), to obtain

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}\mathscr{A}(\tau,\rho)d\tau d\rho \\
&= \mathscr{H}_k^{\beta}[p(t)]\ \mathscr{H}_k^{\alpha}[p(t)f(t)g(t)] - \mathscr{H}_k^{\beta}[p(t)f(t)]\ \mathscr{H}_k^{\alpha}[p(t)g(t)] \\
&\quad - \mathscr{H}_k^{\beta}[p(t)g(t)]\ \mathscr{H}_k^{\alpha}[p(t)f(t)] + \mathscr{H}_k^{\beta}[p(t)f(t)g(t)]\ \mathscr{H}_k^{\alpha}[p(t)].
\end{aligned} \tag{2.15}$$

Using our obtained previously inequality (2.7), we get

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho \\
&\le \frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1} \\
&\quad \times \frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|\left|\int_\tau^\rho |f'(y)|^n dy\right|^{\frac{1}{n}}\left|\int_\tau^\rho |g'(z)|^m dz\right|^{\frac{1}{m}} d\tau d\rho.
\end{aligned} \tag{2.16}$$

If we take Hölder's integral inequality, we easily get following inequality:

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho \\
&\le \frac{\|f'\|_n\|g'\|_m}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\tau-\rho|\,d\tau d\rho.
\end{aligned} \tag{2.17}$$

The left-sided inequality of Theorem 2 can be easily seen from inequalities (2.15) and (2.17). Furthermore, for $0\le\tau\le t$, $0\le\rho\le t$, we have

$$0\le|\tau-\rho|\le t.$$

Therefore, from (2.17), we obtain

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t \left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}|\mathscr{A}(\tau,\rho)|d\tau d\rho$$

$$\leq \frac{t\,\|f'\|_p\|g'\|_q}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_0^t\int_0^t\left[\ln\left(\frac{t}{\tau}\right)\right]^{\frac{\alpha}{k}-1}\left[\ln\left(\frac{t}{\rho}\right)\right]^{\frac{\beta}{k}-1}\frac{p(\tau)}{\tau}\frac{p(\rho)}{\rho}\,d\tau d\rho$$
$$= \|f'\|_n\|g'\|_m\; t\;\mathscr{H}_k^{\alpha}[p(t)]\;\mathscr{H}_k^{\beta}[p(t)]$$

which finish proof process of Theorem 2. □

Remark 2.1. If it is taken as $\beta = \alpha$ in above theorem, Theorem 2 reduces to Theorem 1.

3. Conclusion

In this paper, we have done a considerable work associated with classical and variants of Grüss type inequality, which actually connects the integral of the product of two functions with the product of their integrals. We have also presented the Chebyshev-Grüss type inequalities for Hadamard fractional integrals in the framework of parameter $k > 0$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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