Communications in Mathematics and Applications
Vol. 8, No. 2, pp. 155–169, 2017
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Special Issue on
Numerical Analysis and Applications
Editor: Prof. Dr. Abdalah Rababah

Parameterized Gregory Formula

Research Article

N. Khelil[1,*] and L. Djerou[2]

[1] *Department of Mathematics, M.K. University, Biskra, Algeria*
[2] *Department of Computer Science, M.K. University, Biskra, Algeria*
***Corresponding author:** khelilna@yahoo.fr

Abstract. In this article we prove that the Gregory Formula (G) can be optimized by minimizing some of their coefficients in the remainder term. Experimental tests prove that obtained Formula can be rendered a powerful formula for library use.

Keywords. Umbral calculus; Numerical integration; Gregory formula; Series expansions

MSC. 05A40; 65D32; 41A58

Received: February 6, 2017 **Revised:** April 29, 2017 **Accepted:** May 30, 2017

1. Introduction

Solving numerical integration is an important question in scientific calculations and engineering. Gregory's method is among the very first quadrature formulas ever described in the literature, dating back to James Gregory (1638-1675) ([2], [3], [6], [7]). It seems to have been highly regarded for centuries.

Consider the Gregory integration formula:

$$\int_0^n f(x)dx = \sum_{k=0}^{n} f(k) + \sum_{k\geq 0} \frac{\alpha_k}{k!}(\Delta_1^k f(0) + \Delta_{-1}^k f(n)), \tag{1.1}$$

where Δ_1 is the forward difference operator with step size 1. This formula has a sense so $n \geq 1$, In the contrary case an appropriate variable change will permit us to do the integral without no difficulty.

Our work is based on the observation that the spacing in (1.1) can be made arbitrary. This results in a formula of the form

$$\int_0^n f(x)dx = \sum_{k=0}^{n} f(k) + \sum_{k\geq 0} \frac{\alpha_k(h_1,h_2,\dots,h_k)}{k!}\left(\Delta_{h_k}^k f(0) + \Delta_{-h_k}^k f(n)\right), \tag{1.2}$$

where Δ_h is the forward difference operator with step size h. To justify the formula (1.2) we shall use the umbral methods developed by Rota and his school [8]- [11], instead of classical generating function technique. Our goal is to find parameters h_k minimizing the absolute values of the coefficients $\alpha_k(h_1,h_2,\dots,h_k)$ from certain row k.

This paper is organized as follows: after introduction in Section 1, we recall some basic definitions related to this article in Section 2. And we discuss the theorem of expansion a formal series by a series delta. In Section 2.2 we will prove, if $p_k(x)$ is associated sequence for any $f(t) \in F$, then for any $h(t) \in F$ is written, $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|p_k(x)\rangle}{k!} f_k(t)$, this result generalizes, $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|p_k(x)\rangle}{k!} f^k(t)$. Finally, an example is given to illustrated our theoretical result.

Section 3 the most important part of our work, it is to propose and justify a generalization of the Gregory formula. In Section 4, the proposed algorithm is described, tested on various functions reputed badely integrate. Finally, conclusions are presented in Section 5.

2. Preliminary

This section reviews some of the basic definitions related to this article; we start by discussing what the algebra of formel power series, and what linear functionals are also, we discuss the theorem of expansion a formal series by a series delta. Finally, we give an example of application of this theorem (see [1, 5, 8–13]).

2.1 The Algebra of Formal Power Series

We note F the K-Algebra ($K = R$ or C) of the formal series

$$f(t) = \sum_{n=0}^{\infty} \alpha_k t^k. \tag{2.1}$$

Its support is the set of indices k such that $\alpha_k \neq 0$. The smallest element of this set is called the order of $f(t)$. The subalgebra of F, of the polynomials of one undeterminate, will be noted P. The degree $\deg(p(x))$ of a polynomial $p(x)$ is the largest k such that $\alpha_k \neq 0$.

Let $\mathbf{P}^*$ be the vector space of all linear functional on $\mathbf{P}$.

Therefore, the formal power series $f(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k$ defines a linear functional on $\mathbf{P}$ by setting

$$\langle f(t) \mid x^n \rangle = \alpha_n, \quad \text{for all } n \geq 0. \tag{2.2}$$

In particular

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k} = \begin{cases} n!, & n = k \\ 0, & n \neq k. \end{cases}$$

Actually, any linear functional L in P^* has the form (2.1). If

$$f_L(t) = \sum_{n=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k. \tag{2.3}$$

then from (2.2) we get $\langle f_L \mid x^n \rangle = \langle L \mid x^n \rangle$ and so linear functionals $L = f_L$.

The application $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbf{P}^*$ onto $\mathbf{F}$ [9].

As example, the functional $f(t)$ that satisfies

$$\langle f(t) \mid p(x) \rangle = \int_0^n p(u)du.$$

for all polynomial $p(x)$ can be determined as:

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k = \sum_{k=0}^{\infty} \frac{\int_0^y u^k du}{k!} t^k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} t^k = \frac{e^{nt}-1}{t}.$$

So

$$f(t) = \frac{e^{nt}-1}{t}.$$

2.2 Expansion a Formal Series by a Series Delta

Following Roman [9] we will prove, if $p_k(x)$ is associated sequence for any $f(t) \in F$, then for any $h(t) \in F$ Is written, $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | p_k(x) \rangle}{k!} f_k(t)$, this result generalizes, $h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | p_k(x) \rangle}{k!} f^k(t)$.

Proposition 2.1. *If $f(t) \in \boldsymbol{F}$, then*

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k.$$

We have

$$\left\langle \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} t^k \,\middle|\, x^n \right\rangle = \sum_{k=0}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{k!} \langle t^k \mid x^n \rangle = \langle f(t) \mid x^n \rangle.$$

A sequence $g_k(t)$ for which $0(g_k(t)) = k$ forms pseudobasis for $\mathbf{F}$. In other words, for each series $f(t)$ there is a unique sequence of constants α_k for which

$$f(t) = \sum_{k=0}^{\infty} \alpha_k g_k(t).$$

In particular, the powers of delta series form a pseudobasis for $\mathbf{F}$.

Proposition 2.2. *If $p(x) \in \boldsymbol{P}$, then*

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x) \rangle}{k!} x^k.$$

We have

$$\left\langle \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x)\rangle}{k!} x^k \,\middle|\, t^n \right\rangle = \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x)\rangle}{k!} \langle x^k \mid t^n\rangle$$
$$= \sum_{k=0}^{\infty} \frac{\langle t^k \mid x^n\rangle}{k!} \langle x^k \mid t^n\rangle$$
$$= \langle x^n \mid t^n\rangle, \quad n \geq 0$$
$$= \langle p(x) \mid t^n\rangle.$$

Proposition 2.3. *If* $0(f_k(t)) = k$, *for all* $k \geq 0$, *then*

$$\left\langle \sum_{k=0}^{\infty} \alpha_k f_k(t) \,\middle|\, p(x) \right\rangle = \sum_{k=0}^{\infty} \alpha_k \langle f_k(t) \mid p(x)\rangle,$$

for all $p(x)$ *in* $\boldsymbol{P}$.

Suppose that $deg(p(x)) = d$, then

$$\left\langle \sum_{k=0}^{\infty} \alpha_k f_k(t) \,\middle|\, p(x) \right\rangle = \left\langle \sum_{k=0}^{d} \alpha_k f_k(t) \,\middle|\, p(x) \right\rangle + \left\langle \sum_{k=d+1}^{\infty} \alpha_k f_k(t) \,\middle|\, p(x) \right\rangle$$
$$= \left\langle \sum_{k=0}^{d} \alpha_k f_k(t) \,\middle|\, p(x) \right\rangle$$
$$= \sum_{k=0}^{d} \alpha_k \langle f_k(t) \mid p(x)\rangle$$
$$= \sum_{k=0}^{\infty} \alpha_k \langle f_k(t) \mid p(x)\rangle.$$

Proposition 2.4. *If* $0(f_k(t)) = k$ *(if* $f_k(t)$ *is a delta series), for all* $k \geq 0$ *and if*

$$\langle f_k(t) \mid p(x)\rangle = \langle f_k(t) \mid q(x)\rangle,$$

for all k, *then* $p(x) = q(x)$.

Since the sequence $f_k(t)$, forms a pseudobasis for $\mathbf{F}$, for all $n \geq 0$ there exist constants $\alpha_{n,k}$ for which

$$t^n = \sum_{k=0}^{\infty} \alpha_{n,k} f_k(t).$$

Thus

$$\langle t^n \mid p(x)\rangle = \left\langle \sum_{k=0}^{\infty} \alpha_{n,k} f_k(t) \,\middle|\, p(x) \right\rangle$$
$$= \sum_{k=0}^{\infty} \alpha_{n,k} \langle f_k(t) \mid p(x)\rangle$$
$$= \sum_{k=0}^{\infty} \alpha_{n,k} \langle f_k(t) \mid q(x)\rangle$$

$$= \left\langle \sum_{k=0}^{\infty} \alpha_{n,k} f_k(t) \middle| q(x) \right\rangle$$
$$= \langle t^n \mid q(x) \rangle$$

and so Proposition 2.2 shows that

$$\begin{aligned} p(x) &= \sum_{k=0}^{\infty} \frac{\langle t^k \mid p(x) \rangle}{k!} x^k \\ &= \sum_{k=0}^{\infty} \frac{\langle t^k \mid q(x) \rangle}{k!} x^k \\ &= q(x). \end{aligned}$$

Proposition 2.5. *If* $\deg(p_k(x)) = k$, *for all* $k \geq 0$ *and if*

$$\langle f(t) \mid p_k(x) \rangle = \langle g(t) \mid p_k(x) \rangle,$$

for all k, *then* $f(t) = g(t)$.

For each $n \geq 0$ there exist constants $\alpha_{n,k}$ for which

$$x^n = \sum_{k=0}^{N} \alpha_{n,k} p_k(x).$$

Thus

$$\begin{aligned} \langle f(t) \mid x^n \rangle &= \sum_{k=0}^{N} \alpha_{n,k} \langle f(t) \mid p_k(x) \rangle \\ &= \sum_{k=0}^{N} \alpha_{n,k} \langle g(t) \mid p_k(x) \rangle \\ &= \langle g(t) \mid x^n \rangle. \end{aligned}$$

and so Proposition 2.1 shows that $f(t) = g(t)$.

By a sequence $p_n(x)$ in $\mathbf{P}$ we shall always imply that $\deg(p_n(x)) = n$.

Theorem 2.1. *Let* $f_k(t)$ *be a delta series. Then exists a unique sequence* $p_n(x)$ *of polynomials satisfying the orthogonality conditions*

$$\langle f_k(t) \mid p_n(x) \rangle = n! \delta_{n,k}, \tag{2.4}$$

for all $n, k \geq 0$.

The uniqueness follows from Proposition 2.5.

If $\langle f_k(t) \mid p_n(x) \rangle = \langle f_k(t) \mid q_n(x) \rangle$ then $p_n(x) = q_n(x)$.

For the existence, suppose

$$p_n(x) = \sum_{j=0}^{n} \alpha_{n,j} x^j,$$

where $\alpha_{n,n} \neq 0$, and

$$f_k(t) = \sum_{i=k}^{\infty} \beta_{k,i} t^i,$$

where $\beta_{n,n} \neq 0$, then (2.4) becomes

$$n!\delta_{n,k} = \left\langle \sum_{i=k}^{\infty} \beta_{k,i} t^i \,\middle|\, \sum_{j=0}^{n} \alpha_{n,j} x^j \right\rangle = \sum_{i=k}^{\infty} \sum_{j=0}^{n} \beta_{k,i} \alpha_{n,j} \langle t^i \mid x^j \rangle,$$

since $\langle t^i \mid x^j \rangle = i!$ for $i = j$, therefore,

$$n!\delta_{n,k} = \sum_{i=k}^{n} \alpha_{n,j} i!.$$

Taking $k = n$, we obtain $n! = \beta_{n,n} \alpha_{n,n} n!$. Therefore,

$$\alpha_{n,n} = \frac{1}{\beta_{n,n}}.$$

Taking $k = n-1$,

$$n!\delta_{n,n-1} = \sum_{i=n-1}^{n} \beta_{n-1,i} \alpha_{n,i} i!$$

$$0 = \beta_{n-1,n-1} \alpha_{n,n-1} (n-1)! + \beta_{n-1,n} \alpha_{n,n} (n)!$$

so,

$$\alpha_{n,n-1} = -\frac{\beta_{n-1,n} \alpha_{n,n}}{\beta_{n-1,n-1}}$$

By successively taking $k = n, n-1, \dots, 0$. We obtain a triangular system of equations that can be solved for $\alpha_{n,k}$.

Definition 2.1. We say that the sequence $p_n(x)$ in Theorem 2.1 is the sequence of polynomials associated for $f_k(t)$.

Theorem 2.2 (Expansion theorem)**.** *Let $f_k(t)$ be a delta series. Then for any $h(t)$ in $\boldsymbol{F}$*

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid p_k(x) \rangle}{k!} f_k(t).$$

From Proposition 2.3 we have

$$\left\langle \sum_{k=0}^{\infty} \frac{\langle h(t) \mid p_k(x) \rangle}{k!} f_k(t) \,\middle|\, p_n(x) \right\rangle = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid p_k(x) \rangle}{k!} \langle f_k(t) \mid p_n(x) \rangle$$
$$= \frac{\langle h(t) \mid p_n(x) \rangle}{n!} n!$$
$$= \langle h(t) \mid p_n(x) \rangle.$$

From Proposition 2.5 we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid p_k(x) \rangle}{k!} f_k(t).$$

Corollary 2.1. *Let $f_k(t)$ be a delta series and let $p_n(x)$ be the sequence of polynomials associated for $f_k(t)$. Then*

$$p_n(x) = \overline{f_n^c}(x),$$

where $\overline{f_n^c}(x)$ is the compositional inverse of $f_n^c(x)$ (conjugate of $f_n(x)$).

From the expansion theorem, for $a \in R$ we have

$$e^{at} = \sum_{n=0}^{\infty} \frac{\langle e^{at} \mid p_n(x)\rangle}{n!} f_n(t) = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} f_n(t)$$

then, we have,

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} t^k = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} f_n(t)$$

therefore,

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \overline{f_k}(t) = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} t^n$$

so,

$$\sum_{n=0}^{\infty} \frac{\overline{f_n^c}(a)}{k!} t^n = \sum_{n=0}^{\infty} \frac{p_n(a)}{n!} t^n$$

we get,

$$\overline{f_n^c}(a) = p_n(a)$$

so,

$$p_n(x) = \overline{f_n^c}(x).$$

In other words,

$$M(p_n) = M\left(\overline{f_n^c}\right).$$

2.3 Illustration

From The Expansion Theorem, the functional $f(t) = \frac{e^{nt}-1}{t}$ can be developed by using the delta series

$$f_k(t) = \left(e^{h_k t} - 1\right)^k, \quad k \geq 0,$$

where h_k non-zero parameters.

We have

$$\frac{e^{nt}-1}{t} = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} \left(e^{h_k t} - 1\right)^k,$$

where

$$\alpha_k = \left\langle \frac{e^{nt}-1}{t} \middle| p_k(x) \right\rangle,$$

$p_k(x)$ is the sequence of polynomials associated for $f_k(t)$.

$p_k(x)$ can be determined by using Corollary 2.1;

$$M(p_n) = M\left(\overline{f_n^c}\right).$$

We have,

$$f_1(t) = \left(e^{h_1 t} - 1\right) = \sum_{n=1}^{\infty} \frac{h_1^n}{n!} t^n$$

$$f_2(t) = \left(e^{h_2 t} - 1\right)^2 = \left(\sum_{n=1}^{\infty} \frac{h_2^n}{n!} t^n\right)^2$$

$$f_3(t) = \left(e^{h_3 t} - 1\right)^3 = \left(\sum_{n=1}^{\infty} \frac{h_3^n}{n!} t^n\right)^3$$

$$\vdots$$

$$f_k(t) = \left(e^{h_k t} - 1\right)^k = \left(\sum_{n=1}^{\infty} \frac{h_k^n}{n!} t^n\right)^k.$$

Suppose $\frac{h_k^n}{n!} = C_k^n$, for $k, n = 1, 2, \dots$, so

$$M(f_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^2 & C_2^1 \cdot C_2^1 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^3 & C_2^2 \cdot C_2^2 & C_3^1 \cdot C_3^1 \cdot C_3^1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & C_1^k & C_2^{k-1} \cdot C_2^{k-1} & C_3^{k-2} \cdot C_3^{k-2} \cdot C_3^{k-2} & \cdots & C_k^1 \cdot C_k^1 \cdot C_k^1 \cdot \cdots \cdot C_k^1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Which can be written

$$M(f_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^1 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^2 & C_2^1(2) & 0 & \cdots & 0 & \cdots \\ 0 & C_1^3 & C_2^2(2) & C_3^1(3) & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & C_1^k & C_2^{k-1}(2) & C_3^{k-2}(3) & \cdots & C_k^1(k) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Where $C_k^n(i) = C_k^n \cdot C_k^n \cdot \dots \cdot C_k^n$, i times. So,

$$M\left(f_k^c\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & C_1^1 & C_1^2 & C_1^3 & \cdots & C_1^k & \cdots \\ 0 & 0 & C_2^1(2) & C_2^2(2) & \cdots & C_2^{k-1}(2) & \cdots \\ 0 & 0 & 0 & C_3^1(3) & \cdots & C_3^{k-2}(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_k^1(k) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Thus,

- $M\left(f_0^c\right) = 1$. So, $M\left(\overline{f_0^c}\right) = 1$. And $p_0(x) = 1$, and consequently

  $$\alpha_0 = \int_0^n dx = n\,.$$

- $M\left(f_1^c\right) = \begin{pmatrix} 1 & 0 \\ 0 & C_1^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h_1 \end{pmatrix}.$

  So, $M\left(\overline{f_1^c}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{h_1} \end{pmatrix}$.

  And $p_1(x) = \frac{1}{h_1}x$, and consequently

  $$\alpha_1 = \int_0^n \frac{1}{h_1} x dx = \frac{n^2}{2h_1}.$$

- $M\left(f_2^c\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_1^1 & C_1^2 \\ 0 & 0 & C_2^1(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h_1 & \frac{h_1^2}{2!} \\ 0 & 0 & h_2^2 \end{pmatrix}.$

  So, $M\left(\overline{f_2^c}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{h_1} & \frac{-h_1}{2h_2^2} \\ 0 & 0 & \frac{1}{h_2^2} \end{pmatrix}$.

  And $p_2(x) = \dfrac{-h_1}{2h_2^2}x + \dfrac{1}{h_2^2}x^2$, and consequently

  $$\alpha_2 = \int_0^n \left(\frac{-h_1}{2h_2^2}x + \frac{1}{h_2^2}x^2\right) dx = \frac{-h_1}{4h_2^2}n^2 + \frac{1}{3h_2^2}n^3.$$

In the same way, we calculate $\alpha_3, \alpha_4 \cdots$.

3. Parameterized Gregory Formula

3.1 Plausibility of the formula

Consider the parametized Gregory formula (PG) [4],

$$\int_0^n f(x)dx = \sum_{k=0}^{n} f(k) + \sum_{k\geq 0} \frac{\alpha_k}{k!}\left(\Delta_{h_k}^k f(0) + \Delta_{-h_k}^k f(n)\right). \tag{3.1}$$

with end corrections where Δ_h is the forward difference operator with step size h.

Note that for $h_k = 1$ ($k = 1,2,\dots$), the formula (3.1) reduces to the classical Gregory integration formula [1],

$$\int_0^n f(x)dx = \sum_{k=0}^{n} f(k) + \sum_{k\geq 0} \frac{\alpha_k}{k!}\left(\Delta_{1}^k f(0) + \Delta_{-1}^k f(n)\right). \tag{3.2}$$

The formula (3.1) has a sense so $n \geq 1$. In the contrary case an appropriate variable change will permit us to do the integral without no difficulty.

To justify the formula (3.1) we shall use the umbral methods developed by Rota and his school ([5], [8], [9], [10], [11]), instead of classical generating function technique.

So, we shall replace $f(x)$ by e^{tx} (e^{tx} is the generating function of the sequence $\frac{t^n}{n!}$).

We have,

$$\int_0^n e^{tx}dx = \frac{e^{nt}-1}{t}.$$

$$\sum_{k=0}^{n} e^{tk} = \frac{\left(e^t\right)^{n+1}-1}{e^t-1} = \frac{e^{nt}.e^t-1}{e^t-1}.$$

$$\Delta_{h_k}^k e^{tx} = e^{tx}\left(e^{th_k}-1\right)^k.$$

Then (3.1) becomes

$$\frac{e^{nt}-1}{t} = \frac{e^{nt}\cdot e^t-1}{e^t-1} + \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{th_k}-1\right)^k + \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{-th_k}-1\right)^k e^{tn},$$

so,

$$\frac{e^{nt}}{t} - \frac{1}{t} = \frac{e^{nt}\cdot e^t}{e^t-1} - \frac{1}{e^t-1} + \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{th_k}-1big\right)^k + e^{nt}\sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{-th_k}-1\right)^k,$$

so,

$$e^{nt}\left(\frac{1}{t} - \frac{e^t}{e^t-1} - \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{-th_k}-1\right)^k\right) + \left(\frac{1}{-t} + \frac{1}{e^t-1} - \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{th_k}-1\right)^k\right) = 0,$$

so,

$$e^{nt}\left(\frac{1}{t} + \frac{1}{e^{-t}-1} - \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{-th_k}-1\right)^k\right) + \left(\frac{1}{-t} + \frac{1}{e^t-1} - \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{th_k}-1\right)^k\right) = 0.$$

Suppose that

$$G(t) = \frac{1}{t} + \frac{1}{e^{-t}-1} - \sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{-th_k}-1\right)^k,$$

then (3.1) becomes

$$e^{nt}G(t)+G(-t)=0.$$

we want the formula (3.1) that is independent of n. So $G(t)=0$; from the Theorem 2.2 (Expansion Theorem), we have

$$\frac{1}{e^t-1}-\frac{1}{t}=\sum_{k\geq 0}\frac{\alpha_k}{k!}\left(e^{th_k}-1\right)^k,$$

where

$$\alpha_k=\left\langle \frac{1}{e^t-1}-\frac{1}{t}\,\middle|\, p_k(x)\right\rangle.$$

$p_k(x)$ is the sequence of polynomials associated for $\left(e^{th_k}-1\right)^k$.

Suppose that

$$f(t)=\frac{1}{e^t-1}-\frac{1}{t},$$

so,

$$f(t)=\frac{t-\left(e^t-1\right)}{t(e^t-1)},$$

so,

$$\left(e^t-1\right)\cdot f(t)=1-\frac{\left(e^t-1\right)}{t},$$

and suppose that

$$f(t)=\sum_{n\geq 0}\gamma_n t^n,$$

so,

$$\sum_{k\geq 0}\frac{t^k}{(k+1)!}\sum_{k\geq 0}\gamma_k t^k=-\sum_{k\geq 0}\frac{t^k}{(k+2)!},$$

so,

$$\sum_{k\geq 0}\left(\sum_{n=0}^{k}\frac{1}{(k+1)!}\gamma_{n-k}\right)t^k=-\sum_{k\geq 0}\frac{t^k}{(k+2)!},$$

then the last equality is equivalent to the system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 & \cdots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 & \cdots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\cdot\begin{bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\\ \gamma_3\\ \vdots\\ \gamma_n\\ \vdots\end{bmatrix}=\begin{bmatrix}-\frac{1}{2!}\\ -\frac{1}{3!}\\ -\frac{1}{4!}\\ -\frac{1}{5!}\\ \vdots\\ -\frac{1}{(n+2)!}\\ \vdots\end{bmatrix}.$$

Therefore

$$\gamma_0 = \frac{-1}{2},\ \gamma_1 = \frac{1}{12},\ \gamma_2 = 0,\ \gamma_3 = \frac{-1}{720},\ \gamma_4 = 0, \cdots.$$

So,

$$\begin{aligned}
\alpha_0 = \langle f(t) \mid p_0(x)\rangle &= \langle f(t) \mid 1\rangle \\
&= \gamma_0 \\
&= -\frac{1}{2}, \\
\alpha_1 = \langle f(t) \mid p_1(x)\rangle &= \left\langle f(t) \middle| \frac{1}{h_1}x \right\rangle \\
&= \frac{1}{h_1}\gamma_1 \\
&= \frac{1}{12h_1}, \\
\alpha_2 = \langle f(t) \mid p_2(x)\rangle &= \left\langle f(t) \middle| -\frac{h_1}{2h_2^2}x + \frac{1}{h_2^2}x^2 \right\rangle \\
&= -\frac{h_1}{2h_2^2}\gamma_1 + \frac{1}{h_2^2}\gamma_2 \\
&= -\frac{h_1}{24h_2^2}.
\end{aligned}$$

In the same way, we find

$$\begin{aligned}
\alpha_3 &= \frac{1}{720h_3^3}\left(-10h_1^2 + 30h_2h_1 - 1\right), \\
\alpha_4 &= \frac{-1}{480h_4^4}\left(\frac{5}{3}h_1^3 - 10h_3h_1^2 - \frac{35}{3}h_2^2h_1 + 30h_3h_2h_1 - h_3\right), \\
\alpha_5 &= \frac{1}{60480h_5^5}(-42h_1^4 + 630h_2^3h_1 + 1050h_3^2h_2^2 - 3150h_3^2h_2h_1 + 420h_4h_1^3 - 2520h_4h_2^3h_1 \\
&\quad - 2940h_4h_2^2h_1 + 7560h_4h_3h_2h_1 + 105h_3^2 - 252h_4h_3 + 2).
\end{aligned}$$

4. Improvement of Gregory Formula

Recall that our goal is to prove that the Gregory Formula can be optimized by minimizing some of their coefficients in the remainder term. Truncate the right member of (3.1) at the 5th term, we get the approximation:

$$\begin{aligned}
\int_0^n f(x)dx \approx \sum_{k=0}^{n} f(k) + \alpha_0\big(f(0)+f(n)\big) + \alpha_1(h_1)\big(f(h_1) - f(0) + f(n-h_1) - f(n)\big) \\
+ \frac{\alpha_2(h_1,h_2)}{2!}\big(f(2h_2) - 2f(h_2) + f(0) + f(n-2h_2) - 2f(n-h_2) + f(n)\big)
\end{aligned}$$

$$+\frac{\alpha_3(h_1,\cdots,h_3)}{3!}\big(\Delta^3_{h_3}f(0)+\Delta^3_{-h_3}f(n)\big)+\frac{\alpha_4(h_1,\cdots,h_4)}{4!}\big(\Delta^4_{h_4}f(0)+\Delta^4_{-h_4}f(n)\big)$$
$$+\frac{\alpha_5(h_1,\cdots,h_5)}{5!}\big(\Delta^5_{h_5}f(0)+\Delta^5_{-h_5}f(n)\big). \tag{4.1}$$

For $\alpha_3(h_1,\ldots,h_3)$, $\alpha_4(h_1,\ldots,h_4)$ and $\alpha_5(h_1,\ldots,h_5)$ smallest possible the formula will have a simple form with a number limited of evaluations

$$\int_0^n f(x)dx \approx \sum_{k=0}^{n} f(k)-\frac{1}{2}\big(f(0)+f(n)\big)+\frac{1}{12h_1}\big(f(h_1)-f(0)+f(n-h_1)-f(n)\big)$$
$$-\frac{h_1}{48h_2^2}\big(f(2h_2)-2f(h_2)+f(0)+f(n-2h_2)-2f(n-h_2)+f(n)\big).$$

To this end; we try to determine h_1,h_2,h_3,h_4 and h_5; we take $h_4,h_5=1$, in this study as parameters and let's solve this non linear system.

The problem is reduced to solve the system:

$$\begin{cases}\alpha_3(h_1,\ldots,h_3)=0,\\ \alpha_4(h_1,\ldots,h_4)=0,\\ \alpha_5(h_1,\ldots,h_5)=0.\end{cases}$$

is about problem resolving:

$$S\begin{cases}\big(1/720h_3^3\big)\big(-10h_1^2+30h_2h_1-1\big)=0,\\ \big(-1/480h_4^4\big)\big(5/3h_1^3-10h_3h_1^2-35/3h_2^2h_1+30h_3h_2h_1-h_3\big)=0,\\ \big(1/60480h_5^5\big)\big(-42h_1^4+630h_2^3h_1+1050h_3^2h_2^2-3150h_3^2h_2h_1+420h_4h_1^3-2520h_4h_3h_1^2\\ \quad -2940h_4h_2^2h_1+7560h_4h_3h_2h_1+105h_3^2-252h_4h_3+2)\big)=0.\end{cases} \tag{4.2}$$

Thus, for $S\approx 0$, we have:

$$\int_0^n f(x)dx \approx \sum_{k=0}^{n} f(k)+a_0\big(f(0)+f(n)\big)+\frac{a_1(h_1)}{1!}\big(f(h_1)-f(0)+f(n-h_1)-f(n)\big)$$
$$+\frac{a_2(h_1,h_2)}{2!}\big(f(2h_2)-2f(h_2)+f(0)+f(n-2h_2)-2f(n-h_2)+f(n)\big).$$

The system (4.2) provides us the solution:

$$h_1=0.2633,\quad h_2=0.2144,\quad h_3=0.2113\quad (h_4=h_5=1).$$

Finally PG:

$$\int_0^n f(x)dx \approx \frac{1}{2}f(0)+f(1)+\ldots+f(n-1)+1/2f(n)$$
$$+0.1\big(6f(0.2)-5f(0)-f(0.4)+6f(n-0.2)-f(n-0.4)-5f(n)\big). \tag{4.3}$$

To test the performance of this algorithm we took various functions and we looked for an approximation with Gregory formula (G) and Parameterized Gregory's Formula (PG) Table 1.

Table 1. Comparison

Function	Interval	Exact valor	Formula	Approx. Valor	Rel. Error
$\exp(x)$	$[0,5]$	147.4131591025766	G	149.2289234815334	0.012317518938003
			PG	148.0530705285946	0.004340938284707
$\frac{1}{1+x^2}$	$[0,5]$	1.373400766945016	G	1.328883861236802	0.032413631024276
			PG	1.366857351423234	0.004764389011037
$\sqrt{x}$	$[0,10]$	21.081900000000001	G	20.98174461996889	0.004750775785442
			PG	21.06869053573215	0.000626578452029
$\frac{1}{1+x}$	$[0,2]$	1.098612288668110	G	1.111111111111111	0.011376918474264
			PG	1.104018629290736	0.004921063307221
$\frac{e^x}{\sqrt{e^x+1}}$	$[0,20]$	44050.10320780000	G	44099.75181831744	0.001127094079286
			PG	44076.98817671046	0.000610327035640

5. Conclusions

This paper has presented a new numerical integration formula PG. Experimental results on several well-known functions (badly to integrate by the classic methods ($\exp(x),\dots$) show that the proposed formula give good results and prove that obtained formula can be rendered a powerful formula for library use.

Acknowledgement

The authors wish to thank Dr. K.M. Belbahri for fruitful discussions on the Umbral Calculus. This research was supported in part, by the CNEPRU Under the contract N° B01420140100.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M.K. Belbahri, *Generalized Gregory formula*, Doctoral Thesis, Stevens Institute of Technology (1982).

[2] C.E. Froberg, *Introduction to Numerical Analysis*, Addison-Wesley, 2nd Edition (1969).

[3] R.W. Hamming and R.S. Pinkhan, A class of integration formulas, *The Association for Computing Machinery* **13** (1966), 430–438.

[4] N. Khelil, *Integration de Gregory avec Correction Parametisee*, Magister Thesis, University of Batna (1991).

[5] I. Niven, Formal power series, *The American Mathematical Monthly* **8** (76) (1969), 871–889.

[6] G.M. Phillips, Gregory's method for numerical integration, *The American Mathematical Society* **79** (3) (1972), 270 – 274.

[7] A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, 2nd edition, McGraw-Hill, New York (1978).

[8] S. Roman, *The theory of the Umbral Calculus, Mathematical Analysis and Applications*, Academic Press, Inc., New York **87** (1) (1982), 58–115.

[9] S. Roman, *The Umbral Calculus*, Academic Press, Inc., New York (1984).

[10] S. Roman and G.C. Rota, The umbral calculus, *Computers and Mathematics with Applications* **27** (1978), 95–188.

[11] G.C. Rota, *Finite Operator Calculus*, Academic Press, Inc., New York (1975).

[12] O.S. Zariski and P. Samuel, *Commutative Algebra*, Springer, D. Van Nostrand Company, Inc., **1** (1960).

[13] O.S. Zariski and P. Samuel, *Commutative Algebra*, Springer, D. Van Nostrand Company, Inc., **2** (1960).