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Application of Adaptation HAM for Nonlinear Oscillator Typified as A Mass Attached to A Stretched Elastic Wire

Research Article

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Abstract. This paper applies the *adaptation of homotopy analysis method* (AHAM) for the first time to obtained the periodic solutions for the oscillation of a mass attached to a stretched elastic wire. The AHAM approach can be applied directly to the governing equation without rewrite it in a form that does not contain the square-root expression. More precisely, with the help of the homotopy polynomials procedure the nonlinear term of the problem can be decompose as a series of polynomials to overcomes the difficulty arising in calculating complicated integrals. A comparative study between AHAM and other existing solutions obtained by several authors is conducted to demonstrate the simplicity and the efficiency of AHAM. The approximate frequency and periodic solution for both small and large amplitude of oscillations show a good agreement with the numerical solution.

Keywords. Homotopy analysis method; Homotopy polynomials; Nonlinear oscillation; Periodic solutions

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1. Introduction

In this work, we consider the non-dimensional equation of motion for a mass attached to a stretched elastic wire is [7]

$$\frac{d^2x}{dt^2}+x-\frac{\lambda x}{\sqrt{1+x^2}}=0, \quad 0\le\lambda\le 1, \tag{1.1}$$

subject to the initial conditions

$$x(0)=A, \quad \frac{dx}{dt}(0)=0. \tag{1.2}$$

Assume that the solutions (1.1) are periodic with the period $T=\frac{2\pi}{\Omega}$, where Ω is the frequency of oscillation. Substituting $\tau=\Omega t$, and $x(t)=X(\tau)$ in to (1.1), then we have

$$\Omega^2\frac{d^2X}{d\tau^2}+X-\frac{\lambda X}{\sqrt{1+X^2}}=0, \quad 0\le\lambda\le 1, \tag{1.3}$$

$$X(0)=A, \quad \frac{dX}{d\tau}(0)=0. \tag{1.4}$$

Note that for both the small and large X respectively, (1.3) becomes [21]

$$\Omega\approx\sqrt{1-\lambda} \quad \text{for } A\ll 1, \quad \text{and} \quad \Omega\approx 1 \ \text{ for } A\gg 1. \tag{1.5}$$

Problems of form (1.1)-(1.2) are encountered in the field of engineering because many practical engineering components consist of vibrating systems that can be modeled using oscillator systems. Exact/approximate solutions of these problems are of great importance due to its wide application in scientific research. Strongly nonlinear systems have been studied by several authors. J.H. He [5–7] used the modified Lindstedt–Poincare method and homotopy perturbation method (HPM) to search for approximate solutions of a certain class of Strongly nonlinear systems. L. Xu [22, 23] used parameter-expanding method. A. Belendez et al. [1] provided *modified homotopy perturbation method* (MHPM) to the solution of the above problems. Recently, some authors used energy balance method to obtain higher-order approximations for strongly nonlinear oscillators [2, 3, 8, 20].

The aim of this paper is to apply for the first time the *adaptation of homotopy analysis method* (AHAM), proposed by Odabit and Bataineh [19] to obtain the approximate solutions of the strongly nonlinear systems (1.3)–(1.4). The AHAM can decompose the nonlinear term of the problem as a series of polynomials to overcome the difficulty arising in calculating complected integrals.

2. The Adaptation of HAM

In this section, we will briefly describe the use of the MHAM for differential equation

$$\mathcal{N}[X(\tau)]=0, \tag{2.1}$$

where $\mathcal{N}$ is nonlinear operator, τ denotes the independent variable, $X(\tau)$ is an unknown function to be determined.

By means of generalizing the standard HAM, Liao [10] constructs the so-called *zeroth-order deformation equation*

$$(1-q)\mathcal{L}[\Phi(\tau;q)-X_0(\tau)] = q\hbar H(\tau)\{\mathcal{N}[\Phi(\tau;q)]\}, \tag{2.2}$$

such that

$$\Phi(\tau;q) = \sum_{m=0}^{\infty} q^m X_m(\tau), \tag{2.3}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, X_0 is an initial guess of $X(\tau)$ and $\mathcal{L}$ is an auxiliary linear operator. Differentiating (2.2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called *mth-order deformation equation*

$$\mathcal{L}[X_m(\tau) - \chi_m X_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{X}_{m-1}(\tau)), \tag{2.4}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1, \end{cases}$$

$$\vec{X}_{m-1}(\tau) = \{X_0(\tau), X_1(\tau), \ldots, X_{m-1}(\tau)\},$$

and

$$R_m(\vec{X}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\{\mathcal{N}[\phi(\tau;q)]\}}{\partial q^{m-1}}\bigg|_{q=0}. \tag{2.5}$$

Respectively, for more details about HAM please refer to [9–16] and others related works.

Now, the AHAM suggest that the nonlinear operator $\mathcal{N}$ can be expressed in Taylor series expansion as

$$\mathcal{N}(X) = \sum_{n=0}^{\infty} a_n X^n, \tag{2.6}$$

where $a_n's \in \mathbb{R}$. In view to construct the new *zeroth-order deformation equation*, define the homotopy map as $\Phi(\tau;q) : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$, so, (2.2) becomes as

$$(1-q)\mathcal{L}[\Phi(\tau;q)-X_0(\tau)] = q\hbar H(\tau)\mathcal{N}[q\Phi(\tau;q)]. \tag{2.7}$$

That is

$$(1-q)\mathcal{L}[\Phi(\tau;q)-X_0(\tau)] = q\hbar H(\tau)\sum_{n=0}^{\infty} a_n \left(\Phi(\tau;q)\right)^n q^n. \tag{2.8}$$

Obviously, when $q = 0$ and $q = 1$ both $\Phi(\tau;0) = X_0(\tau)$ and $\Phi(\tau;1) = X(\tau)$ hold. Thus as q increases from 0 to 1, the solutions $\Phi(\tau;0)$ vary from the initial guesses $X_0(\tau)$ to the solution $X(\tau)$. Differentiating (2.8) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called new *mth-order deformation equations*

$$\mathcal{L}[X_m(\tau) - \chi_m X_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{X}_{m-1}), \tag{2.9}$$

where

$$R_m(\vec{X}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\{\mathcal{N}[q\phi(\tau;q)]\}}{\partial q^{m-1}}\bigg|_{q=0}. \tag{2.10}$$

Therefor, the solution of the strongly nonlinear problem $\mathcal{N}(X(\tau))=0$, can be easily obtained as

$$X(\tau)=\sum_{m=0}^{\infty}X_m(\tau). \tag{2.11}$$

3. Solution Procedure by AHAM

To find the periodic motions Ω of (1.3) subject to initial conditions (1.4) by means of AHAM [19], we choose the initial approximation

$$X_0(\tau)=\mathrm{A}\cos(\tau), \tag{3.1}$$

and the linear operator

$$\mathcal{L}[\Phi(\tau;q)]=\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2}+\Phi(t;q), \tag{3.2}$$

with the property

$$\mathcal{L}[c_1\cos(\tau)+c_2\sin(\tau)]=0, \tag{3.3}$$

where c_i $(i=1,2)$ are constants of integration. Eq. (1.3) suggests that we define a nonlinear operator

$$\begin{aligned}\mathcal{N}[q\Phi(\tau;q),\Omega(\mathrm{A};q)]&=(\Omega(\mathrm{A};q))^2\frac{\mathrm{d}^2\Phi(\tau;q)}{\mathrm{d}\tau^2}+\Phi(\tau;q)\\&\quad-\lambda\left(\Phi(\tau;q)+\sum_{j=1}^{\infty}(-1)^j\frac{(2j-1)!\Phi(\tau,q)^{2j+1}}{2^{2j-1}j!(j-1)!}q^{2j+1}\right),\end{aligned} \tag{3.4}$$

where $\Omega(\mathrm{A};q)=\sum_{m=0}^{\infty}\Omega_m q^m$ and $\Phi(t;q)=\sum_{m=0}^{\infty}X_m(\tau)q^m$.

Using the above definition, we construct the *zeroth-order deformation equation* as in (2.8) and the *mth-order deformation equation* for $m\geq 1$ is as in (2.9) with c_i $(i=1,2)$ are zero, where

$$\begin{cases}R_1(X_0)=-\lambda X_0+\Omega_0^2X_0''+X_0,\\ R_2(X_1)=-\lambda X_1+\Omega_0^2X_1''+X_1+2\Omega_1\Omega_0X_0'',\\ R_3(X_2)=-\lambda X_2+\Omega_0^2X_2''+X_2+2\Omega_1\Omega_0X_1''+2\Omega_2\Omega_0X_0''+\Omega_1^2X_0'',\\ R_m(X_3)=-\lambda X_3+\Omega_0^2X_3''+X_3+2\Omega_1\Omega_0X_2''+2\Omega_2\Omega_0X_1''+\Omega_1^2X_1''\\ \qquad\qquad +2\Omega_1\Omega_2X_0''+2\Omega_0\Omega_3X_0''+\frac{1}{2}{X_0}^3\\ \qquad\qquad\vdots\end{cases} \tag{3.5}$$

Now, the solution of (2.9) for $m\geq 1$ becomes

$$X_m(\tau)=\chi_m X_{m-1}(\tau)+\hbar L^{-1}R_m(\vec{X}_{m-1}(\tau)). \tag{3.6}$$

We now successively obtain for $\tau = \frac{\pi}{2}$, $X_m(\tau) = 0$ and $0 < \lambda < 1$

$$
\begin{aligned}
X_1(\tau) &= -\frac{1}{4}\pi A\hbar\left(\lambda + \Omega_0^2 - 1\right), && \Rightarrow \quad \Omega_0 = \sqrt{1-\lambda},\\
X_2(\tau) &= \frac{1}{2}\pi A\hbar\sqrt{1-\lambda}\,\Omega_1, && \Rightarrow \quad \Omega_1 = 0,\\
X_3(\tau) &= \frac{1}{2}\pi A\hbar\sqrt{1-\lambda}\,\Omega_2, && \Rightarrow \quad \Omega_2 = 0,\\
X_4(\tau) &= \frac{1}{32}\pi A\hbar\left(3A^2\lambda + 16\sqrt{1-\lambda}\Omega_3\right), && \Rightarrow \quad \Omega_3 = -\frac{3\lambda}{16\sqrt{1-\lambda}}A^2,\\
X_5(\tau) &= \frac{1}{2}\pi A\hbar\sqrt{1-\lambda}\Omega_4, && \Rightarrow \quad \Omega_4 = 0, \text{ etc.}
\end{aligned}
$$

Then the series solutions expression by AHAM is

$$X(\tau) = \sum_{m=0}^{\infty} X_m(\tau) = A\cos(\tau) = A\cos(\Omega t), \tag{3.7}$$

where the approximate frequencies $\Omega(A)$ is

$$
\begin{aligned}
\Omega(A) &= \sum_{m=0}^{\infty} \Omega_m(A)\\
&= \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}}A^2 + \frac{3\lambda(17\lambda-20)}{512(1-\lambda)^{3/2}}A^4 + \frac{\lambda\left(547\lambda^2 - 1220\lambda + 700\right)}{8192(1-\lambda)^{5/2}}A^6\\
&\quad + \frac{15\lambda\left(1741\lambda^3 - 5672\lambda^2 + 6256\lambda - 2352\right)}{524288(1-\lambda)^{7/2}}A^8 - \frac{21\lambda(821\lambda-1386)}{524288(1-\lambda)^{3/2}}A^{10} + \cdots .
\end{aligned} \tag{3.8}
$$

Eq. (3.8) can be written as a closed form of complete elliptic integrals of the first and second kind $E(-A^2)$ and $K(-A)$, respectively as

$$\Omega(A) = \sqrt{1 - \frac{4\lambda}{\pi A^2}[E(-A^2) - K(-A^2)]} . \tag{3.9}$$

For more details about the complete elliptic integrals of the first and second kind, $K(m)$ and $E(m)$ (see [18]). The exact angular frequency, $\Omega_e(A)$, of Eq. (1.3) subject to (1.4) was found in [1] as

$$\Omega_e(A) = \frac{\pi}{2A}\left[\int_0^1 \frac{du}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{A^2+1} - \sqrt{A^2u^2+1}\,)}}\right]^{-1} . \tag{3.10}$$

Moreover, the approximate frequencies $\Omega(A)$ for $\lambda = 1$ obtained by AHAM is

$$
\begin{aligned}
\Omega(A) &= \sum_{m=0}^{\infty} \Omega_m(A)\\
&= \frac{1}{2}\sqrt{\frac{3}{2}}A - \frac{5}{8}\sqrt{\frac{3}{2}}A^3 - \frac{25}{64}\sqrt{\frac{3}{2}}A^5 + \cdots .
\end{aligned} \tag{3.11}
$$

4. Results and Discussions

To check the accuracy of the present method, we plot the approximate frequency obtained by AHAM Eq. (3.9) with the corresponding exact frequency Eq. (3.10) for different values of λ in the interval $[0,100]$, it is clear that the result agree very well withe exact frequency (considered to be exact). Tables 1–2 shows in details the comparison of the approximate frequencies obtained by the present method Eq. (3.9) with the exact frequency $\Omega_e(\mathrm{A})$ and other existing frequencies that are obtained by [4,17,24] for $\lambda = 0.5, 0.95$. The results of these tables are demonstrated that the approximate frequencies given by AHAM agree very well with withe exact frequency which is better result than those obtained in [4,17,24]. The comparison of the approximate solution obtained by the present method (Eq. (3.5)) with the numerical solution obtained by the classical fourth-order Runge-Kutta method (RK4) at the stepsize $\Delta t = 0.001$ and other existing solutions that are obtained by [4,17,24] for $\lambda = 0.95$ and $A = 10$ are show in Table 3. The approximate solution given by AHAM agree very well with RK4 better than those obtained in [4,17,24].

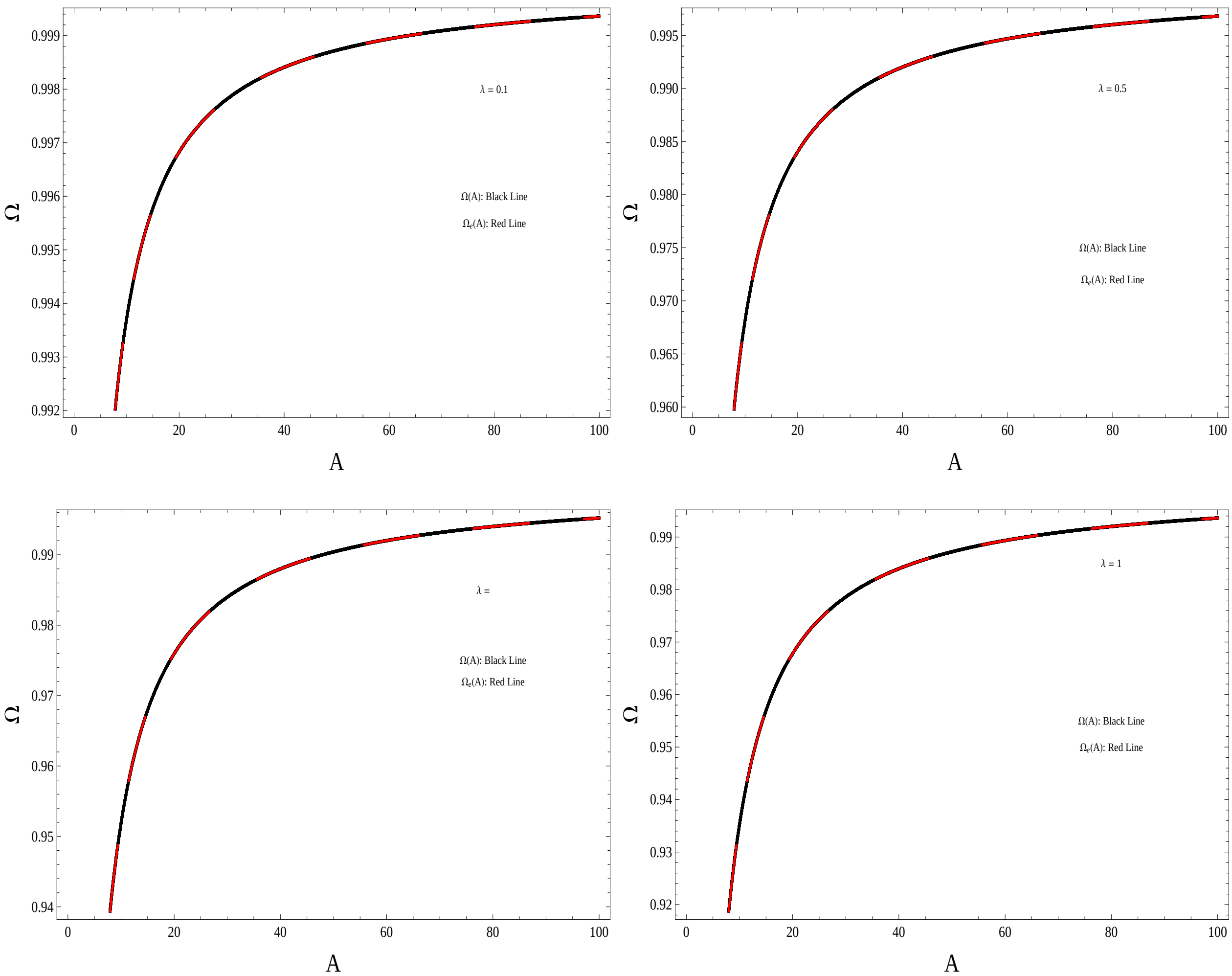

Figure 1. Comparison of the approximate frequencies $\Omega(\mathrm{A})$ with the corresponding exact frequency $\Omega_e(\mathrm{A})$ for different values of λ.

Table 1. Comparison of the approximate frequencies obtained by the present method Eq. (3.9) with the exact frequency $\Omega_e(A)$ and other existing frequencies that are obtained by [4,17,24] for $\lambda = 0.5$.

A	$\Omega_e(A)$	Mickens [17]	Ganji et al. [4]	Zhao [24]	Present study
0.01	0.707120	0.707116	0.707120	0.866037	0.707120
0.1	0.708423	0.707987	0.708424	0.867170	0.708423
0.2	0.712259	0.710582	0.712271	0.870489	0.712262
0.4	0.726126	0.720330	0.726271	0.882252	0.726162
0.6	0.745140	0.734651	0.745683	0.897720	0.745269
0.8	0.765907	0.751536	0.767072	0.913595	0.766147
1	0.786171	0.769254	0.788075	0.927961	0.786524
2	0.860447	0.843401	0.864865	0.969782	0.860963
3	0.899904	0.887017	0.904671	0.984638	0.900303
4	0.922727	0.912871	0.927153	0.990901	0.923016
5	0.937317	0.929471	0.941285	0.994030	0.937529
10	0.968102	0.964358	0.970480	0.998456	0.968168
100	0.996812	0.996459	0.997067	0.999984	0.996813
500	0.999363	0.999293	0.999414	0.999999	0.999363
1000	0.999682	0.999646	0.999707	0.999999	0.999682

Table 2. Comparison of the approximate frequencies obtained by the present method Eq. (3.9) with the exact frequency$\Omega_e(A)$ and other existing frequencies that are obtained by [4,17,24] for $\lambda = 0.95$.

A	$\Omega_e(A)$	Mickens [17]	Ganji et al. [4]	Zhao [24]	Present study
0.01	0.223686	0.223660	0.223686	0.312365	0.223686
0.1	0.231367	0.228836	0.231391	0.323516	0.231388
0.2	0.252549	0.243639	0.252836	0.354238	0.252792
0.4	0.317642	0.293022	0.319674	0.447114	0.319204
0.6	0.391035	0.354195	0.395577	0.547084	0.394094
0.8	0.459947	0.416090	0.46686	0.634908	0.463965
1	0.520335	0.473633	0.529168	0.706124	0.524765
2	0.709629	0.671950	0.721931	0.886069	0.713013
3	0.797913	0.771310	0.809330	0.943366	0.800024
4	0.846399	0.826640	0.856307	0.966748	0.847774
5	0.876561	0.861071	0.885117	0.978276	0.87751
10	0.938333	0.931114	0.943122	0.994413	0.938597
100	0.993933	0.993260	0.994420	0.999944	0.993935
500	0.998790	0.998656	0.998886	0.999998	0.998790
1000	0.999395	0.999328	0.999443	0.999999	0.999395

Table 3. Comparison of the approximate solution obtained by the present method Eq. (3.7) with the numerical solution obtained by fourth-order Runge–Kutta method (RK4) and other existing solutions that are obtained by [4, 17, 24] for $\lambda = 0.95$ and $A = 10$.

t	x_{RK4}	Mickens [17]	Ganji et al. [4]	Present study
0.00	10.000000	1.0000000	1.0000000	1.0000000
0.25	9.718510	9.730290	9.723320	9.72596
0.50	8.891522	8.935720	8.908602	8.91886
0.75	7.570393	7.659130	7.600920	7.62293
1.00	5.837117	5.969410	5.872640	5.9092
1.25	3.799078	3.957680	3.819390	3.8716
1.50	1.581589	1.732470	1.554800	1.62181
1.75	−0.691240	−0.586197	−0.795828	−0.716875
2.00	−0.2943400	−2.873240	−2.102420	−3.01627
2.25	−5.067420	−5.005300	−5.237340	−5.15034

5. Conclusions

In this paper, the *adaption homotopy analysis methods* (AHAM) was applied to determine the second-order approximation of strongly nonlinear oscillator systems. The AHAM Firstly, modify the standard HAM in order to reduce the required computational work and to overcome the difficulty arising in calculating complected integrals. Secondly, decompose the nonlinear term of the problem as a series of polynomials. The computation of the homotopy polynomials is a key procedure for the presented method. We conclude that the present technique is very effective and convenient for solving strongly nonlinear oscillator problems.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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