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Existence of Fixed Points for An Integral Operator via Fixed Point Theorem on Gauge Spaces

Research Article

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Abstract. In this paper we have discussed the existence of fixed points for an integral operator using a new fixed point theorem in the setting of gauge spaces.

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1. Introduction

As we know, without any doubt the Banach contraction principle is considered as the most fundamental entity in the metric fixed point theory and by research point of view, it won't be false to state that its inception has opened many closed doors. Experts in the relevant field have been continuously extending this famous contraction condition which is quite promising. For example Frigon [10], and Chis and Precup [7] generalized the Banach contraction principle on gauge spaces. One can characterize gauge spaces by the fact that the distance between two distinct points in a space may be zero, which has attracted researchers to gain interest in this field of metric fixed point theory. To go with this we also suggest some interesting results obtained by different authors in [1,4–6,9,11,13].

Wardowski [21] introduced a new family of mappings called F or $\mathfrak{F}$ family. Using the mappings from $\mathfrak{F}$ family he introduced a new contraction condition called F-contraction. This F-contraction nicely generalize the most famous Banach contraction condition. Later on, many researchers worldwide generalized this result (for example, see [2,3,8,12,14–20]).

The purpose of this paper is to introduce some F-type-contraction in the setting of gauge spaces and obtain some fixed point theorems for such mappings in gauge spaces. As an application of our result we establish an existence theorem for integral equations.

Wardowski [21] introduced the $\mathfrak{F}$ family in the way that: $\mathfrak{F}$ is the class of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the following three assumptions:

(F_1) F is strictly increasing, that is, for each $a_1,a_2\in(0,\infty)$ with $a_1<a_2$, we have $F(a_1)<F(a_2)$.

(F_2) For each sequence $\{\eth_n\}$ of positive real numbers we have $\lim_{n\to\infty}\eth_n=0$ if and only if $\lim_{n\to\infty}F(\eth_n)=-\infty$.

(F_3) There exists $k\in(0,1)$ such that $\lim_{\eth\to0^+}\eth^kF(\eth)=0$.

Following are some examples of such functions.

- $F_a=\ln x$ for each $x\in(0,\infty)$.
- $F_b=x+\ln x$ for each $x\in(0,\infty)$.
- $F_c=-\frac{1}{\sqrt{x}}$ for each $x\in(0,\infty)$.

Further, Wardowski [21] introduced F-contraction and related fixed point theorem in the following way:

Theorem 1.1 ([21])**.** *Let (X,d) be a complete metric space and let $T:X\to X$ is F-contraction, that is, there exist $F\in\mathfrak{F}$ and $\tau>0$ such that for each $x,y\in X$ with $d(Tx,Ty)>0$, we have*

$$\tau+F(d(Tx,Ty))\le F(d(x,y)).$$

Then T has a unique fixed point.

This theorem reduces to Banach contraction principle if T is F-contraction with $F(x)=\ln x$. Minak et al. [14] generalized this result as follows:

Theorem 1.2 ([14]). *Let (X,d) be a complete metric space and let $T:X\to X$. Assume that there exist $F\in\mathfrak{F}$ and $\tau>0$ such that*

$$\tau+F(d(Tx,Ty))\le F\left(\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}\right),$$

for each $x,y\in X$ with $d(Tx,Ty)>0$. If T or F is continuous, then T has a unique fixed point.

Now, we explain the gauge spaces and its terminologies due to [9].

Definition 1.3 ([9]). Let X be a nonempty set. A function $d:X\times X\to[0,\infty)$ is called pseudo metric on X if for each $x,y,z\in X$, we have

(i) $d(x,x)=0$ for each $x\in X$;

(ii) $d(x,y)=d(y,x)$;

(iii) $d(x,z)\le d(x,y)+d(y,x)$.

Let X be a nonempty set endowed with the pseudo metric d. The d-ball of radius $\epsilon>0$ centered at $x\in X$ is the set

$$B(x,d,\epsilon)=\{y\in X: d(x,y)<\epsilon\}.$$

Definition 1.4 ([9]). A family $\mathfrak{F}=\{d_\nu:\nu\in\mathfrak{A}\}$ of pseudo metrics is said to be separating if for each pair (x,y) with $x\neq y$, there exists $d_\nu\in\mathfrak{F}$ with $d_\nu(x,y)\neq 0$.

Definition 1.5 ([9]). Let X be a nonempty set and $\mathfrak{F}=\{d_\nu:\nu\in\mathfrak{A}\}$ be a family of pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases the family

$$\mathfrak{B}(\mathfrak{F})=\left\{B(x,d_\nu,\epsilon): x\in X, d_\nu\in\mathfrak{F}\text{ and }\epsilon>0\right\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of pseudo metrics. The pair $(X,\mathfrak{T}(\mathfrak{F}))$ is called a gauge space.

Definition 1.6 ([9]). Let $(X,\mathfrak{T}(\mathfrak{F}))$ be a gauge space with respect to the family $\mathfrak{F}=\{d_\nu:\nu\in\mathfrak{A}\}$ of pseudo metrics on X. Let $\{x_n\}$ is a sequence in X and $x\in X$. Then:

(i) The sequence $\{x_n\}$ converges to x if for each $\nu\in\mathfrak{A}$ and $\epsilon>0$, there exists $N_0\in\mathbb{N}$ such that $d_\nu(x_n,x)<\epsilon$ for each $n\ge N_0$.

(ii) The sequence $\{x_n\}$ is a cauchy sequence if for each $\nu\in\mathfrak{A}$ and $\epsilon>0$, there exists $N_0\in\mathbb{N}$ such that $d_\nu(x_n,x_m)<\epsilon$ for each $n,m\ge N_0$.

(iii) $(X,\mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X,\mathfrak{T}(\mathfrak{F}))$ is convergent in X.

(iv) A subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.

2. Main Results

Through out this paper, $\mathfrak{A}$ is directed set and X is a nonempty set endowed with a separating complete gauge structure $\{d_\nu:\nu\in\mathfrak{A}\}$.

Theorem 2.1. *Let $T : X \to X$ be a mapping for which we have F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\alpha(x,y) \geq 1 \Rightarrow \tau + F(d_\nu(Tx,Ty)) \leq F(a_\nu d_\nu(x,y) + b_\nu d_\nu(x,Tx) + c_\nu d_\nu(y,Ty) + e_\nu d_\nu(x,Ty) + L_\nu d_\nu(y,Tx)) \text{ for all } \nu \in \mathfrak{A} \tag{1}$$

for each $x, y \in X$, whenever $d_\nu(Tx,Ty) \neq 0$, where $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $a_\nu + b_\nu + c_\nu + 2e_\nu = 1$ for all $\nu \in \mathfrak{A}$. Further, assume that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(ii) *for each $x, y \in X$ with $\alpha(x,y) \geq 1$, we have $\alpha(Tx,Ty) \geq 1$;*

(iii) *for any sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By hypothesis (i), there exists $x_0 \in X$ with $\alpha(x_0, x_1) \geq 1$. Take $x_1 = Tx_0$. From (1), we have

$$\begin{aligned}
\tau + F(d_\nu(x_1,x_2)) &= \tau + F(d_\nu(Tx_0,Tx_1)) \\
&\leq F\big(a_\nu d_\nu(x_0,x_1) + b_\nu d_\nu(x_0,Tx_0) + c_\nu d_\nu(x_1,Tx_1), e_\nu d_\nu(x_0,Tx_1) + L_\nu d_\nu(x_1,Tx_0)\big) \\
&= F\big(a_\nu d_\nu(x_0,x_1) + b_\nu d_\nu(x_0,x_1) + c_\nu d_\nu(x_1,x_2), e_\nu d_\nu(x_0,x_2) + L_\nu .0\big) \\
&\leq F\big(a_\nu d_\nu(x_0,x_1) + b_\nu d_\nu(x_0,x_1) + c_\nu d_\nu(x_1,x_2), e_\nu (d_\nu(x_0,x_1) + d_\nu(x_1,x_2))\big) \\
&= F\big((a_\nu + b_\nu + e_\nu) d_\nu(x_0,x_1) + (c_\nu + e_\nu) d_\nu(x_1,x_2)\big) \text{ for all } \nu \in \mathfrak{A}.
\end{aligned} \tag{2}$$

Since F is strictly increasing, we get from above that

$$d_\nu(x_1,x_2) < (a_\nu + b_\nu + e_\nu) d_\nu(x_0,x_1) + (c_\nu + e_\nu) d_\nu(x_1,x_2) \text{ for all } \nu \in \mathfrak{A}.$$

That is,

$$(1 - c_\nu - e_\nu) d_\nu(x_1,x_2) < (a_\nu + b_\nu + e_\nu) d_\nu(x_0,x_1) \text{ for all } \nu \in \mathfrak{A}.$$

As $a_\nu + b_\nu + c_\nu + 2e_\nu = 1$, thus we have

$$d_\nu(x_1,x_2) < d_\nu(x_0,x_1) \text{ for all } \nu \in \mathfrak{A}.$$

Now, from (2), we have

$$\tau + F(d_\nu(x_1,x_2)) \leq F(d_\nu(x_0,x_1)), \text{ for all } \nu \in \mathfrak{A}.$$

By hypothesis (ii), we have $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$. From (1), we have

$$\begin{aligned}
\tau + F(d_\nu(x_2,x_3)) &= \tau + F(d_\nu(Tx_1,Tx_2)) \\
&\leq F\big(a_\nu d_\nu(x_1,x_2) + b_\nu d_\nu(x_1,Tx_1) + c_\nu d_\nu(x_2,Tx_2), e_\nu d_\nu(x_1,Tx_2) + L_\nu d_\nu(x_2,Tx_1)\big) \\
&= F\big(a_\nu d_\nu(x_1,x_2) + b_\nu d_\nu(x_1,x_2) + c_\nu d_\nu(x_2,x_3), e_\nu d_\nu(x_1,x_3) + L_\nu .0\big) \\
&\leq F\big(a_\nu d_\nu(x_1,x_2) + b_\nu d_\nu(x_1,x_2) + c_\nu d_\nu(x_2,x_3), e_\nu (d_\nu(x_1,x_2) + d_\nu(x_2,x_3))\big) \\
&= F\big((a_\nu + b_\nu + e_\nu) d_\nu(x_1,x_2) + (c_\nu + e_\nu) d_\nu(x_2,x_3)\big) \text{ for all } \nu \in \mathfrak{A}.
\end{aligned} \tag{3}$$

Since F is strictly increasing, we get from above that

$$d_\nu(x_2,x_3) < (a_\nu + b_\nu + e_\nu) d_\nu(x_1,x_2) + (c_\nu + e_\nu) d_\nu(x_2,x_3) \text{ for all } \nu \in \mathfrak{A}.$$

That is,

$$(1-c_\nu-e_\nu)d_\nu(x_2,x_3)<(a_\nu+b_\nu+e_\nu)d_\nu(x_1,x_2) \text{ for all } \nu\in\mathfrak{A}.$$

As $a_\nu+b_\nu+c_\nu+2e_\nu=1$, thus we have

$$d_\nu(x_2,x_3)<d_\nu(x_1,x_2) \text{ for all } \nu\in\mathfrak{A}.$$

Now from (3), we have

$$\tau+F(d_\nu(x_2,x_3))\le F(d_\nu(x_1,x_2)) \text{ for all } \nu\in\mathfrak{A}.$$

So we have

$$F(d_\nu(x_2,x_3))\le F(d_\nu(x_1,x_2))-\tau\le F(d_\nu(x_0,x_1))-2\tau \text{ for all } \nu\in\mathfrak{A}.$$

Continuing in the same way, we get a sequence $\{x_n\}\subset X$ such that

$$x_n\in Tx_{n-1},\ x_{n-1}\ne x_n \text{ and } \alpha(x_{n-1},x_n)\ge 1 \text{ for each } n\in\mathbb{N}.$$

Furthermore,

$$F(d_\nu(x_n,x_{n+1}))\le F(d_\nu(x_0,x_1))-n\tau \text{ for each } n\in\mathbb{N} \text{ and } \nu\in\mathfrak{A}. \tag{4}$$

Letting $n\to\infty$ in (4), we get $\lim_{n\to\infty}F(d_\nu(x_n,x_{n+1}))=-\infty$ for all $\nu\in\mathfrak{A}$. Thus, by property (F_2), we have $\lim_{n\to\infty}d_\nu(x_n,x_{n+1})=0$. Let $(d_\nu)_n=d_\nu(x_n,x_{n+1})$ for all $\nu\in\mathfrak{A}$ for each $n\in\mathbb{N}$. From (F_3) there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}(d_\nu)_n^kF((d_\nu)_n)=0 \text{ for all } \nu\in\mathfrak{A}.$$

From (4) we have

$$(d_\nu)_n^kF((d_\nu)_n)-(d_\nu)_n^kF((d_\nu)_0)\le-(d_\nu)_n^kn\tau\le 0 \text{ for each } n\in\mathbb{N} \text{ and } \nu\in\mathfrak{A}. \tag{5}$$

Letting $n\to\infty$ in (5), we get

$$\lim_{n\to\infty}n(d_\nu)_n^k=0 \text{ for all } \nu\in\mathfrak{A}. \tag{6}$$

This implies that there exists $n_1\in\mathbb{N}$ such that $n(d_\nu)_n^k\le 1$ for each $n\ge n_1$ and $\nu\in\mathfrak{A}$. Thus, we have

$$(d_\nu)_n\le\frac{1}{n^{1/k}}, \text{ for each } n\ge n_1 \text{ and } \nu\in\mathfrak{A}. \tag{7}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m,n\in\mathbb{N}$ with $m>n>n_1$. By using the triangular inequality and (7), we have

$$\begin{aligned}d_\nu(x_n,x_m)&\le d_\nu(x_n,x_{n+1})+d_\nu(x_{n+1},x_{n+2})+\cdots+d_\nu(x_{m-1},x_m)\\&=\sum_{i=n}^{m-1}(d_\nu)_i\le\sum_{i=n}^{\infty}(d_\nu)_i\le\sum_{i=n}^{\infty}\frac{1}{i^{1/k}} \text{ for all } \nu\in\mathfrak{A}.\end{aligned}$$

Since $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty}d_\nu(x_n,x_m)=0$ for all $\nu\in\mathfrak{A}$. Which implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. By condition (iii), we have $\alpha(x_n,x^*)\ge 1$ for each $n\in\mathbb{N}$. We claim that $d_\nu(x^*,Tx^*)=0$ for all $\nu\in\mathfrak{A}$. On contrary suppose that $d_\nu(x^*,Tx^*)>0$ for some ν, there exists $n_0\in\mathbb{N}$ such that $d_\nu(x_n,Tx^*)>0$ for each $n\ge n_0$. Thus for each $n\ge n_0$ by using triangular property and (1), we have

$$\begin{aligned} d_\nu(x^*,Tx^*) &\le d_\nu(x^*,x_{n+1}) + d_\nu(x_{n+1},Tx^*) \\ &= d_\nu(x^*,x_{n+1}) + d_\nu(Tx_n,Tx^*) \\ &< d_\nu(x^*,x_{n+1}) + a_\nu d_\nu(x_n,x^*) + b_\nu d_\nu(x_n,x_{n+1}) + c_\nu d_\nu(x^*,Tx^*), \\ &\quad + e_\nu d_\nu(x_n,Tx^*) + L_\nu d_\nu(x^*,x_{n+1}). \end{aligned} \tag{8}$$

Letting $n \to \infty$ in (8), we have

$$d_\nu(x^*,Tx^*) \le (c_\nu + e_\nu) d_\nu(x^*,Tx^*) < d_\nu(x^*,Tx^*).$$

Which is a contradiction. Thus $d_\nu(x^*,Tx^*) = 0$ for each $\nu \in \mathfrak{A}$. As X is separating we have $x^* = Tx^*$. □

Theorem 2.2. *Let $T : X \to X$ be a mapping for which we have continuous F in $\mathfrak{F}$ and $\tau > 0$ such that*

$$\alpha(x,y) \ge 1 \Rightarrow \tau + F(d_\nu(Tx,Ty)) \le F\Big(\max\Big\{d_\nu(x,y), d_\nu(x,Tx), d_\nu(y,Ty), \frac{d_\nu(x,Ty)+d_\nu(y,Tx)}{2}\Big\} + L d_\nu(y,Tx)\Big) \text{ for all } \nu \in \mathfrak{A} \tag{9}$$

for each $x,y \in X$, whenever $d_\nu(Tx,Ty) \ne 0$, where $L \ge 0$. Further assume that the following conditions hold:

(i) *there exists $x_0 \in X$ with $\alpha(x_0,Tx_0) \ge 1$;*

(ii) *for $x,y \in X$ with $\alpha(x,y) \ge 1$ we have $\alpha(Tx,Ty) \ge 1$;*

(iii) *for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n,x) \ge 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. By hypothesis (i), there exists $x_0 \in X$ with $\alpha(x_0,Tx_0) \ge 1$. Take $x_1 = Tx_0$. From (9), we have

$$\begin{aligned} \tau + F(d_\nu(x_1,x_2)) &= \tau + F(d_\nu(Tx_0,Tx_1)) \\ &\le F\Big(\max\Big\{d_\nu(x_0,x_1), d_\nu(x_0,Tx_0), d_\nu(x_1,Tx_1), \\ &\qquad \frac{d_\nu(x_1,Tx_0)+d_\nu(x_0,Tx_1)}{2}\Big\} + L d_\nu(x_1,Tx_0)\Big) \\ &= F\Big(\max\{d_\nu(x_0,x_1), d_\nu(x_1,x_2)\}\Big) \text{ for all } \nu \in \mathfrak{A}. \end{aligned} \tag{10}$$

If we assume that $\max\{d_\nu(x_0,x_1), d_\nu(x_1,x_2)\} = d_\nu(x_1,x_2)$, then we have a contradiction to (10). Thus, $\max\{d_\nu(x_0,x_1), d_\nu(x_1,x_2)\} = d_\nu(x_0,x_1)$ for all $\nu \in \mathfrak{A}$. From (10), we have

$$\tau + F(d_\nu(x_1,x_2)) \le F(d_\nu(x_0,x_1)) \text{ for all } \nu \in \mathfrak{A}. \tag{11}$$

As $\alpha(x_0,x_1) \ge 1$, then we have $\alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \ge 1$. From (9), we have

$$\begin{aligned} \tau + F(d_\nu(x_2,x_3)) &= \tau + F(d_\nu(Tx_1,Tx_2)) \\ &\le F\Big(\max\Big\{d_\nu(x_1,x_2), d_\nu(x_1,Tx_1), d_\nu(x_2,Tx_2), \end{aligned}$$

$$\left.\frac{d_\nu(x_2,Tx_1)+d_\nu(x_1,Tx_2)}{2}\right\}+Ld_\nu(x_2,Tx_1)\Big)$$

$$=F\Big(\max\{d_\nu(x_1,x_2),d_\nu(x_2,x_3)\}\Big)\ \text{ for all } \nu\in\mathfrak{A}. \tag{12}$$

If we assume that $\max\{d_\nu(x_1,x_2),d_\nu(x_2,x_3)\}=d_\nu(x_2,x_3)$, then we have a contradiction to (12). Thus, $\max\{d_\nu(x_1,x_2),d_\nu(x_2,x_3)\}=d_\nu(x_1,x_2)$ for all $\nu\in\mathfrak{A}$. From (12), we have

$$\tau+F(d_\nu(x_2,x_3))\le F(d_\nu(x_1,x_2))\ \text{ for all } \nu\in\mathfrak{A}. \tag{13}$$

From (11) and (13), we have

$$F(d_\nu(x_2,x_3))\le F(d_\nu(x_0,x_1))-2\tau\ \text{ for all } \nu\in\mathfrak{A}. \tag{14}$$

Continuing in the same way, we get a sequence $\{x_n\}\subset X$ such that

$$x_n\in Tx_{n-1},\ x_{n-1}\ne x_n \text{ and } \alpha(x_{n-1},x_n)>1 \text{ for each } n\in\mathbb{N}.$$

Moreover,

$$F(d_\nu(x_n,x_{n+1}))\le F(d_\nu(x_0,x_1))-n\tau \text{ for each } n\in\mathbb{N} \text{ and } \nu\in\mathfrak{A}. \tag{15}$$

Letting $n\to\infty$ in (15), we get $\lim_{n\to\infty}F(d_\nu(x_n,x_{n+1}))=-\infty$ for all $\nu\in\mathfrak{A}$. Thus, by property (F_2), we have $\lim_{n\to\infty}d_\nu(x_n,x_{n+1})=0$ $\forall\nu\in\mathfrak{A}$. Let $(d_\nu)_n=d_\nu(x_n,x_{n+1})$ for each $n\in\mathbb{N}$ and $\nu\in\mathfrak{A}$. From (F_3) there exists $k\in(0,1)$ such that

$$\lim_{n\to\infty}(d_\nu)_n^kF((d_\nu)_n)=0\ \text{ for all } \nu\in\mathfrak{A}.$$

From (15) we have

$$(d_\nu)_n^kF((d_\nu)_n)-(d_\nu)_n^kF((d_\nu)_0)\le-(d_\nu)_n^kn\tau\le 0 \text{ for each } n\in\mathbb{N} \text{ and } \nu\in\mathfrak{A}. \tag{16}$$

Letting $n\to\infty$ in (16), we get

$$\lim_{n\to\infty}n(d_\nu)_n^k=0\ \text{ for all } \nu\in\mathfrak{A}. \tag{17}$$

This implies that there exists $n_1\in\mathbb{N}$ such that $n(d_\nu)_n^k\le 1$ for each $n\ge n_1$ and $\nu\in\mathfrak{A}$. Thus, we have

$$(d_\nu)_n\le\frac{1}{n^{1/k}},\quad \text{for each } n\ge n_1 \text{ and } \nu\in\mathfrak{A}. \tag{18}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m,n\in\mathbb{N}$ with $m>n>n_1$. By using the triangular inequality and (18), we have

$$\begin{aligned}d_\nu(x_n,x_m)&\le d_\nu(x_n,x_{n+1})+d_\nu(x_{n+1},x_{n+2})+\cdots+d_\nu(x_{m-1},x_m)\\&=\sum_{i=n}^{m-1}(d_\nu)_i\le\sum_{i=n}^{\infty}(d_\nu)_i\le\sum_{i=n}^{\infty}\frac{1}{i^{1/k}}\ \text{ for all } \nu\in\mathfrak{A}.\end{aligned}$$

Since $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n\to\infty}d_\nu(x_n,x_m)=0$ for all $\nu\in\mathfrak{A}$. Which implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there is $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. By condition (iii), we have $\alpha(x_n,x^*)\ge 1$ for each $n\in\mathbb{N}$. We claim that $d_\nu(x^*,Tx^*)=0$ for all $\nu\in\mathfrak{A}$. On contrary suppose that $d_\nu(x^*,Tx^*)>0$ for some ν, there exists $n_0\in\mathbb{N}$ such that $d_\nu(x_n,Tx^*)>0$ for each $n\ge n_0$. From (9), for each $n\ge n_0$, we have

$$\tau+F(d_\nu(x_{n+1},Tx^*))\le F\Big(\max\Big\{d_\nu(x_n,x^*),d_\nu(x_n,Tx_n),d_\nu(x^*,Tx^*),$$

$$\frac{d_\nu(x^*,Tx_n)+d_\nu(x_n,Tx^*)}{2}\Big\}+Ld_\nu(x^*,Tx_n)\Big) \text{ for all } \nu\in\mathfrak{A}.$$

Letting $n\to\infty$ in above inequality and by continuity of F, we get

$$\tau+F(d_\nu(x^*,Tx^*))\le F(d_\nu(x^*,Tx^*)).$$

This implies $\tau\le 0$. Which is a contradiction. Thus $d_\nu(x^*,Tx^*)=0$ for each $\nu\in\mathfrak{A}$. As X is separating we have $x^*=Tx^*$. □

In the following corollaries we assume that X is a nonempty set endowed with a separating complete gauge structure, further a directed graph $G=(V,E)$ is defined on X such that the set of its vertices V coincides with X (i.e., $V=X$) and the set of its edges E is such that $E\supseteq\triangle$, where $\triangle=\{(x,x):x\in X\}$. Let us also assume that G has no parallel edges.

The following corollaries can be obtained from our results by defining $\alpha:X\times X\to[0,\infty)$ as:

$$\alpha(x,y)=\begin{cases}1 & \text{if } (x,y)\in E\\ 0 & \text{otherwise.}\end{cases}$$

Corollary 2.3. *Let $T:X\to X$ be a mapping for which we have F in $\mathfrak{F}$ and $\tau>0$ such that*

$$(x,y)\in E\Rightarrow \tau+F(d_\nu(Tx,Ty))\le F(a_\nu d_\nu(x,y)+b_\nu d_\nu(x,Tx)+c_\nu d_\nu(y,Ty)$$
$$+e_\nu d_\nu(x,Ty)+L_\nu d_\nu(y,Tx)) \text{ for all } \nu\in\mathfrak{A} \tag{19}$$

for each $x,y\in X$ whenever $d_\nu(Tx,Ty)\ne 0$, where $a_\nu,b_\nu,c_\nu,e_\nu,L_\nu\ge 0$, and $a_\nu+b_\nu+c_\nu+2e_\nu=1$ for all $\nu\in\mathfrak{A}$. Further, assume that the following conditions hold:

(i) *there exists $x_0\in X$ such that $(x_0,Tx_0)\in E$;*

(ii) *for each $x,y\in X$ with $(x,y)\in E$, we have $(Tx,Ty)\in E$;*

(iii) *for any sequence $\{x_n\}\subseteq X$ such that $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E$ for each $n\in\mathbb{N}$, we have $(x_n,x)\in E$ for each $n\in\mathbb{N}$.*

Then T has a fixed point.

Corollary 2.4. *Let $T:X\to X$ be a mapping for which we have continuous F in $\mathfrak{F}$ and $\tau>0$ such that*

$$(x,y)\in E\Rightarrow \tau+F(d_\nu(Tx,Ty))\le F\Big(\max\Big\{d_\nu(x,y),d_\nu(x,Tx),d_\nu(y,Ty),$$
$$\frac{d_\nu(x,Ty)+d_\nu(y,Tx)}{2}\Big\}+Ld_\nu(y,Tx)\Big) \text{ for all } \nu\in\mathfrak{A} \tag{20}$$

for each $x,y\in X$ whenever $d_\nu(Tx,Ty)\ne 0$, where $L\ge 0$. Further assume that the following conditions hold:

(i) *there exists $x_0\in X$ with $(x_0,Tx_0)\in E$;*

(ii) *for each $x,y\in X$ with $(x,y)\in E$ we have $(Tx,Ty)\in E$;*

(iii) *for any sequence $\{x_n\}\subseteq X$ such that $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E$ for each $n\in\mathbb{N}$, we have $(x_n,x)\in E$ for each $n\in\mathbb{N}$.*

Then T has a fixed point.

3. Application

Consider the Volterra integral equation of the form:

$$x(t)=\int_0^{f(t)} K(t,s,x(s))ds, \quad t\in I=[0,\infty) \tag{21}$$

where $f:I\to\mathbb{R}$ is continuous function with $0\le f(t)$ for each $t\in I$ and $K:I\times I\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing function. Let $X=(C[0,\infty),\mathbb{R})$ be the space of all realvalued continuous functions. Define the family of pseudonorms by $\|x\|_n=\max_{t\in[0,n]}\left\{|x(t)|e^{-|\tau t|}\right\}$, for each $n\in\mathbb{N}$, where $\tau>0$ is arbitrary. By using this family of pseudonorms we get a family of pseudo metrics as $d_n(x,y)=\|x-y\|_n$. Clearly, $\mathfrak{F}=\left\{d_n:n\in\mathbb{N}\right\}$ defines gauge structure on X, which is complete and separating. Define the graph as $V=X$ and $E=\{(x,y):x(t)\le y(t) \text{ for all } t\}$.

Theorem 3.1. *Let $X=(C[0,\infty),\mathbb{R})$ and let the operator $T:X\to X$ is defined by*

$$Tx(t)=\int_0^{f(t)} K(t,s,x(s))ds, \quad t\in I=[0,\infty),$$

where $f:I\to\mathbb{R}$ is continuous function with $0\le f(t)$ for each $t\in I$ and $K:I\times I\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing function. Assume that the following conditions hold:

(i) *there exist $\tau>0$ and $\beta:X\to(0,\infty)$ such that for each $(x,y)\in E$ and $t,s\in[0,n]$, we have*

$$|K(t,s,x)-K(t,s,y)|\le\frac{e^{-\tau}}{\beta(x+y)}d_n(x,y) \text{ for each } n\in\mathbb{N};$$

moreover,

$$\left|\int_0^{f(t)}\frac{1}{\beta(x(s)+y(s))}ds\right|\le e^{|\tau t|}$$

for each $t\in I$;

(ii) *there exists $x_0\in X$ such that $(x_0,Tx_0)\in E$;*

(iii) *for $x,y\in X$ with $(x,y)\in E$ we have $(Tx,Ty)\in E$;*

(iv) *for any sequence $\{x_n\}\subseteq X$ such that $x_n\to x$ as $n\to\infty$ and $(x_n,x_{n+1})\in E$ for each $n\in\mathbb{N}$, we have $(x_n,x)\in E$ for each $n\in\mathbb{N}$.*

Then the integral equation (21) *has atleast one solution.*

Proof. For any $(x,y)\in E$ and $t\in[0,n]$ for each $n\ge1$, we have

$$\begin{aligned}|Tx(t)-Ty(t)| &\le\int_0^{f(t)}|K(t,s,x(s))-K(t,s,y(s))|ds\\ &\le\int_0^{f(t)}\frac{e^{-\tau}}{\beta(x(s)+y(s))}d_n(x,y)ds\\ &=e^{-\tau}d_n(x,y)\int_0^{f(t)}\frac{1}{\beta(x(s)+y(s))}ds\\ &\le e^{|\tau t|}e^{-\tau}d_n(x,y).\end{aligned}$$

Thus, we have

$$|Tx(t)-Ty(t)|e^{-|\tau t|}\le e^{-\tau}d_n(x,y).$$

Equivalently

$$d_n(Tx,Ty) \le e^{-\tau} d_n(x,y).$$

As natural logarithm belongs to $\mathfrak{F}$. Applying it on above inequality, we get

$$\ln(d_n(Tx,Ty)) \le \ln(e^{-\tau} d_n(x,y)).$$

After some simplification, we get

$$\tau + \ln(d_n(Tx,Ty)) \le \ln(d_n(x,y)) \quad \text{for each } n \in \mathbb{N}.$$

Thus, T satisfies (19) with $a_n = 1$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \mathbb{N}$ and $F(x) = \ln x$. As K is nondecreasing, for each $(x,y) \in E$, we have $(Tx,Ty) \in E$. Further, all the other conditions of the Corollary 2.3, follows immediately from the hypothesis of the theorem. Thus, there exists a fixed point of the operator T, that is, integral equation (21) has atleast one solution. □

4. Conclusions

In this work we obtained the existence existence of fixed points for an integral operator by using a new fixed point theorem in the setting of gauge spaces. Also we used our fixed point results apply to find a solution of the Volterra integral equations.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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