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Research Article

On F-α-Geraghty Contractions

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Abstract. In this paper, we introduce the notion of F-α-Geraghty contraction type mappings and establish some common fixed point theorems for an admissible pair mappings under the notion of F-α-Geraghty contractive type in the setting of metric spaces. We give example for support results.

Keywords. F-α-Geraghty contractions type; Common fixed point

MSC. 47H09; 47H10

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1. Introduction

Fixed point problems in non-linear analysis was study and very important tool in the last 60 years. In fact, the techniques of fixed point have been apply to many fields of sciences such as Chemistry, Biology, Physics and Engineering. Over the years, fixed point theory has been generalized by several mathematicians (see [1–11]).

Throughout this article, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote that set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

Wardowski [1] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Firstly, let Im be the set of functions $F:\mathbb{R}^+\to\mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha,\beta\in\mathbb{R}^+$ such that $\alpha<\beta$ implies that $F(\alpha)<F(\beta)$.

(F2) For any sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.1 ([1]). Let (X,d) be a metric space and $F \in Im$. A self-mapping $S : X \to X$ is said to be an F-contraction, if there exists $\tau > 0$ such that $d(Sx,Sy) > 0 \to \tau + F(d(Sx,Sy)) \le F(d(x,y))$ for all $x,y \in X$.

Theorem 1.2 ([1]). *Let (X,d) be a complete metric space and $S : X \to X$ be an F-contraction. Then, S has a unique fixed point.*

Geraghty [6] studied a generalized of Banach contraction principle. We denote by Ω the family of all functions $\beta : [0,\infty) \to [0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 1.3 ([6]). *Let be a metric space and $S : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \Omega$ such that for all $x,y \in X$,*

$$d(Sx,Sy) \le \beta(d(x,y))d(x,y).$$

On the other hand, Samet *et al*. [9] introduced the class of α-admissible mappings.

Definition 1.4 ([9]). For a nonempty set X, let $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be given mappings. We say that S is α-admissible, if for all $x,y \in X$ we have $\alpha(x,y) \ge 1$ implies $\alpha(Sx,Sy) \ge 1$.

Definition 1.5 ([2]). For a nonempty set X, let $S,f : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be given mappings. We say that (S,f) is triangular α-admissible, if

(S1) $\alpha(x,y) \ge 1 \Rightarrow \alpha(Sx,fy) \ge 1$ and $\alpha(fx,Sy) \ge 1$, $x,y \in X$.

(S2) $\alpha(x,z) \ge 1$, $\alpha(z,y) \ge 1 \Rightarrow \alpha(x,y) \ge 1$, $x,y,z \in X$.

2. Main Results

We introduce the concept of an F-α-Geraghty contraction as follows:

Definition 2.1. Let $S,f : X \to X$ be a metric space and be a self-mappings. (S,f) is said to be an F-α-Geraghty contraction, if there exists $\tau > 0$ such that, for all $x,\in X$ with $\alpha(x,y) \ge 1$ we have

$$d(Sx,fy) > 0 \Rightarrow \tau + F(d(Sx,fy)) \le F\left(\beta(M(x,y))\right)M(x,y), \tag{2.1}$$

where $F \in, \beta \in \Omega$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,fy), \frac{d(x,fy)+d(y,Sx)}{2}\right\}. \tag{2.2}$$

In case where $F(x) = \ln(x)$ for $x > 0$, (2.1) becomes

$$d(Sx, fy) \leq e^{-\tau}\beta(M(x,y))M(x,y) \\ \leq \beta(M(x,y)M(x,y)) \tag{2.3}$$

for all $x, y \in X$ with $\alpha(x,y) \leq 1$ and $Sx \neq fy$ Note that (2.3) is satisfied for all $x, y \in X$ with $\alpha(x,y) \leq 1$ and $Sx = fy$.

If $S = f$ then is called generalized F-α-Geraghty contraction mapping, if there exists $\tau > 0$ such that, for all $x, y \in X$, with $d(x,y) \geq 1$ we have

$$d(fx, fy) \geq 0 \Rightarrow \tau + F(d(fx, fy)) \leq F\left(\beta(M(x,y))\right)M(x,y),$$

where $F \in Im, \beta \in \Omega$ and

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{d(x,Sy)+d(y,Sx)}{2}\right\}.$$

Theorem 2.2. *Let (X,d) be a complete metric space and $S, f : X \to X$ be such that (S,f) is an F-α-Geraghty contraction. Suppose that the following holds:*

(1) *(S,f) is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(3) *S and f are continuous.*

Then, (S,f) have common fixed point.

Proof. By (2.2), there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Let $x_1 \in X$ be such that $x_1 = Sx_0$ and $x_2 = Sx_1$. Continuous this process, we construct a sequence $\{x_n\}$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = fx_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By assumption $\alpha(x_0, x_1) \geq 1$ and a pair (S,f) is triangular α-admissible, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

From (2.1), (2.4) and the triangular inequality, we get

$$\tau + F(d(x_{2n+1}, x_{2n+2})) = \tau + F(d(Sx_{2n}, fx_{2n+1})) \\ \leq F\left(\beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1})\right),$$

where

$$\begin{aligned} &M(x_{2n}, x_{2n+1}) \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, fx_{2n+1}), \frac{d(x_{2n}, fx_{2n+1}) + d(x_{2n+1}, Sx_{2n})}{2}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\right\} \\ &\leq \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, fx_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\right\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right\}. \end{aligned}$$

So, we get

$$\tau + F(d(x_{2n+1},x_{2n+2})) \le F(\beta(\max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\})) \times (\max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}). \tag{2.5}$$

If $d(x_{2n+1},x_{2n+2}) > d(x_{2n},x_{2n+1})$, then

$$\max\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\} = d(x_{2n+1},x_{2n+2}).$$

So (2.5) becomes

$$\tau + F(d(x_{2n+1},x_{2n+2})) \le F\left(\beta(d(x_{2n+1},x_{2n+2}))d(x_{2n+1},x_{2n+2})\right).$$

This yields,

$$F(d(x_{2n+1},x_{2n+2})) < F\left(\beta(d(x_{2n+1},x_{2n+2}))d(x_{2n+1},x_{2n+2})\right).$$

From (F1) and $\beta \in \omega$, we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &< \beta(d(x_{2n+1},x_{2n+2}))d(x_{2n+1},x_{2n+2}) \\ &< d(x_{2n+1},x_{2n+2}) \end{aligned}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}\cup\{0\}$, we get

$$F(d(x_{2n+1},x_{2n+2})) \le F\left(\beta(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1})\right) - \tau. \tag{2.6}$$

Let $d(x_{2n+1},x_{2n+2}) = \gamma_n$. From (2.6) implies that

$$F(\gamma_n) \le F(\beta(\gamma_0)\gamma_0) - \tau. \tag{2.7}$$

Taking $n \to \infty$ in (2.7), we obtain

$$\lim_{n\to\infty} \gamma_n = 0. \tag{2.8}$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. From (2.7) and (F3) , there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0. \tag{2.9}$$

By (2.7), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \gamma_n^k F(\gamma_n) - \gamma_n^k F(\beta(\gamma_0)\gamma_0) &\le \gamma_n^k (F(\beta(\gamma_0)\gamma_0) - n\tau) - \gamma_n^k F(\beta(\gamma_0)\gamma_0) \\ &\le 0. \end{aligned} \tag{2.10}$$

Taking $n \to \infty$ in (2.10), by (2.8) and (2.9), we get

$$\lim_{n\to\infty} \gamma_n^k = 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\gamma_n \le \frac{1}{n^{\frac{1}{k}}}.$$

Then, for all $n \ge n_0$ and $q \in \mathbb{N}$

$$\begin{aligned} d(x_n,x_{n+q}) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+q-1},x_{n+q}) \\ &\le \sum_{i=n}^{n+q-1} d(x_i,x_{i+1}) \end{aligned}$$

$$= \sum_{i=n}^{n+q-1} \gamma_i$$
$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} < \infty$, then $\lim_{n\to\infty} d\left(x_n, x_{n+q}\right) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence. By (X,d) is a complete metric space, there exists $x^* \in X$ such that $d(x_n, x^*) = 0$.

Finally, we shall prove that is a common fixed point of and Since $d(x_n, x^*) = 0$. So, we have $\lim_{n\to\infty} d(x_{2n}, x^*) = \lim_{n\to\infty} d(x_{2n+1}, x^*) = 0$. By continuity of S and f. We get that $\lim_{n\to\infty} d(x_{2n+1}, Sx^*) = \lim_{n\to\infty} d(Sx_{2n}, Sx^*) = 0$ and $\lim_{n\to\infty} d(x_{2n+2}, fx^*) = \lim_{n\to\infty} d(fx_{2n+1}, fx^*) = 0$. Thus, $Sx^* = x^* = fx^*$ and hence x^* is a common fixed point of S and f. □

In the following, we have some corollary of our result.

Corollary 2.3. *Let (X,d) be a complete metric space and $S,f : X \to X$ be given mappings. Suppose there exist a function $\alpha : X \times X \to \mathbb{R}$ and $\tau > 0$ such that*

$$d(Sx, fy) > 0 \Rightarrow \tau + F(\alpha(x,y)d(Sx, fy)) \leq F\left(\beta(M(x,y))M(x,y)\right), \tag{2.11}$$

for all $x,y \in X$, where $F \in Im, \beta \in \Omega$ and $M(x,y)$ is defined by (2.2)*. Suppose that the following holds:*

(1) *(S,f) is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(3) *S and f are continuous.*

Then, (S,f) have common fixed point.

Proof. Let $x,y \in X$, with $\alpha(x,y) \geq 1$. By (F1), if $d(Sx, fy) > 0$ and (2.11) holds, the proof is concluded by Theorem 2.2. □

Corollary 2.4. *Let (X,d) be a complete metric space and $S,f : X \to X$ be given mappings. Suppose there exist $\tau > 0$ such that*

$$d(Sx, fy) > 0 \Rightarrow \tau + F(d(Sx, fy)) \leq F\left(\beta(M(x,y))\right)M(x,y), \tag{2.12}$$

for all $x,y \in X$, where $F \in Im, \beta \in \Omega$ and $M(x,y)$ is defined by (2.2)*. Then, (S,f) have common fixed point.*

Proof. It suffices to take $\alpha(x,y) = 1$ in Theorem 2.2. □

Corollary 2.5. *Let (X,d) be a complete metric space and $S,f : X \to X$ be given mappings. Suppose there exist $\tau > 0$ such that*

$$d(Sx, fy) > 0 \Rightarrow \tau + F(d(Sx, fy)) \leq F(M(x,y))M(x,y), \tag{2.13}$$

for all $x,y\in X$, where $F\in Im,\beta\in\Omega$ and $M(x,y)$ is defined by (2.2). *Then, (S,f) have common fixed point.*

Proof. It suffices to take $\alpha(x,y)=1$ and $\beta(M(x,y))=M(x,y)$ in Theorem 2.2. □

Theorem 2.6. *Let (X,d) be a complete metric space and $S,f:X\to X$ be such that (S,f) is an F-α-Geraghty contraction. Suppose that the following holds:*

(1) *(S,f) is triangular α-admissible;*

(2) *there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\geq 1$;*

(3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $x_n\to x^*\in X$ as $n\to\infty$, then there exist a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n_k},x^*\right)\geq 1$ for all k.*

Then, (S,f) have common fixed point.

Proof. Following the proof of Theorem 2.2, we know that define $x_{2n+1}=Sx_{2n}$ and $x_{2n+2}=fx_{2n+1}$ for all $n\in\mathbb{N}\cup\{0\}$ converges to $x^*\in X$. By the hypotheses of (2.3), there exists a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{2n_k},x^*\right)\geq 1$ for all k. Then, by (2.1), we have

$$\begin{aligned}\tau+F\left(d\left(x_{2n_k+1},fx^*\right)\right)&=\tau+F\left(d\left(Sx_{2n_k},fx^*\right)\right)\\&\leq F\left(\beta\left(M\left(x_{2n_k},x^*\right)\right)M\left(x_{2n_k},x^*\right)\right),\end{aligned}$$

where

$$M\left(x_{2n_k},x^*\right)=\max\left\{d\left(x_{2n_k},x^*\right),d\left(x_{2n_k},Sx_{2n_k}\right),d\left(x^*,fx^*\right),\frac{d\left(x_{2n_k},fx^*\right)+d\left(x^*,Sx_{2n_k}\right)}{2}\right\}.$$

Taking $k\to\infty$, we get

$$\lim_{k\to\infty}M\left(x_{2n_k},x^*\right)=d\left(x^*,fx^*\right). \tag{2.14}$$

Suppose that $d(x^*,fx^*)>0$. From (2.14) there exists $N\in\mathbb{N}$ such that for all $k\geq N$, we have $M\left(x_{2n_k},x^*\right)>0$, which implies that

$$\beta\left(M\left(x_{2n_k},x^*\right)\right)<M\left(x_{2n_k},x^*\right).$$

This is,

$$d\left(x_{2n_k},fx^*\right)<M\left(x_{2n_k},x^*\right). \tag{2.15}$$

Taking $k\to\infty$ in (2.15), we get $d(x^*,fx^*)<d(x^*,fx^*)$, which is a contradiction. Hence, we find that x^* is a common of f. Similarly, we find that x^* is a common of S. Thus, x^* is a common of S and f. □

Corollary 2.7. *Let (X,d) be a complete metric space and $S,f:X\to X$ be given mappings. Suppose there exist a function $\alpha:X\times X\to\mathbb{R}$ and $\tau>0$ such that*

$$d(Sx,fy)>0\Rightarrow\tau+F(\alpha(x,y)d(Sx,fy))\leq F\left(\beta(M(x,y))M(x,y)\right), \tag{2.16}$$

for all $x,y\in X$, where $F\in Im$, $\beta\in\Omega$ and $M(x,y)$ is defined by (2.2). *Suppose that the following holds:*

(1) *(S,f) is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;*

(3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exist $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n_k},x^*\right) \geq 1$ for all k.*

Then, (S,f) have common fixed point.

Proof. Let $x,y \in X$, with $\alpha(x,y) \geq 1$ By (F1), if $d(Sx,fy) > 0$ and (2.16) holds, the proof is concluded by Theorem 2.6. If

$$M(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Sy), \frac{d(x,Sy)+d(y,Sx)}{2}\right\}$$

and in Theorem 2.2 and Theorem 2.6, we have the following corollaries. □

Corollary 2.8. *Let (X,d) be a complete metric space and $S,f : X \to X$ be generalized F-α-Geraghty contraction mapping such that the following holds:*

(1) *S is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;*

(3) S and f are continuous.

Then, S has a fixed point.

Corollary 2.9. *Let (X,d) be a complete metric space and $S,f : X \to X$ be generalized F-α-Geraghty contraction mapping such that the following holds:*

(1) *S is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;*

(3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exist a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n_k},x^*\right) \geq 1$ for all k.*

Then, S has a fixed point.

If $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Sy)\}$ and $S = f$ in Theorem 2.2, Theorem 2.6, we obtain the following corollaries.

Corollary 2.10. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $f : X \to X$ be generalized F-α-Geraghty contraction mapping such that the following holds:*

(1) *S is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$;*

(3) *S and f are continuous.*

Then, S has a fixed point.

Corollary 2.11. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $f : X \to X$ be generalized F-α-Geraghty contraction mapping such that the following holds:*

(1) *S is triangular α-admissible;*

(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exist a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(x_{n_k}, x^*\right) \geq 1$ for all k.*

Then, S has a fixed point.

3. Application on A Dynamic Programming

In this subsection, we present an application on a dynamic programming. The existence of solutions of functional equations and system of functional equations arising in dynamic programming which have been studied by using various fixed point theorems (more details, the reader can see [3–5]. We assume that U and V are Banach spaces, $W \subset U$ is a state space and $D \subset V$ is a decision space. In particular, we are interested in solving the following two functional equations arising in dynamic programming:

$$g(x) = \sup_{y \in D}\{r(x,y) + P(x,y,g(\tau(x,y)))\}, \quad x \in W, \tag{3.1}$$

$$g(x) = \sup_{y \in D}\{r(x,y) + Q(x,y,g(\tau(x,y)))\}, \quad x \in W, \tag{3.2}$$

where $\tau : W \times D \to W, r : W \times D \to \mathbb{R}$ and $P,Q : W \times D \times \mathbb{R} \to \mathbb{R}$. We study the existence and uniqueness of $h_* \in B(W)$ a common solution of the functional equations (3.1) and (3.2).

Let $B(W)$ denote the set of all bounded real-valued functions on W. We know that $B(W)$ endowed with the metric

$$d(h,k) = \sup_{x \in W}|h(x) - k(x)|, h,k \in B(W), \tag{3.3}$$

is a complete metric space. Consider the mappings $S, f : B(W) \to B(W)$

$$S(h)(x) = \sup_{y \in D}\{r(x,y) + P(x,y,h(\tau(x,y)))\}, \quad x \in W, \tag{3.4}$$

$$f(h)(x) = \sup_{y \in D}\{r(x,y) + Q(x,y,h(\tau(x,y)))\}, \quad x \in W. \tag{3.5}$$

It's clear that, if r, P and Q are bounded, then the operators S and f are well-defined. We shall prove the following theorem.

Theorem 3.1. *Let $0 < \alpha < 1$. Suppose there exists $k \in (0,\alpha)$ such that for every $(x,y) \in W \times D$ and $h_1, h_2 \in B(W)$, we obtain*

$$P(x,y,h_1(\tau(x,y))) - Q(x,y,h_2(\tau(x,y)))| \leq kM(h_1,h_2), \tag{3.6}$$

where

$$M(h_1,h_2) = \max\left\{d(h_1,h_2), d(h_1,Sh_2), d(h_2,fh_2), \frac{d(h_1,fh_2) + d(h_2,Sh_1)}{2}\right\}. \tag{3.7}$$

Then, S and f have a unique common fixed point in $B(W)$.

Proof. Let $\xi > 0$ be an arbitrary positive real number, $x \in W$, $h_1, h_2 \in B(W)$. By using (3.4) and (3.5), there exist $y_1, y_2 \in D$ such that

$$S(h_1)(x) < r(x,y_1) + P(x,y_1,h_1(\tau(x,y_1))) + \xi, \tag{3.8}$$

$$f(h_2)(x) < r(x,y_2) + Q(x,y_2,h_2(\tau(x,y_2))) + \xi, \tag{3.9}$$

$$S(h_1)(x) \geq r(x,y_2) + P(x,y_2,h_1(\tau(x,y_2))), \tag{3.10}$$

$$f(h_2)(x) \geq r(x,y_1) + Q(x,y_1,h_2(\tau(x,y_1))). \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$\begin{aligned} S(h_1)(x) - f(h_2)(x) &\leq P(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1))) + \xi \\ &\leq |P(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1)))| + \xi \\ &\leq kM(h_1,h_2) + \xi. \end{aligned}$$

Similarly, from (3.9) and (3.10), we obtain that

$$f(h_2)(x) - S(h_1)(x) \leq kM(h_1,h_2) + \xi. \tag{3.12}$$

Consequently, we deduce that

$$|S(h_1)(x) - f(h_2)(x)| \leq kM(h_1,h_2) + \xi. \tag{3.13}$$

Since the inequality (3.13) is true for any $x \in W$, we get that

$$d(S(h_1), f(h_2)) \leq kM(h_1,h_2) + \xi. \tag{3.14}$$

Finally, ξ is arbitrary, so

$$d(S(h_1), f(h_2)) \leq kM(h_1,h_2) \leq kM(h_1,h_2)M(h_1,h_2), \tag{3.15}$$

by taking $\tau = -\ln(\frac{k}{\alpha}), \beta(t) = \alpha t$ and $F(t) = \ln(t)$. Applying Corollary 2.5, the mappings S and f have a unique common fixed point, that is, the functional equations (3.1) and (3.2) have a unique common solution $h_* \in B(W)$. □

Conclusion

This paper presents some common fixed point theorems for a pair of F-α-Geraghty contraction. The presented theorems extend, generalize and inprove classical results in fixed point theory and Banach contraction principle.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] T.M.-K. Abdeljawad, α-contractive fixed point and common fixed point theorems, *Fixed Point Theorem Appl.* **2013**(2013), DOI: 10.1186/1687-1812-2013-19.

[2] M. Arshad, A. Hussain and A. Azam, Fixed point of α-Geraghty contraction with application, *U.P.B. Sci. Bull., Series A* **78**(2) (2016), 67 – 78.

[3] R. Baskaran and P.V. Subrahmanyam, A note on the solution of a class of functional equations, *Appl. Anal.* **22** (1986), 235 – 241.

[4] R. Bellman, *Methods of Nonliner Analysis*, Vol. II, Academic Press, New York – London (1973).

[5] P.C. Bhakta and S. Mitra, Some existence theorems for functional equations arising in dynamic programming, *J. Math. Anal. Appl.* **98** (1984), 348 – 362.

[6] M. Geraghty, On contractive mapping, *Proc. Amer. Math. Soc.* **40** (1973), 604 – 608.

[7] D. Jain, A. Padcharoen, P. Kumam and D. Gopal, A new approach to study fixed point of multivalued mappings in modular metric spaces and applications, *Mathematics* **4**(2016), 51.

[8] A. Padcharoen, D. Gopal, P. Chaipunya and P. Kumam, Fixed point and periodic point results for α-type F-contractions in modular metric spaces, *Fixed Point Theory Appl.* **2006**(2016), 39.

[9] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal.* **75** (2012), 2154 – 2165.

[10] P. Sumalai, P. Kumam, Y.J. Cho and A. Padcharoen, The (CLRg)-property for coincidence point theorems and Fredholm integral equations in modular metric spaces, *European Journal of Pure and Applied Mathematics* **10**(2) (2017), 238 – 254.

[11] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl*. **2012** (2012), 94.