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Research Article

On Vertex-transitive Cayley Graphs of Finite Transformation Semigroups with Restricted Range

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Abstract. Let $T(X)$ be the semigroup of all transformations on a set X. For a non-empty subset Y of X, denoted by $T(X,Y)$ the subsemigroup of $T(X)$ consisting of all transformations whose range is contained in Y. Kelarev and Praeger in [9] gave necessary and sufficient conditions for all vertex-transitive Cayley graphs of semigroups. In this paper, we give similar descriptions for all vertex-transitive Cayley graphs of $T(X,Y)$.

Keywords. Cayley graph; Vertex-transitive graph; Transformation semigroup; Restricted range

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1. Introduction

Let S be a semigroup and $A \subseteq S$. The *Cayley graph* $\mathrm{Cay}(S,A)$ of the semigroup S with the connection set A is defined as the digraph with vertex set S and arc set $E(\mathrm{Cay}(S,A))$ containing of all ordered pairs (x,y) such that $y = xa$ for some $a \in A$. The Cayley graphs of groups and semigroups have been considered by many authors, see for example in [1], [2], [4], [5], [8], [9], [10], [11], [13], [14], [15]. Lately, Cayley graphs play an important role in theoretical computer science as one of powerful tools in this subject (see [3], [7]). In addition, the notion of quantum walks in physics also employs Cayley graphs (for example, see [6], [12]).

Let $D = (V, E)$ be a digraph. A mapping $\phi : V(D) \to V(D)$ is called an *endomorphism* of the digraph D if $(u\phi, v\phi) \in E(D)$ for all $(u,v) \in E(D)$, i.e., ϕ is an *arc-preserving*. If $\phi : V(D) \to V(D)$ is a bijective endomorphism and ϕ^{-1} is also an endomorphism, then ϕ is called an *automorphism*. Let $\mathrm{End}(D)$ denote the set of all endomorphisms of D and $\mathrm{Aut}(D)$ denote the set of all automorphisms of D. For a Cayley digraph $\mathrm{Cay}(S,A)$, let $\mathrm{End}(S,A)$ denote $\mathrm{End}(\mathrm{Cay}(S,A))$ and $\mathrm{Aut}(S,A)$ denote $\mathrm{Aut}(\mathrm{Cay}(S,A))$.

A digraph $D = (V, E)$ is $\mathrm{Aut}(D)$-*vertex-transitive* or *vertex-transitive* if for any two vertices $x, y \in V$, there is an automorphism $\psi \in \mathrm{Aut}(D)$ such that $x\psi = y$. In general, a subset C of $\mathrm{End}(D)$ is called *vertex-transitive on* D, and D is called C-*vertex-transitive* if, for any two vertices $x, y \in V$, there is an endomorphism $\psi \in C$ such that $x\psi = y$. For a Cayley digraph $\mathrm{Cay}(S,A)$, an element $\varphi \in \mathrm{End}(S,A)$ is said to be a *colour-preserving endomorphism* if $xa = y$ implies $(x\varphi)a = y\varphi$, for all $x, y \in S$ and $a \in A$. We will denote by $\mathrm{ColEnd}(S,A)$ and $\mathrm{ColAut}(S,A)$ the sets of all colour-preserving endomorphisms and all colour-preserving automorphisms of $\mathrm{Cay}(S,A)$, respectively.

It is well known that for every group G and $A \subseteq G$, $\mathrm{Cay}(G,A)$ is $\mathrm{Aut}(G,A)$-vertex-transitive. In 2003, Kelarev and Praeger [9] characterized all $\mathrm{Aut}(S,A)$-vertex-transitive and all $\mathrm{ColAut}(S,A)$-vertex-transitive Cayley graphs of semigroups. Since then many authors have given similar descriptions for all vertex-transitive Cayley graphs of some specific semigroups, see for example in, [4], [10], [11], [14].

The *full transformation semigroup on the set* X denoted by $T(X)$ is the set of all functions from X into itself. It is well known that every semigroup can be embedded in some transformation semigroup. Therefore, some properties of semigroups might be obtained from the study on transformation semigroups. Now, we consider a subsemigroup of $T(X)$, namely $T(X,Y)$, given by

$$T(X,Y) = \{\alpha \in T(X) : \mathrm{im}(\alpha) \subseteq Y\}$$

where $Y \neq \emptyset$ and $\mathrm{im}(\alpha)$ denotes the image of α. generalization of $T(X)$. In this paper, we show under which conditions Cayley graphs of $T(X,Y)$ satisfy the properties of being $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive and $\mathrm{ColAut}(T(X,Y),A)$-vertex-transitive.

2. Preliminaries

For $\alpha \in T(X,Y)$ and $x \in X$, the image of x under α is written as $x\alpha$. For $Z \subseteq X$, $Z\alpha$ denotes the set of all images of elements in Z under α. The kernel of α is defined by

$$\ker(\alpha) = \{(a,b) \in X \times X : a\alpha = b\alpha\}.$$

In 2008, Sanwong and Sommanee [17] described Green's relations on $T(X,Y)$ and then we have the following lemma about an $\mathcal{R}$-relation.

Lemma 2.1 ([17]). *Let* $\alpha, \beta \in T(X,Y)$. *Then* $\beta = \alpha\mu$ *for some* $\mu \in T(X,Y)$ *if and only if* $\ker(\alpha) \subseteq \ker(\beta)$. *Consequently,* $\alpha\mathcal{R}\beta$ *if and only if* $\ker(\alpha) = \ker(\beta)$.

The symbol π_α denotes the partition of X induced by the transformation α, namely

$$\pi_\alpha = \{y\alpha^{-1} : y \in \mathrm{im}(\alpha)\},$$

where $y\alpha^{-1}$ is the set of all $x \in X$ such that $x\alpha = y$. It is easily seen that, for all $\alpha, \beta \in T(X,Y)$,

$$\ker(\alpha) = \ker(\beta) \text{ if and only if } \pi_\alpha = \pi_\beta.$$

The following results are obtained in [16], [18] and [19].

Lemma 2.2 ([16])**.** *Let $\alpha, \beta \in T(X,Y)$. Then $X\beta \subseteq Y\alpha$ if and only if there exists $\gamma \in T(X,Y)$ such that $\gamma\alpha = \beta$.*

Lemma 2.3 ([19])**.** *Let $A \subseteq T(X,Y)$. Then $T(X,Y)A = T(X,Y)$ if and only if $Y\alpha = Y$ for some $\alpha \in A$.*

Lemma 2.4 ([19])**.** *Let $A \subseteq T(X,Y)$ and $Y\alpha = Y$ for all $\alpha \in A$. If elements β and γ are in the same component in* $\mathrm{Cay}(T(X,Y),A)$*, then $\beta \mathcal{R} \gamma$.*

Theorem 2.5 ([19])**.** *Let $A \subseteq T(X,Y)$. Then $\langle A\rangle$ is a completely simple semigroup and $T(X,Y)A = T(X,Y)$ if and only if $Y\alpha = Y$ for all $\alpha \in A$.*

Lemma 2.6 ([18])**.** *Let X be a finite set and Y_1, Y_2 be non-empty subsets of X. Then $T(X,Y_1) \cong T(X,Y_2)$ if and only if $|Y_1| = |Y_2|$.*

By the above lemma, there is no loss of generality in assuming $X = \{1,2,\dots,n\}$ and $Y = \{1,2,\dots,r\}$. For convenience, if $\alpha \in T(X,Y)$ where $\alpha = \begin{pmatrix} 1 & \dots & n \\ a_1 & \dots & a_n \end{pmatrix}$, we write $\alpha = [a_1,\dots,a_n]$.

Recall that a semigroup S is *left* (*right*) *zero semigroup* if $ab = a (ab = b)$ for all $a,b \in S$. A *left simple* (*right simple*) is a semigroup which has no proper left (right) ideal. A *left group* (*right group*) is a semigroup which is left (right) simple and right (left) cancellative. It is well known that a semigroup is a left (right) group if and only if it is isomorphic to the direct product of a group and a left (right) zero semigroup. A *completely simple* is a semigroup which has no proper ideals and contains a minimal idempotent with respect to the partial order $f \le g \Leftrightarrow f = fg = gf$.

A subgraph H of a digraph G is called an *induced subgraph* of G if for every $u,v \in V(H)$, (u,v) is an edge in H whenever (u,v) is an edge in G. For a non-empty set A of vertices of a graph G, the *subgraph of G induced by A* is the induced subgraph with vertex set A and denoted by $G[A]$ or simply $[A]$.

The following two theorems are the characterizations of all $\mathrm{ColAut}(S,A)$-vertex-transitive and $\mathrm{Aut}(S,A)$-vertex-transitive Cayley graphs such that all principal left ideals of $\langle A\rangle$ are finite.

Theorem 2.7 ([9])**.** *Let S be a semigroup, and $A \subseteq S$ such that all principal left ideals of the subsemigroup $\langle A\rangle$ are finite. Then, the Cayley graph* $\mathrm{Cay}(S,A)$ *is* $\mathrm{ColAut}(S,A)$*-vertex-transitive if and only if the following conditions hold:*

(1) *$Sa = S$ for all $a \in A$;*

(2) *$\langle A\rangle$ is isomorphic to a direct product of a left zero band and a group;*

(3) *$|s\langle A\rangle|$ is independent of the choice of $s \in S$.*

Theorem 2.8 ([9])**.** *Let S be a semigroup, and $A \subseteq S$ such that all principal left ideals of the subsemigroup $\langle A\rangle$ are finite. Then, the Cayley graph* $\mathrm{Cay}(S,A)$ *is* $\mathrm{Aut}(S,A)$*-vertex-transitive if and only if the following conditions hold:*

(1) *$SA = S$;*

(2) *$\langle A\rangle$ is a completely simple semigroup;*

(3) *The Cayley graph* $\mathrm{Cay}(\langle A\rangle, A)$ *is Aut$(\langle A\rangle, A)$-vertex-transitive;*

(4) *$|s\langle A\rangle|$ is independent of the choice of $s \in S$.*

3. Vertex-transitivity

Throughout of this section we assume that A is non-empty set. In this section, we show under which conditions Cayley graphs of $T(X,Y)$ enjoy the property of being $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive and the property of being $\mathrm{ColAut}(T(X,Y),A)$-vertex-transitive.

Write $A_Y = \{\alpha_{|_Y} : \alpha \in A\}$. Let $\alpha, \beta, \in T(X,Y)$ and $Y\alpha = Y = Y\beta$. We get, $y \in Y$, $y\alpha = y\alpha_{|_Y}$ and so

$$y(\alpha\beta) = (y\alpha)\beta = (y\alpha_{|_Y})\beta_{|_Y} = y(\alpha_{|_Y}\beta_{|_Y}).$$

Hence $(\alpha\beta)_{|_Y} = \alpha_{|_Y}\beta_{|_Y}$.

Lemma 3.1. *Let $A \subseteq T(X,Y)$ and $Y\alpha = Y$ for all $\alpha \in A$. Then* $\mathrm{Cay}(\langle A\rangle, A)$ *is* $\mathrm{Aut}(\langle A\rangle, A)$*-vertex-transitive.*

Proof. Let $\alpha \in A$. Since $\langle A_Y\rangle$ is a subgroup of $T(Y)$, $\mathrm{Cay}(\langle A_Y\rangle, A_Y)$ is $\mathrm{Aut}(\langle A_Y\rangle, A_Y)$-vertex-transitive and hence $\mathrm{Cay}(\langle A_Y\rangle, A_Y)$ is a strongly connected component. Moreover, $\mathrm{Cay}(\langle A_Y\rangle, A_Y)$ $= [\langle A_Y\rangle]$.

We claim that $[\alpha\langle A\rangle] \cong [\langle A_Y\rangle]$. Define $f : V([\alpha\langle A\rangle]) \to V([\langle A_Y\rangle])$ by $\beta f = \beta_{|_Y}$ for all $\beta \in V([\alpha\langle A\rangle]) = \alpha\langle A\rangle$. Let $\beta, \gamma \in V([\alpha\langle A\rangle])$. Then $\beta = \alpha\beta_1\cdots\beta_k$ and $\gamma = \alpha\gamma_1\cdots\gamma_l$ for some $\beta_1,\dots,\beta_k,\gamma_1,\dots,\gamma_l \in A$. We have $(\lambda\mu)_{|_Y} = \lambda_{|_Y}\mu_{|_Y}$ since $Y\lambda = Y = Y\mu$ for all $\lambda, \mu \in A$. Hence

$$\begin{aligned}(\beta\gamma)f &= (\beta\gamma)_{|_Y}\\ &= [(\alpha\beta_1\cdots\beta_k)(\alpha\gamma_1\cdots\gamma_l)]_{|_Y}\\ &= (\alpha\beta_1\cdots\beta_k)_{|_Y}(\alpha\gamma_1\cdots\gamma_l)_{|_Y}\\ &= \beta_{|_Y}\gamma_{|_Y}\\ &= (\beta f)(\gamma f).\end{aligned}$$

Thus f is a group homomorphism. Now, we show that f is 1-1. Let $\lambda, \mu \in V([\alpha\langle A\rangle])$ be such that $\lambda f = \beta = \mu f$ for some $\beta \in V([\langle A_Y\rangle])$. Then $\lambda_{|_Y} = \beta = \mu_{|_Y}$ and by Lemma 2.4, it implies that $\pi_\lambda = \pi_\mu$. If $y \in Y$, we get $y\lambda = y\mu$. Let $x \in X \setminus Y$. Then $x \in y_i\lambda^{-1}$ for some $y_i \in Y$. Thus there exists $z \in Y \cap y_i\lambda^{-1}$ and so $x\lambda = z\lambda = z\mu = x\mu$ since $\pi_\mu = \pi_\lambda$. Hence $\lambda = \mu$. This implies that f is 1-1.

Now, let $\gamma' \in V([\langle A_Y\rangle])$. Since $[\langle A_Y\rangle]$ is a strongly connected component, there exists a dipath from $\alpha_{|_Y}$ to γ'. That is $\gamma' = \alpha_{|_Y}(\gamma_1)_{|_Y}\cdots(\gamma_k)_{|_Y} = (\alpha\gamma_1\cdots\gamma_k)_{|_Y}$ where $\gamma_1,\dots,\gamma_k \in A$. Therefore, there exists $\alpha(\gamma_1\cdots\gamma_k) \in \alpha\langle A\rangle$ such that $(\alpha\gamma_1\cdots\gamma_k)f = \gamma'$, and hence f is onto.

Next, we show that f preserves arcs. Let $(\beta,\gamma) \in E([\alpha\langle A\rangle])$ for some $\beta, \gamma \in V([\alpha\langle A\rangle])$. Then $\gamma = \beta\delta$ for some $\delta \in A$ and so

$$\gamma_{|_Y} = \gamma f = (\beta\delta)f = (\beta f)(\delta f) = \beta_{|_Y}\delta_{|_Y}.$$

Hence $(\beta_{|_Y},\gamma_{|_Y}) \in E([\langle A_Y\rangle])$. Let $(\lambda',\mu') \in E([\langle A_Y\rangle])$ for some $\lambda',\mu' \in V([\langle A_Y\rangle])$ be such that $\mu f = \mu'$ and $\lambda f = \lambda'$ for some $\mu,\lambda \in V([\alpha\langle A\rangle])$. Then $\lambda'\beta' = \mu'$ for some $\beta' \in A_Y$. Thus $\beta' = \beta f = \beta_{|_Y}$ for some $\beta \in V([\alpha\langle A\rangle])$. Thus

$$(\lambda\beta)f = (\lambda f)(\beta f) = (\lambda_{|_Y})(\beta_{|_Y}) = \lambda'\beta' = \mu' = \mu_{|_Y} = \mu f.$$

This implies that $\lambda\beta = \mu$ since f is $1-1$ and so $(\lambda,\mu) \in E([\alpha\langle A\rangle])$. Therefore, $[\alpha\langle A\rangle] \cong [\langle A_Y\rangle]$ for all $\alpha \in A$. Since $[\langle A_Y\rangle]$ is $\mathrm{Aut}(\langle A_Y\rangle, A_Y)$-vertex-transitive, $[\alpha\langle A\rangle]$ is $\mathrm{Aut}([\alpha\langle A\rangle])$-vertex-transitive for all $\alpha\langle A\rangle \subseteq \langle A\rangle$.

Next, we show $\langle A\rangle = \dot{\bigcup}_{\alpha\in A}\alpha\langle A\rangle$ and $[\langle A\rangle] = \dot{\bigcup}_{\alpha\in A}[\alpha\langle A\rangle]$. Clearly, $\dot{\bigcup}_{\alpha\in A}\alpha\langle A\rangle \subseteq \langle A\rangle$. Let $\beta \in \langle A\rangle$. Then $\beta = \beta_1\cdots\beta_k$ where $\beta_1,\dots,\beta_k \in A$ and so $\beta \in \beta_1\langle A\rangle$. Hence $\langle A\rangle \subseteq \bigcup_{\alpha\in A}\alpha\langle A\rangle$ and thus $\langle A\rangle = \bigcup_{\alpha\in A}\alpha\langle A\rangle$. Let $\gamma \in \alpha\langle A\rangle \cap \delta\langle A\rangle$. By the proof of Lemma 2.4, there are dipaths from γ to α and δ to γ, i.e., $\alpha = \gamma\alpha_1\cdots\alpha_l$ and $\gamma = \delta\gamma_1\cdots\gamma_h$ where $\alpha_1,\dots,\alpha_l,\gamma_1,\dots,\gamma_h \in A$. Let $\lambda \in \alpha\langle A\rangle$. Then

$$\begin{aligned}\lambda &= \alpha\lambda_1\cdots\lambda_s \qquad \text{where } \lambda_1,\dots,\lambda_s \in A\\ &= \gamma\alpha_1\dots\alpha_l\lambda_1\cdots\lambda_s\\ &= \delta\gamma_1\cdots\gamma_h\alpha_1\cdots\alpha_l\lambda_1\cdots\lambda_s \in \delta\langle A\rangle.\end{aligned}$$

Therefore, $\alpha\langle A\rangle \subseteq \delta\langle A\rangle$. Similarly, $\delta\langle A\rangle \subseteq \alpha\langle A\rangle$. Hence $\delta\langle A\rangle = \alpha\langle A\rangle$. Therefore, $\langle A\rangle = \dot{\bigcup}_{\alpha\in A}\alpha\langle A\rangle$.

Finally, we show that $[\langle A\rangle] = \dot{\bigcup}_{\alpha\in A}[\alpha\langle A\rangle]$ where $[\langle A\rangle]$ is the induced subgraph of $\mathrm{Cay}(T(X,Y),A)$. By the above proof, we have $V([\langle A\rangle]) = V(\dot{\bigcup}_{\alpha\in A}[\alpha\langle A\rangle])$. It remains to prove that $E([\langle A\rangle]) = \bigcup_{\alpha\in A}E([\alpha\langle A\rangle])$. It is obvious that $\bigcup_{\alpha\in A}E([\alpha\langle A\rangle]) \subseteq E([\langle A\rangle)]$. Now, let $(\beta,\gamma) \in E([\langle A\rangle])$ where $\beta,\gamma \in \langle A\rangle$. Then $\gamma = \beta\delta$ for some $\delta \in A$ and $\beta = \beta_1\cdots\beta_k$ where $\beta_1,\dots,\beta_k \in A$. So $\beta,\gamma \in \beta_1\langle A\rangle$ and thus $(\beta,\gamma) \in E([\beta_1\langle A\rangle]) \subseteq \bigcup_{\alpha\in A}E([\alpha\langle A\rangle])$. Hence

$$E([\langle A\rangle]) \subseteq \bigcup_{\alpha\in A}E([\alpha\langle A\rangle]).$$

Thus $[\langle A\rangle] = \dot{\bigcup}_{\alpha\in A}[\alpha\langle A\rangle]$. It is clear that $\mathrm{Cay}(\langle A\rangle,A) = [\langle A\rangle]$. Therefore, $\mathrm{Cay}(\langle A\rangle,A) = \dot{\bigcup}_{\alpha\in A}[\alpha\langle A\rangle]$ is $\mathrm{Aut}(\langle A\rangle,A)$-vertex-transitive as required. □

Example 3.2. Let $X = \{1,2,\dots,7\}$, $Y = \{1,2,\dots,6\}$, $A = \{\alpha,\beta\}$ a subset of $T(X,Y)$, $\alpha = [2,3,1,5,6,4,2]$ and $\beta = [4,5,6,1,2,3,1]$.

We have $\mathrm{Cay}(\langle A\rangle,A)$ as shown in Figure 1. We get that $Y\alpha = Y = Y\beta$ and $\mathrm{Cay}(\langle A\rangle,A)$ is $\mathrm{Aut}(\langle A\rangle,A)$-vertex-transitive.

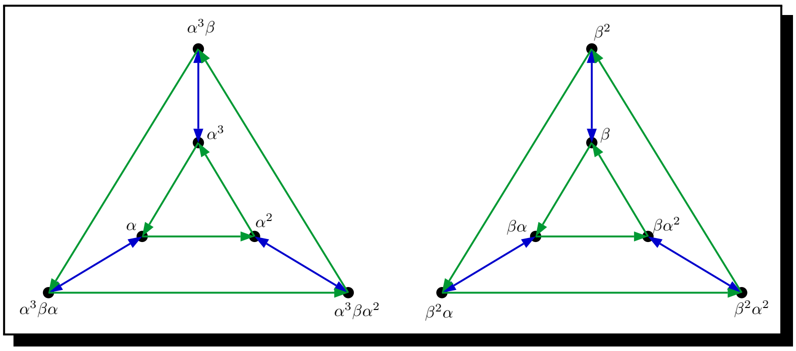

Figure 1. $\mathrm{Cay}(\langle A\rangle,A)$ where $A = \{[2,3,1,5,6,4,2],[4,5,6,1,2,3,1]\}$

In the following theorems, we shall give the descriptions for all vertex-transitive Cayley graphs of $T(X,Y)$.

Theorem 3.3. *Let* $A \subseteq T(X,Y)$. *Then* $\mathrm{Cay}(T(X,Y),A)$ *is* $\mathrm{Aut}(T(X,Y),A)$*-vertex-transitive if and only if the following conditions hold:*

(1) $Y\alpha = Y$ *for all* $\alpha \in A$;

(2) $|\beta\langle A\rangle|$ *is independent of the choice of* $\beta \in T(X,Y)$.

Proof. ($\Rightarrow$) Assume that $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive. By Theorem 2.8, the condition (2) holds and $T(X,Y)A = T(X,Y)$. Thus $Y\alpha = Y$ for all $\alpha \in A$ by Theorem 2.5.
($\Leftarrow$) We will prove by using Theorem 2.8. It remains to prove that $T(X,Y)A = T(X,Y)$, $\mathrm{Cay}(\langle A\rangle,A)$ is $\mathrm{Aut}(\langle A\rangle,A)$-vertex-transitive, and $\langle A\rangle$ is a completely simple semigroup. By Lemma 2.3 and Lemma 3.1, we get $T(X,Y)A = T(X,Y)$ and $\mathrm{Cay}(\langle A\rangle,A)$ is $\mathrm{Aut}(\langle A\rangle,A)$-vertex-transitive. Since $Y\alpha = Y$ for all $\alpha \in A$, $\langle A\rangle$ is a completely simple semigroup by Theorem 2.5 which completes the proof. □

Example 3.4. Let $X = \{1,2,3,4\}$, $Y = \{1,2\}$, and $A = \{[2,1,1,2],[1,2,1,1]\}$. Then $Y\alpha = Y$ for all $\alpha \in A$ and $|\beta\langle A\rangle|$ is independent of the choice of $\beta \in T(X,Y)$. By Theorem 3.3, $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive and shown in the following figure.

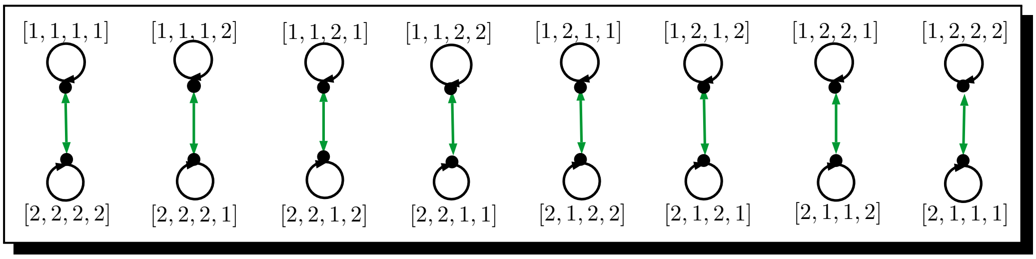

Figure 2. $\mathrm{Cay}(T(X,Y),A)$ where $A = \{[2,1,1,2],[1,2,1,1]\}$

Corollary 3.5. *Let* $A \subseteq T(X,Y)$. *Then* $\mathrm{Cay}(T(X,Y),A)$ *is* $\mathrm{Aut}(T(X,Y),A)$*-vertex-transitive if and only if* $\mathrm{Cay}(T(Y),A_Y)$ *is* $\mathrm{Aut}(T(Y),A_Y)$*-vertex-transitive.*

Proof. ($\Rightarrow$) Assume that $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive. By Theorem 3.3(1), we have $Y\alpha_{|_Y} = Y$ for all $\alpha_{|_Y} \in A_Y$. We claim that $|\gamma'\langle A_Y\rangle| = |\gamma\langle A\rangle|$ where $\gamma' = \gamma_{|_Y}$ in $T(Y)$, $\gamma \in T(X,Y)$. Now, define $g : \gamma\langle A\rangle \longrightarrow \gamma'\langle A_Y\rangle$ by $(\gamma\beta)g = \gamma'\beta_{|_Y}$. We can prove that g is 1-1 and onto which is similar to prove that f is 1-1 and onto in Lemma 3.1. Hence $|\gamma\langle A\rangle| = |\gamma'\langle A_Y\rangle|$ for all $\gamma' \in T(Y)$ where $\gamma' = \gamma_{|_Y}$ for some $\gamma \in T(X,Y)$. For $\beta',\delta' \in T(Y)$, there exist $\beta,\delta \in T(X,Y)$ such that $\beta' = \beta_{|_Y}, \delta' = \delta_{|_Y}$. From Theorem 3.3(2), we have $|\beta'\langle A_Y\rangle| = |\beta\langle A\rangle| = |\delta\langle A\rangle| = |\delta'\langle A_Y\rangle|$.

($\Leftarrow$) Assume that $\mathrm{Cay}(T(Y),A_Y)$ is $\mathrm{Aut}(T(Y),A_Y)$-vertex-transitive. Then, by Theorem 3.3, we have

(i) $Y\alpha_{|_Y} = Y$ for all $\alpha_{|_Y} \in A_Y$ and

(ii) $|\beta'\langle A_Y\rangle|$ is independent of the choice of $\beta' \in T(Y)$.

By (i), we get that $Y\alpha = Y$ for all $\alpha \in A$. Finally, let $\gamma \in T(X,Y)$. Then $\gamma' = \gamma_{|_Y} \in T(Y)$. Thus we have a function g from $\gamma\langle A\rangle$ to $\gamma'\langle A_Y\rangle$ which is 1-1 and onto. By (ii), $|\gamma\langle A\rangle| = |\gamma'\langle A_Y\rangle| = |\beta'\langle A_Y\rangle| = |\beta\langle A\rangle|$. □

Example 3.6. From Example 3.2, we let $A_Y = \{\alpha_{|_Y}, \beta_{|_Y}\}$ where $\alpha_{|_Y} = [2,3,1,5,6,4]$ and $\beta_{|_Y} = [4,5,6,1,2,3]$. We have $\mathrm{Cay}(\langle A_Y\rangle, A_Y)$ is $\mathrm{Aut}(\langle A_Y\rangle, A_Y)$-vertex-transitive, as shown in Figure 3 and $\mathrm{Cay}(\langle A\rangle, A)$ also is, as in Figure 1.

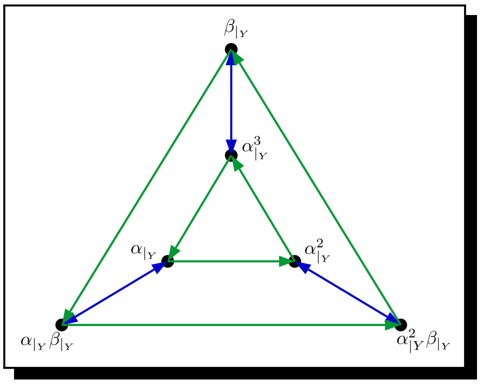

Figure 3. $\mathrm{Cay}(\langle A_Y\rangle, A_Y)$

Now we consider the set A such that $Y\alpha = Y$ for all $\alpha \in A$. We get that

$$\langle A\rangle = \{\beta \in T(X,Y) : \beta_{|_Y} \in \langle A_Y\rangle \text{ and } \beta\mathcal{R}\alpha \text{ for some } \alpha \in A\}.$$

It is well known that for $\phi \in T(X)$, ϕ is an idempotent if and only if $x\phi = x$ for all $x \in \mathrm{im}(\phi)$. Then the set of all idempotents of $\langle A\rangle$ is

$$E(\langle A\rangle) = \{\varepsilon = [1,2,\dots,r,z_1,z_2,\dots,z_{n-r}] \in T(X,Y) : \varepsilon\mathcal{R}\alpha \text{ for some } \alpha \in A\}.$$

Moreover, $E(\langle A\rangle)$ is a left zero semigroup.

Lemma 3.7. *Let $A \subseteq T(X,Y)$. If $Y\alpha = Y$ for all $\alpha \in A$, then $\langle A\rangle$ is a left group.*

Proof. Let $Y\alpha = Y$ for all $\alpha \in Y$ and $E = E(\langle A\rangle)$ be the set of all idempotents in $\langle A\rangle$. Then $\langle A_Y\rangle$ is a subgroup of $T(Y)$ and E is a left zero band. Now, we show that $\langle A\rangle \cong \langle A_Y\rangle \times E$.

Define $f : \langle A_Y\rangle \times E \to \langle A\rangle$ by for $\alpha' = [y_1,\dots,y_r] \in \langle A_Y\rangle$ and $\varepsilon = [1,2,\dots,r,z_1,\dots,z_{n-r}] \in E$,

$$([y_1,\dots,y_r],[1,2,\dots,r,z_1,\dots,z_{n-r}]) \mapsto [y_1,\dots,y_r,z_1\alpha',\dots,z_{n-r}\alpha'].$$

It easily see that f is a well-defined homomorphism and for $\alpha' \in \langle A_Y\rangle$ and $\varepsilon \in E$, there exists $\alpha \in A$ such that $\alpha_{|_Y} = \alpha'$. It follows that $(\alpha',\varepsilon)f \in \langle A\rangle$. Next, we show that f is $1-1$. Let $(\alpha',\varepsilon_1),(\beta',\varepsilon_2) \in \langle A_Y\rangle \times E$ be such that $(\alpha',\varepsilon_1)f = \gamma = (\beta',\varepsilon_2)f$ where $\varepsilon_1 = [1,2,\dots,r,a_1,\dots,a_{n-r}]$ and $\varepsilon_2 = [1,2,\dots,r,b_1,\dots,b_{n-r}]$. Let $\gamma = [y_1,\dots,y_r,z_1,\dots,z_{n-r}]$. Then $\alpha' = [y_1,\dots,y_r] = \beta'$ and $a_i\alpha' = z_i = b_i\beta'$ for all $i = 1,\dots,n-r$ and it implies that $a_i = b_i$. Thus $\varepsilon_1 = \varepsilon_2$ and consequently, $(\alpha',\varepsilon_1) = (\beta',\varepsilon_2)$. Finally, we prove that f is onto. Let $\delta \in \langle A\rangle$ be such that $\delta = [y_1,\dots,y_r,z_1,\dots,z_{n-r}]$. Then $\delta' = \delta_{|_Y} = [y_1,\dots,y_r] \in \langle A_Y\rangle$. Set $\varepsilon = [1,2,\dots,r,a_1,\dots,a_{n-r}]$ where $a_i \in z_i\delta^{-1} \cap Y$. So $\varepsilon \in E$ and $(\delta',\varepsilon)f = \delta$. Therefore, f is onto. □

Theorem 3.8. *The Cayley graph* $\mathrm{Cay}(T(X,Y),A)$ *is* $\mathrm{ColAut}(T(X,Y),A)$*-vertex-transitive if and only if it is* $\mathrm{Aut}(T(X,Y),A)$*-vertex-transitive.*

Proof. Obviously, if $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{ColAut}(T(X,Y),A)$-vertex-transitive, then it is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive. Conversely, assume that $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive. Then $Y\alpha = Y$ for all $\alpha \in A$ and so $T(X,Y)\alpha = T(X,Y)$ for all $\alpha \in A$ by Lemma 2.3. Lemma 3.7 implies that $\langle A \rangle$ is isomophic to a direct product of a group and a left zero band. It follows that $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{ColAut}(T(X,Y),A)$-vertex-transitive by Theorem 2.7 and the proof is complete. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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