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Research Article

New Fixed Point Results via C-class Functions in b-Rectangular Metric Spaces

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Abstract. In this paper, we prove some fixed point results in the setting of b-rectangular metric space via C-class functions. Moreover, in the last part of the paper, we point out that there is a slight flaw in the proof of Erhan *et al*. [14, Theorem 4] and present a correct version of the theorem.

Keywords. b-rectangular metric space; Fixed point; C-class functions

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1. Introduction and preliminaries

Brianciari [11] generalized the concept of a metric where the triangular inequality is replaced by a rectangular one. Using this concept, many papers have been done in order to prove (common) fixed point results (for more details, see [5, 6, 15–17, 22–24] and [25]). On the other hand, the idea of a b-metric has been introduced in the papers [12] and [13] (for other results, see [1, 2, 7–10] and [20]). Extending the above concepts, the following definition was given by Roshan et al. [21, Lemma 1.10] (see also [4]).

Definition 1.1. Let X be a nonempty set, $s \geq 1$ be a given real number and let $d : X \times X \to [0,+\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ (b-rectangular inequality).

Then (X,d) is called a b-rectangular or a b-generalized metric space (b-g.m.s.).

The following is an easy example of a b-g.m.s.

Example 1.2. Let $X = A \cup B$, where $A = \{0,1\}$ and $B = \{\frac{1}{n} : n = 2,3,4,\dots\}$.
Define $d : X \times X \to [0,\infty)$ by

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y \\ 4, & \text{if } x \neq y \text{ and } \{x,y\} \subseteq B \\ 1, & \text{if } x \in B,\ y \in A \text{ and } x \neq y \ \text{ or } \ \{x,y\} \subseteq A \text{ and } x \neq y. \end{cases}$$

Then (X,d) is a b-g.m.s with coefficient $s = 2 > 1$, but (X,d) is not a g.m.s, as $d\left(\frac{1}{2},\frac{1}{4}\right) = 4 > 3 = d\left(\frac{1}{2},0\right) + d(0,1) + d\left(1,\frac{1}{4}\right)$.

The following lemma dif and only ifers from [15, Lemma 1.10] and [17, Lemma 1]. We need it in the sequel.

Lemma 1.3 ([21, Lemma 1]). *Let (X,d) be a b-g.m.s. and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.*

The following lemma is also useful for the rest.

Lemma 1.4 ([21, Example 1.1]). *Let (X,d) be a b-g.m.s.*

(a) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, and $x_n \neq x$, $y_n \neq y$ for $n \in \mathbb{N}$. Then, we have*

$$\frac{1}{s} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s d(x,y).$$

(b) *If $y \in X$ and $\{x_n\}$ is a nonconstant Cauchy sequence in X with $x_n \neq x_m$ for all $n \neq m$, converging to $x \neq y$, then*

$$\frac{1}{s} d(x,y) \leq \liminf_{n\to\infty} d(x_n,y) \leq \limsup_{n\to\infty} d(x_n,y) \leq s d(x,y),$$

for all $x \in X$.

Definition 1.5 ([3]). A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called a *C-class* function if it is continuous and satisfies the following axioms:

(1) $f(s,t) \leq s$ for all $s,t \in [0,\infty)$;

(2) $f(s,t) = s$ implies that either $s = 0$, or $t = 0$.

We will denote the family of C-class functions as $\mathcal{C}$ (see also, [19]). Note that for some $F \in \mathcal{C}$, we add the condition $F(0,0) = 0$.

Example 1.6 ([3]). The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:

(1) $F(s,t) = s - t$;

(2) $F(s,t) = ms$ for $0 < m < 1$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$ for $r \in (0,\infty)$.

Definition 1.7 ([18]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.8. We let Ψ denote the class of the altering distance functions.

Definition 1.9 ([3]). An ultra altering distance function is a continuous and nondecreasing mapping $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) > 0$ for all $t > 0$.

Remark 1.10. Let Φ_u denote the set of all ultra altering distance functions.

Definition 1.11. Let $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$. The tripled (ψ,φ,F) is said to be monotone if for any $x, y \in [0,\infty)$

$$x \le y \Longrightarrow F(\psi(x),\varphi(x)) \le F(\psi(y),\varphi(y)).$$

Example 1.12. Let $F(s,t) = s - t$, $\phi(x) = \sqrt{x}$ and

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then (ψ,ϕ,F) is monotone.

Example 1.13. Let $F(s,t) = s - t$, $\phi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2, & \text{if } x > 1, \end{cases}$$

then (ψ,ϕ,F) is not monotone.

Example 1.14. Let $F(s,t) = \frac{s}{1+t}$, $\phi(x) = \sqrt[3]{x}$ and

$$\psi(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \le x \le 1, \\ x^3, & \text{if } x > 1, \end{cases}$$

then (ψ,ϕ,F) is monotone.

Example 1.15. Let $F(s,t) = s - t$, $\phi(x) = x^3$ and

$$\psi(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \le x \le 1, \\ x^3, & \text{if } x > 1, \end{cases}$$

then (ψ, ϕ, F) is not monotone.

Example 1.16. Let $F(s,t) = \log\left(\frac{t+e^s}{1+t}\right)$, $\psi(x) = x$ and $\phi(x) = e^x$, then (ψ, ϕ, F) is monotone.

2. Main Results

Our first main result is

Theorem 2.1. *Let $(X, \preceq, d)$ be a complete b-g.m.s. and $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\psi(sd(fx, fy)) \le F(\psi(M(x,y)), \varphi(M(x,y))) \tag{2.1}$$

for all comparable elements $x, y \in X$, where $\psi \in \Psi$, $\varphi \in \Phi_u$, $F \in C$, such that (ψ, φ, F) is monotone. Assume also that $\psi(r+t) \le \psi(r) + \psi(t)$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(x,f^2y)}{1+s[d(x,fx)+d(y,fy)+d(fy,f^2y)]}, \frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\right\}.$$

If f is continuous, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Let $x_0 \in X$. Taking $x_n = f^n x_0$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = fx_n$, i.e., x_n is a fixed point of f. From now on, we suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq fx_0$ and f is an increasing function, we get

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^n x_0 \preceq f^{n+1}x_0 \preceq \cdots.$$

Step 1: We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Having in mind $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by Definition 1.11 in (2.1), we get

$$\begin{aligned} \psi(sd(x_n, x_{n+1})) &= \psi(sd(fx_{n-1}, fx_n)) \\ &\le F(\psi(M(x_{n-1}, x_n)), \varphi(M(x_{n-1}, x_n))) \\ &\le F(\psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\})), \varphi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\})). \end{aligned} \tag{2.3}$$

We used that

$$M(x_{n-1}, x_n) = \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(fx_{n-1},fx_n)}, \frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},fx_{n-1})d(x_{n-1},f^2x_n)}{1+s[d(x_{n-1},fx_{n-1})+d(x_n,fx_n)+d(fx_n,f^2x_n)]},\right.$$

$$
\begin{aligned}
&\left.\frac{d(x_{n-1},fx_{n-1})d(x_{n-1},fx_n)}{1+d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}\right\} \\
&= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)},\right. \\
&\quad \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+2})}{1+s[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}, \\
&\quad \left.\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_{n+1})+d(x_n,x_n)}\right\} \\
&\le \max\left\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}),\right. \\
&\quad \frac{d(x_{n-1},x_n)s[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}{1+s[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]}, \\
&\quad \left.\frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1})}{1+d(x_{n-1},x_{n+1})}\right\} \\
&\le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}.
\end{aligned}
$$

If for some $n \ge 1$, $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then from (2.3) we obtain

$$
\begin{aligned}
\psi(sd(x_n,x_{n+1})) &\le F(\psi(d(x_n,x_{n+1})), \varphi(d(x_n,x_{n+1}))) \\
&\le \psi(d(x_n,x_{n+1})) \le \psi(sd(x_n,x_{n+1})).
\end{aligned} \tag{2.4}
$$

Thus $\psi(d(x_n,x_{n+1})) = 0$ or $\varphi(d(x_n,x_{n+1})) = 0$. This implies that $d(x_n,x_{n+1}) = 0$, which is a contradiction. We deduce that $\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$ for all $n \ge 1$. Again by (2.3), we have

$$
\begin{aligned}
\psi(sd(x_n,x_{n+1})) &\le F(\psi(d(x_{n-1},x_n)), \varphi(d(x_{n-1},x_n))) \\
&\le \psi(d(x_{n-1},x_n)) \le \psi(sd(x_{n-1},x_n)),
\end{aligned}
$$

which implies that

$$
d(x_n,x_{n+1}) \le d(x_{n-1},x_n), \quad \text{for all } n \ge 1. \tag{2.5}
$$

The sequence $\{d(x_n,x_{n+1})\}$ is decreasing, and so there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. Assume that $r > 0$. Letting with $n \to \infty$ in (2.3),

$$
\psi(sr) \le F(\psi(r), \varphi(r)) \le \psi(r) \le \psi(sr).
$$

So $\psi(r) = 0$ or $\varphi(r) = 0$. This implies that $r = 0$, which is a contradiction. Hence (2.2) is proved.

Step 2: We have $x_n \ne x_m$ for all $n, m \in \mathbb{N}$.

We argue by contradiction. Assume that $x_n = x_m$ for some $n > m$, so $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process, $x_{n+k} = x_{m+k}$ for each $k \in \mathbb{N}$. Then (2.1) implies that

$$
\begin{aligned}
\psi(d(x_m,x_{m+1})) &= \psi(d(x_n,x_{n+1})) \\
&\le \psi(sd(x_n,x_{n+1})) = \psi(sd(fx_{n-1},fx_n)) \\
&\le F(\psi(M(x_{n-1},x_n)), \varphi(M(x_{n-1},x_n))) \\
&\le F\big(\psi(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})), \varphi\big(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\big).
\end{aligned}
$$

If $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_n,x_{n+1})$ for some $n\geq 1$, then

$$\begin{aligned}\psi(d(x_m,x_{m+1}))&=\psi(d(x_n,x_{n+1}))\\&\leq\psi(sd(x_n,x_{n+1}))\\&=\psi(sd(fx_{n-1},fx_n))\\&\leq F(\psi(d(x_n,x_{n+1})),\varphi(d(x_n,x_{n+1})))\\&\leq\psi(d(x_n,x_{n+1}))\\&=\psi(d(x_m,x_{m+1})).\end{aligned}$$

So, $\psi(d(x_n,x_{n+1}))=0$ or $\varphi(d(x_n,x_{n+1}))=0$. This implies that $d(x_n,x_{n+1})=0$, which is a contradiction. Thus $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_{n-1},x_n)$ for all $n\geq 1$. Then we obtain

$$\begin{aligned}\psi(d(x_m,x_{m+1}))&<\psi(d(x_{n-1},x_n))\\&\leq F(\psi(M(x_{n-2},x_{n-1})),\varphi(M(x_{n-2},x_{n-1})))\\&\leq\psi(d(x_{n-2},x_{n-1}))\\&\vdots\\&\leq F(\psi(M(x_m,x_{m+1})),\varphi(M(x_m,x_{m+1})))\\&=F(\psi(d(x_m,x_{m+1})),\varphi(d(x_m,x_{m+1})))\\&<\psi(d(x_m,x_{m+1})).\end{aligned}$$

Thus $\psi(d(x_m,x_{m+1}))=0$ or $\varphi(d(x_m,x_{m+1}))=0$. This implies that $d(x_m,x_{m+1})=0$, which is a contradiction. That is, we can assume that $x_n\neq x_m$ for all $n\neq m$.

Step 3: We will show that $\{x_n\}$ is a b-g.m.s Cauchy sequence.
Using the b-rectangular inequality and a property of ψ in (2.1),

$$\begin{aligned}\psi(d(x_n,x_m))&\leq\psi(sd(x_n,x_{n+1})+sd(x_{n+1},x_{m+1})+sd(x_{m+1},x_m))\\&\leq\psi(sd(x_n,x_{n+1}))+\psi(sd(x_{n+1},x_{m+1}))+\psi(sd(x_{m+1},x_m))\\&\leq\psi(sd(x_n,x_{n+1}))+F(\psi(M(x_n,x_m)),\varphi(M(x_n,x_m)))+\psi(sd(x_m,x_{m+1})).\end{aligned}\tag{2.6}$$

But,

$$\begin{aligned}d(x_n,x_m)&\leq M(x_n,x_m)\\&=\max\Bigg\{d(x_n,x_m),\frac{d(x_n,fx_n)d(x_m,fx_m)}{1+d(fx_n,fx_m)},\frac{d(x_n,fx_n)d(x_m,fx_m)}{1+d(x_n,x_m)},\\&\qquad\frac{d(x_n,fx_n)d(x_n,f^2x_m)}{1+s[d(x_n,fx_n)+d(x_m,fx_m)+d(fx_m,f^2x_m)]},\\&\qquad\frac{d(x_n,fx_n)d(x_n,fx_m)}{1+d(x_n,fx_m)+d(x_m,fx_n)}\Bigg\}.\end{aligned}\tag{2.7}$$

Therefore, from (2.2) and (2.7)

$$\limsup_{m,n\to\infty}M(x_n,x_m)=\limsup_{m,n\to\infty}d(x_n,x_m).\tag{2.8}$$

Taking lim sup as $m,n \to \infty$ in (2.6) and applying again (2.2), we get

$$\begin{aligned}\psi(\limsup_{m,n\to\infty} d(x_n,x_m)) &\le \limsup_{m,n\to\infty} F(\psi(M(x_n,x_m)),\varphi(M(x_n,x_m)))\\ &\le F(\psi(\limsup_{m,n\to\infty}(M(x_n,x_m)),\varphi(\limsup_{m,n\to\infty} M(x_n,x_m)))\\ &\le \psi(\limsup_{m,n\to\infty} M(x_n,x_m))\\ &\le \psi(\limsup_{m,n\to\infty} d(x_n,x_m))\end{aligned} \tag{2.9}$$

which implies that

$$\psi(\limsup_{m,n\to\infty} d(x_n,x_m)) = 0 \quad \text{or} \quad \varphi(\limsup_{m,n\to\infty} d(x_n,x_m)) = 0.$$

So

$$\lim_{m,n\to\infty} \sup d(x_n,x_m) = 0. \tag{2.10}$$

Consequently, $\{x_n\}$ is a b-g.m.s Cauchy sequence in X.

Step 4: We shall prove that f has a fixed point.
Since (X,d) is b-g.m.s complete, the sequence $\{x_n\}$ b-g.m.s-converges to some $z \in X$, that is, $\lim_{n\to\infty} d(x_n,z) = 0$. We shall show that such z is a fixed point of f. We argue by contradiction. Suppose that $fz \ne z$. From Lemma 1.3, it follows that x_n dif and only ifers from both fz and z for n sufficiently large. Using the b-rectangular inequality,

$$d(fz,z) \le sd(fz,fx_n) + sd(fx_n,fx_{n+1}) + sd(fx_{n+1},z).$$

Taking $n \to \infty$, the continuity of f yields that $fz = z$. Therefore, z is a fixed point of f.

Step 5: We shall show that the set of fixed point of f is well ordered if only if f has a unique fixed point.
Let u and v be two fixed points of f such that $u \ne v$. From (2.1), we obtain

$$\begin{aligned}\psi(d(u,v)) = \psi(d(fu,fv)) &\le F(\psi(M(u,v)),\varphi(M(u,v)))\\ &= F(\psi(d(u,v)),\varphi(d(u,v))).\end{aligned} \tag{2.11}$$

But

$$\begin{aligned}M(u,v) &= \max\left\{d(u,v), \frac{d(u,fu)d(v,fv)}{1+d(fu,fv)}, \frac{d(u,fu)d(v,fv)}{1+d(u,v)},\right.\\ &\qquad \left.\frac{d(u,fu)d(u,f^2v)}{1+s[d(u,fu)+d(v,fv)+d(fv,f^2v)]}, \frac{d(u,fu)d(u,fv)}{1+d(u,fv)+d(v,fu)}\right\}\\ &= \max\{d(u,v),0\} = d(u,v).\end{aligned}$$

Then (2.11) leads to $\psi(d(u,v)) = 0$ or $\varphi(d(u,v)) = 0$. This implies that $d(u,v) = 0$, a contradiction. Hence $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is a singleton and hence it is well ordered. □

The continuity of f in Theorem 2.1 can be dropped and be replaced by the following hypothesis:

(H) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq u$ for all k.

Theorem 2.2. *Assume that all hypotheses of Theorem 2.1 hold, except that the continuity assumption on f is replaced by (H). Then f has a fixed point.*

Proof. From the proof of Theorem 2.1, we construct an increasing Cauchy sequence $\{x_n\}$ with $x_n \neq x_m$ for all $m \neq n$ in X such that $x_n \to z \in X$. By using (H), we obtain $x_{n(k)} \preceq z$. Now, we will show that $fz = z$. On contrary, assume that $fz \neq z$. From Lemma 1.4 and (2.1),

$$\begin{aligned}\psi(d(z,fz)) &= \psi\left(s\frac{1}{s}d(z,fz)\right)\\ &\le \psi\left(s\limsup_{k\to\infty} d(x_{n(k)+1},fz)\right)\\ &= \limsup_{k\to\infty}\psi(sd(x_{n(k)+1},fz))\\ &\le F\left(\psi\left(\lim_{k\to\infty}\sup M(x_{n(k)},z)\right),\varphi\left(\lim_{k\to\infty}\inf M(x_{n(k)},z)\right)\right),\end{aligned}$$

where

$$\begin{aligned}M(x_{n(k)},z) = \max\Bigg\{& d(x_{n(k)},z), \frac{d(x_{n(k)},fx_{n(k)})d(z,fz)}{1+d(fx_{n(k)},fz)}, \frac{d(x_{n(k)},fx_{n(k)})d(z,fz)}{1+d(x_{n(k)},z)},\\ & \frac{d(x_{n(k)},fx_n)d(x_{n(k)},f^2z)}{1+s[d(x_{n(k)},fx_{n(k)})+d(z,fz)+d(fz,f^2z)]},\\ & \frac{d(x_{n(k)},fx_{n(k)})d(x_{n(k)},fz)}{1+d(x_{n(k)},fz)+d(z,fx_{n(k)})}\Bigg\}.\end{aligned}$$

Letting $k \to \infty$ and using (2.2)), we get $\psi(d(z,fz)) \le F(\psi(0),\varphi(0)) = 0$, a contradiction. This implies that $z = fz$. □

By choosing $F(s,t) = rs$, where $0 \le r < 1$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.3 ([21]). *Let $(X,\preceq,d)$ be a complete b-g.m.s. and $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Assume there exists r with $0 \le r < \frac{1}{s}$ such that*

$$d(fx,fy) \le rM(x,y),$$

for all comparable elements $x,y \in X$, where

$$\begin{aligned}M(x,y) = \max\Bigg\{& d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(x,y)},\\ & \frac{d(x,fx)d(x,f^2y)}{1+s[d(x,fx)+d(y,fy)+d(fy,f^2y)]}, \frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\Bigg\}.\end{aligned}$$

If f is continuous or (H) holds, then f has a fixed point. Also, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.4 ([21]). *Let $(X,\preceq,d)$ be a partially ordered complete b-g.m.s. and $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$.*

Assume that

$$d(fx,fy)\le \alpha d(x,y)+\beta\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}+\gamma\frac{d(x,fx)d(y,fy)}{1+d(x,y)} +\delta\frac{d(x,fx)d(x,f^2y)}{1+s[d(x,fx)+d(y,fy)+d(fy,f^2y)]}+\lambda\frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)} \tag{2.12}$$

for all comparable elements $x,y\in X$, where $\alpha,\beta\ge 0$ and $\alpha+\beta+\gamma+\delta+\lambda<\frac{1}{s}$. If f is continuous or (H) holds, then f has a fixed point.

By choosing $F(s,t)=s-t$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.5. *Let $(X,\preceq,d)$ be a complete b-g.m.s. and $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq fx_0$. Assume that*

$$\psi(d(fx,fy))\le\psi(M(x,y))-\varphi(M(x,y)),$$

for all comparable elements $x,y\in X$, where $\psi,\varphi\in\Psi$ with $\psi(r+t)\le\psi(r)+\psi(t)$ and

$$M(x,y)=\max\left\{d(x,y),\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)},\frac{d(x,fx)d(y,fy)}{1+d(x,y)},\frac{d(x,fx)d(x,f^2y)}{1+s[d(x,fx)+d(y,fy)+d(fy,f^2y)]},\frac{d(x,fx)d(x,fy)}{1+d(x,fy)+d(y,fx)}\right\}.$$

If f is continuous or (H) holds, then f has a fixed point. Also, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.6. *Let $(X,\preceq,d)$ be a partially ordered complete b-g.m.s. and $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq fx_0$. Assume that*

$$\psi(sd(fx,fy))\le F\big(\psi(M(x,y)),\varphi(M(x,y))\big) \tag{2.13}$$

for all comparable elements $x,y\in X$, where $\psi\in\Psi$, $\varphi\in\Phi_u$, $F\in C$, such that (ψ,φ,F) is monotone, $\psi(r+t)\le\psi(r)+\psi(t)$ and

$$M(x,y)=\max\left\{d(x,y),\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\},$$

If f is continuous or (H) holds, then f has a fixed point. Also, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. It suffices to consider Definition 1.11 in Theorem 2.2. □

Next, we give some results for almost generalized weakly contractive mappings. For instance, let (X,d) be a b-g.m.s and $f:X\to X$ be a given mapping. For $x,y\in X$, set

$$M(x,y)=\max\{d(x,y),d(x,fx),d(y,fy)\}$$

and

$$N(x,y)=\min\{d(x,fx),d(x,fy),d(y,fx),d(y,fy)\}.$$

Definition 2.7. Let (X,d) be a b-g.m.s. We say that a mapping $f: X \to X$ is an almost generalized $(F,\psi,\varphi)_s$-contractive mapping if there exist $L \geq 0$ and $\psi \in \Psi, \varphi \in \Phi_u$ and $F \in \mathcal{C}$ such that (ψ,φ,F) is monotone and $\psi(r+t) \leq \psi(r)+\psi(t)$ with

$$\psi(sd(fx,fy)) \leq F(\psi(M(x,y)),\varphi(M(x,y))) + L\psi(N(x,y)) \tag{2.14}$$

for all $x,y \in X$.

We state the following result.

Theorem 2.8. *Let $(X,\preceq,d)$ be a partially ordered complete b-g.m.s. and $f: X \to X$ be a continuous mapping which is non-decreasing with respect to $\preceq$. Assume that f is an almost generalized $(F,\psi,\varphi)_s$-contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point. Also, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_{n+1} = fx_n$. Having that $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

Suppose that $x_n = x_m$ for some $n > m$, then we have $x_{n+1} = fx_n = fx_m = x_{m+1}$. By continuing this process there is a positive integer k (indeed, $k = n-m$) such that $x_n = x_{m+k} = x_{n+k}$. So we get $x_n = f(x_{n+k-1}) = f^2(x_{n+k-2}) = \cdots = f^k(x_n)$. If $k=1$, then $fx_n = x_n$, so x_n is a fixed point of f. If $k>1$, according to the proof of Theorem 4 in [14], $f^{k-1}(x_n)$ is a fixed point of f. The proof is completed. From now on, we assume that $x_n \neq x_m$ for $n \neq m$. By (2.14), we obtain that

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\leq \psi(sd(x_n,x_{n+1}))\\ &= \psi(sd(fx_{n-1},fx_n))\\ &\leq F\big(\psi(M(x_{n-1},x_n)),\varphi(M(x_{n-1},x_n))\big) + L\psi(N(x_{n-1},x_n)),\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n),d(x_{n-1},fx_{n-1}),d(x_n,fx_n)\right\}\\ &= \max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}N(x_{n-1},x_n) &= \min\left\{d(x_{n-1},fx_{n-1}),d(x_{n-1},fx_n),d(x_n,fx_{n-1}),d(x_n,fx_n)\right\}\\ &= \min\left\{d(x_{n-1},x_n),d(x_{n-1},x_{n+1}),0,d(x_n,x_{n+1})\right\} = 0.\end{aligned} \tag{2.17}$$

From (2.15)–(2.17) and the properties of ψ and φ, we obtain

$$\psi(d(x_n,x_{n+1})) \leq F\big(\psi\big(\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}\big),\varphi\big(\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\}\big)\big). \tag{2.18}$$

If for some n, $\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1})\right\} = d(x_n,x_{n+1})$, then by (2.18), we have

$$\psi(d(x_n,x_{n+1})) \leq F(\psi(d(x_n,x_{n+1})),\varphi(d(x_n,x_{n+1}))).$$

So $\psi(d(x_n,x_{n+1}))=0$ or $\varphi(d(x_n,x_{n+1}))=0$. This implies that $d(x_n,x_{n+1})=0$, which gives a contradiction. Then for all $n\geq 1$

$$\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}=d(x_{n-1},x_n).$$

Therefore, (2.18) becomes

$$\psi(d(x_n,x_{n+1}))\leq F(\psi(d(x_{n-1},x_n)),\varphi(d(x_{n-1},x_n)))\leq \psi(d(x_{n-1},x_n)). \tag{2.19}$$

Thus, $\{d(x_n,x_{n+1})\}$ is a non-increasing sequence of positive numbers. Hence there exists $r\geq 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=r.$$

Taking the limit $n\to\infty$ in (2.19), we obtain

$$\psi(r)\leq F(\psi(r),\varphi(r))\leq \psi(r).$$

Thus $\psi(r)=0$ or $\varphi(r)=0$. This implies that $r=0$, that is,

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.20}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. Suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \text{ and } d(x_{m_i},x_{n_i})\geq\varepsilon. \tag{2.21}$$

That is

$$d(x_{m_i},x_{n_i-2})<\varepsilon. \tag{2.22}$$

Taking the lim sup as $i\to\infty$ and using (2.22), we obtain

$$\limsup_{n\to\infty} d(x_{m_i},x_{n_i-2})\leq\varepsilon. \tag{2.23}$$

On the other hand, we have

$$d(x_{m_i},x_{n_i})\leq sd(x_{m_i},x_{m_i+1})+sd(x_{m_i+1},x_{n_i-1})+sd(x_{n_i-1},x_{n_i}).$$

Using (2.20), (2.21) and taking the lim sup as $i\to\infty$, we obtain

$$\frac{\varepsilon}{s}\leq\limsup_{n\to\infty} d(x_{m_i+1},x_{n_i-1}). \tag{2.24}$$

From the b-rectangle inequality, we get

$$d(x_{m_i},x_{n_i})\leq sd(x_{m_i},x_{n_i-2})+sd(x_{n_i-2},x_{n_i-1})+sd(x_{n_i-1},x_{n_i}).$$

Taking the lim sup as $i\to\infty$ and using (2.20), (2.21), we have

$$\frac{\varepsilon}{s}\leq\limsup_{n\to\infty} d(x_{m_i},x_{n_i-2}). \tag{2.25}$$

Using (2.14), we get

$$\begin{aligned}\psi(sd(x_{m_i+1},x_{n_i-1}))&=\psi(sd(fx_{m_i},fx_{n_i-2}))\\ &\leq F(\psi(M(x_{m_i},x_{n_i-2})),\varphi(M(x_{m_i},x_{n_i-2})))+L\psi(N(x_{m_i},x_{n_i-2})),\end{aligned} \tag{2.26}$$

where

$$M(x_{m_i},x_{n_i-2}) = \max\{d(x_{m_i},x_{n_i-2}), d(x_{m_i},fx_{m_i}), d(x_{n_i-2},fx_{n_i-2})\}$$
$$= \max\{d(x_{m_i},x_{n_i-2}), d(x_{m_i},x_{m_i+1}), d(x_{n_i-2},x_{n_i-1})\} \tag{2.27}$$

and

$$N(x_{m_i},x_{n_i-2}) = \min\{d(x_{m_i},fx_{m_i}), d(x_{m_i},fx_{n_i-2}), d(x_{n_i-2},fx_{m_i}), d(x_{n_i-2},fx_{n_i-2})\}$$
$$= \min\{d(x_{m_i},x_{m_i+1}), d(x_{m_i},x_{n_i-1}), d(x_{n_i-2},x_{m_i+1}), d(x_{n_i-2},x_{n_i-1})\}. \tag{2.28}$$

Taking the lim sup as $i\to\infty$ in (2.27) and (2.28) and using (2.20), (2.23), we obtain

$$\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2}) = \max\left\{\limsup_{i\to\infty} d(x_{m_i},x_{n_i-2}),0,0\right\} \le \varepsilon.$$

Therefore

$$\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2}) \le \varepsilon, \tag{2.29}$$

and

$$\limsup_{i\to\infty} N(x_{m_i},x_{n_i-2}) = 0. \tag{2.30}$$

Similarly, as $i\to\infty$ in (2.27) and using (2.20) and (2.25), we obtain

$$\frac{\varepsilon}{s} \le \limsup_{i\to\infty} M(x_{m_i},x_{n_i-2}). \tag{2.31}$$

Now, taking the lim sup as $i\to\infty$ in (2.26) and using (2.24), (2.29) and (2.30), we get

$$\begin{aligned}\psi\left(s\cdot\frac{\varepsilon}{s}\right) &\le \psi\left(s\limsup_{i\to\infty} d(x_{m_i+1},x_{n_i-1})\right)\\ &\le F\left(\psi\left(\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2})\right), \limsup_{n\to\infty}\varphi(M(x_{m_i},x_{n_i-2}))\right)\\ &\le F\left(\psi(\varepsilon), \varphi\left(\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2})\right)\right)\\ &\le \psi(\varepsilon),\end{aligned}$$

which implies that

$$\psi(\varepsilon)=0 \quad \text{or} \quad \varphi\left(\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2})\right)=0.$$

Hence $\varepsilon=0$ or $\limsup_{i\to\infty} M(x_{m_i},x_{n_i-2})=0$, which is a contradiction with respect to (2.31). Thus $\{x_{n+1}\}$ is a b-g.m.s. Cauchy sequence in X, which is complete, so there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$, that is,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = z.$$

Now, suppose that f is continuous. We show that z is a fixed point of f. Suppose that $fz\ne z$. By Lemma 1.3, it follows that x_n dif and only ifers from both fz and z for n sufficiently large. From the b-rectangle inequality, we obtain

$$d(z,fz) \le sd(z,fx_n) + sd(fx_n,fx_{n+1}) + sd(fx_{n+1},fz).$$

Taking the limit $n \to \infty$, we have

$$d(z, fz) \le 0.$$

So we get $fz = z$, that is, z is a fixed point of f. □

Note that the continuity of f in Theorem 2.8 is not necessary and can be dropped.

Theorem 2.9. *Under the hypotheses of Theorem 2.8, except that the continuity assumption on f is replaced by the hypothesis (H). Then f has a fixed point in X.*

Proof. From the proof of Theorem 2.8, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z$, for some $z \in X$. From the assumption on X, we get that $x_{n(k)} \preceq z$, for all $k \in \mathbb{N}$. Now, we show that $fz = z$. From (2.14), we obtain

$$\begin{aligned}\psi(sd(x_{n(k)+1}, fz)) &= \psi(sd(fx_{n(k)}, fz)) \\ &\le F(\psi(M(x_{n(k)}, z)), \varphi(M(x_{n(k)}, z))) + L\psi(N(x_{n(k)}, z)),\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}M(x_{n(k)}, z) &= \max\left\{d(x_{n(k)}, z), d(x_{n(k)}, fx_{n(k)}), d(z, fz)\right\} \\ &= \max\left\{d(x_{n(k)}, z), d(x_{n(k)}, x_{n(k)+1}), d(z, fz)\right\}\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}N(x_{n(k)}, z) &= \min\left\{d(x_{n(k)}, fx_{n(k)}), d(x_{n(k)}, fz), d(z, fx_{n(k)}), d(z, fz)\right\} \\ &= \min\left\{d(x_{n(k)}, x_{n(k)+1}), d(x_{n(k)}, fz), d(z, x_{n(k)+1}), d(z, fz)\right\}.\end{aligned} \tag{2.34}$$

Taking the limit as $k \to \infty$ in (2.33) and (2.34), we obtain

$$M(x_{n(k)}, z) \to d(z, fz) \tag{2.35}$$

and

$$N(x_{n(k)}, z) \to 0.$$

Taking the lim sup as $k \to \infty$ in (2.32) and using Lemma 1.4 with (2.35), we obtain

$$\begin{aligned}\psi(d(z, fz)) &= \psi\left(s \cdot \frac{1}{s} d(z, fz)\right) \\ &\le \psi\left(s \limsup_{k\to\infty} d(x_{n(k)+1}, fz)\right) \\ &\le F\left(\psi\left(\limsup_{k\to\infty} M(x_{n(k)}, z)\right), \limsup_{k\to\infty} \varphi(M(x_{n(k)}, z))\right) \\ &\le F\left(\psi(d(z, fz)), \varphi\left(\limsup_{k\to\infty} M(x_{n(k)}, z)\right)\right).\end{aligned}$$

Therefore, $\psi(d(z, fz)) = 0$ or $\varphi\left(\limsup_{k\to\infty} M(x_{n(k)}, z)\right) = 0$. Consequently, $\psi(d(z, fz)) = 0$ or $\limsup_{k\to\infty} M(x_{n(k)}, z) = 0$. Thus from (2.35), we get $z = fz$, that is, z is a fixed point of f. □

By choosing $F(s, t) = \frac{s}{1+t}$ in Theorem 2.8, we obtain the following corollary.

Corollary 2.10. *Let $(X,\preceq,d)$ be a complete b-g.m.s. and $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq fx_0$. Assume that*

$$\psi(sd(fx,fy))\le \frac{\psi(M(x,y))}{1+\varphi(M(x,y))}+L\psi(N(x,y))$$

for all comparable elements $x,y\in X$, where $L\ge 0$, $\psi\in\Psi,\varphi\in\Phi_u$ and $F\in\mathcal{C}$ such that (ψ,φ,F) is monotone and $\psi(r+t)\le\psi(r)+\psi(t)$ with

$$M(x,y)=\max\{d(x,y),d(x,fx),d(y,fy)\}$$

and

$$N(x,y)=\min\{d(x,fx),d(x,fy),d(y,fx),d(y,fy)\}.$$

If f is continuous or (H) holds, then f has a fixed point. Also, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

The following example is inspired from [21, Example 3].

Example 2.11. Let $X=\{a,b,c,\delta,e\}$ be equipped with the order $\preceq$ given by

$$\preceq=(a,a),(b,b),(c,c),(\delta,\delta),(e,e),(\delta,c),(\delta,b),(\delta,a),(\delta,e),(c,a),(b,a),(e,a)$$

and let $d:X\times X\to[0,+\infty)$ be given as $d(x,x)=0$ for $x\in X$,

$$d(x,y)=d(y,x) \text{ for } x,y\in X,$$

$$d(c,b)=1,$$

$$d(a,c)=d(c,e)=d(b,a)=d(a,e)=\frac{1}{8},$$

$$d(c,\delta)=d(b,\delta)=d(b,e)=d(a,\delta)=d(\delta,e)=\frac{1}{2}.$$

Then it is easy to check that $(X,\preceq,d)$ is a (complete) ordered b-g.m.s. with parameter $s=\frac{8}{3}$. Consider the mapping $f:X\to X$ defined as

$$f=\begin{pmatrix} a & b & c & \delta & e\\ a & a & a & c & a\end{pmatrix}.$$

It is easy to check that all the conditions of Corollary 2.3 are fulfilled with

$$d(fx,fy)\le\frac{1}{4}M(x,y).$$

In particular, the contractive condition in Corollary 2.3 is nontrivial only in the case when $x\in\{a,b,c,e\}$ and $y=\delta$ (or vice versa), when it reduces to

$$d(fx,fy)=d(c,a)=\frac{1}{8}=\frac{1}{4}\frac{1}{2}\le\frac{1}{4}M(x,y).$$

It follows that f has a fixed point (which is $z=a$).

The following example is inspired from [21, Example 4].

Example 2.12. Consider the set $X = A \cup [2,3]$, where $A = \{0, 1/3, 1/4, 1/5, 1/6, 1/7\}$ is endowed with the partial order defined as follows:

$$t \preceq 1/4 \preceq 1/7 \preceq 1/6 \preceq 1/3 \preceq 0 \preceq 1/5 \quad \text{for all } t \in [2,3].$$

Define $d : X \times X \to [0, +\infty)$ by

$$\begin{aligned}
&d(0,1/3) = d(1/4,1/5) = d(1/6,1/7) = 0.16,\\
&d(0,1/4) = d(1/3,1/6) = d(1/5,1/6) = 0.09,\\
&d(0,1/5) = d(1/3,1/4) = d(1/5,1/7) = 0.25,\\
&d(0,1/6) = d(1/3,1/7) = d(1/4,1/7) = 0.36,\\
&d(0,1/7) = d(1/3,1/5) = d(1/4,1/6) = 0.49,\\
&d(x,x) = 0 \text{ and } d(x,y) = d(y,x) \text{ for } x,y \in X,\\
&d(x,y) = (x-y)^2 \text{ if } \{x,y\} \cap [2,3] \neq \emptyset.
\end{aligned}$$

Obviously, (X,d) is a b-g.m.s. with $s = 3$. Now, consider the mapping $f : X \to X$ given as

$$fx = \begin{cases} 1/7 & \text{if } x \in [2,3],\\ 1/5 & \text{if } x \in A \setminus \{1/4\},\\ 1/6 & \text{if } x = 1/4. \end{cases}$$

It is easy to check that f is increasing with respect to $\preceq$. Also, there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. In order to show that the contractive condition (2.14) is fulfilled with $\psi(t) = t, \varphi(t) = \frac{1}{1000}$ and $F(s,t) = \frac{s}{1+t}$, we distinguish the following:

1. For $x \in [2,3]$ and $y \in A \setminus \{1/4\}$, we have $fx = 1/7, fy = 1/5$ and $M(x,y) > d(x,fx) > (13/7)^2 > 2$, so
$$\psi(d(fx,fy)) = 0.25 < \frac{\psi(M(x,y))}{1+\varphi(M(x,y))}.$$
2. If $x \in [2,3]$ and $y = 1/4$, then $fx = 1/7, fy = 1/6$ and $M(x,y) > 2$, thus
$$\psi(d(fx,fy)) = 0.16 < \frac{\psi(M(x,y))}{1+\varphi(M(x,y))}.$$
3. For $x \in A \setminus \{1/4\}$ and $y = 1/4, fx = 1/5, fy = 1/6, M(x,y) = 0.49$, we have
$$\psi(d(fx,fy)) = 0.09 < \frac{\psi(M(x,y))}{1+\varphi(M(x,y))}.$$

Hence, all the conditions of Theorem 2.8 are satisfied and f has a unique fixed point (which is $u = 1/5$).

3. A Note on Erhan's paper "Fixed points of (ψ,φ) contractions on rectangular metric spaces"

In 2012, Erhan *et al.* [14] studied existence and uniqueness of fixed points of a general class of (ψ,φ) contractive mappings on complete rectangular metric spaces ($s = 1$). However, there is a slight flaw in the proof of their main result, which is [14, Theorem 4].

Erhan *et al.* [14] obtained the following result:

Theorem 3.1 ([14]). *Let (X,d) be a Hausdorff and complete g.m.s. and let $T : X \to X$ be a self-map satisfying*

$$\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) + Lm(x,y) \tag{3.1}$$

for all $x,y \in X$ where $\psi,\varphi \in \Psi$ and $L \ge 0$ with

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$

with

$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Then T has a unique fixed point in X.

Proof. This above theorem is proved in [14] by the the following steps:

Step 1. Show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Step 2. Show that T has a periodic point, that is, there exist a positive integer p and a point $z \in X$ such that $z = T^p z$.

Step 3. If $p = 1$, then $z = Tz$, so z is a fixed point of T. If $p > 1$, then show that $T^{p-1}z$ is a fixed point of T.

Step 4. Show that the uniqueness of fixed point of T.

In *Step* 2, in order to show that T has a periodic point, the authors used a reduction and absurdum and shown that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, then $\{x_n\}$ converges to a limit $u \in X$.

Authors [14] proved that u is a fixed point of the T. By taking $x = x_n$ and $y = u$ in (3.1), they obtained the following inequality

$$\psi(d(Tx_n,Tu)) \le \psi(M(x_n,u)) - \varphi(M(x_n,u)) + Lm(x_n,u), \tag{3.2}$$

where

$$M(x_n,u) = \max\{d(x_n,u), d(x_n,Tx_n), d(u,Tu)\},$$

and

$$m(x_n,u) = \min\{d(x_n,Tx_n), d(u,Tu), d(x_n,Tu), d(u,Tx_n)\}.$$

From *Step* 1, note that $m(x_n,u) \to 0$ as $n \to \infty$. In the rest of the proof, the authors [14] considered the following three cases:

Case 1. $M(x_n,u) = d(x_n,u)$.

Case 2. $M(x_n,u) = d(x_n,x_{n+1})$.

Case 3. $M(x_n,u) = d(u,Tu)$.

Their proof is true only for *Cases* 1 and 2. But, proof is incorrect and unclear for *Case* 3. When taking limit as $n \to \infty$ in (3.2), the authors [14] used the continuity of function $d(x,y)$ on rectangular metric spaces (while, we know that this function is not continuous in each of its coordinates in general, see [22, Example 1.1]). They also obtained

$$\psi(d(u,Tu)) \le \psi(d(u,Tu)) - \varphi(d(u,Tu)).$$

Then they conclude that $u = Tu$.

Now, we perfect and simplify the proof of *Case* 3. If we assume that $u \neq Tu$, then by Lemma 1.4, for $s = 1$ we get

$$\begin{aligned} d(u,Tu) &\le \liminf_{n\to\infty} d(Tx_n,Tu) \\ &\le \limsup_{n\to\infty} d(Tx_n,Tu) \\ &\le d(u,Tu). \end{aligned}$$

So $\lim_{n\to\infty} d(Tx_n,Tu) = d(u,Tu)$. Letting $n \to \infty$ in (3.2), we get

$$\psi(d(u,Tu)) \le \psi(d(u,Tu)) - \varphi(d(u,Tu)).$$

It follows that $u = Tu$, which is a contradiction. So u is a fixed point of T. □

4. Conclusion

We arrived to correct the proof of Erhan et al. [14, Theorem 4]. We also established some fixed point results in the setting of b-rectangular metric space via C-class functions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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