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Research Article

Note on (i,j)-Distance Graph of a Graph

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Abstract. In this paper, we define the (i,j)-distance graph of a graph and presented several characterizations based on this notion.

Keywords. (i,j)-Distance graph; Distance (i,j)-bipartite graphs; Strongly (i,j)-Bipartite graphs

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1. Introduction

For all terminology and notation in graph theory we refer the reader to consult any one of the standard text-books by Chartrand and Zhang [1], Harary [3] and West [5].

Historically, graphs haven been used as models for studying the structure and relationships in many real-world situations. One relationship that has received and considerable attention is the distance between vertices of a graph. For a connected graph $G = (V,E)$ and a pair u,v of vertices of G, we can define the distance $d(u,v)$ as the length of a shortest $u-v$ path in G. With the aid of an algorithm developed by Dijkstra [2], computing distances is straightforward. The primary purpose of this paper is to study generalization of distance graphs.

Let $G = (V,E)$ be a graph with set of vertices V and set of edges E. For any two vertices $u,v \in V$, the distance between u and v, denoted by $d(u,v)$ is the length of a shortest path between u and v. The diameter of the graph G, denoted by diam(G) (diam$_G(u,v)$, whenever G has to be specified) and is given by, $\text{diam}(G) = \max_{u,v\in V} d(u,v)$.

2. (i,j)-Distance Graph of a Graph

Let $G = (V,E)$ be a connected graph with diameter $d(G)$, and i and j be any two distinct integers such that $1 \le i < j \le d(G)$. The *(i,j)-distance graph* of G, denoted by $G^{(i,j)}$ is a graph having vertex set V and two vertices are adjacent, if the distance between them is i or j. Clearly G is a subgraph of $G^{(i,j)}$ if either $i = 1$ or $j = 1$, otherwise $G^{(i,j)}$ is a subgraph of of G. The $(1,2)$-distance graph of G is nothing but G^2, the square of G.

For example, the cycle C_5 is $(2,3)$-distance graph of P_5, the path on 5 vertices, as shown in Figure 1. The edge labels in Figure 1(b) indicates the distance between corresponding vertices in G.

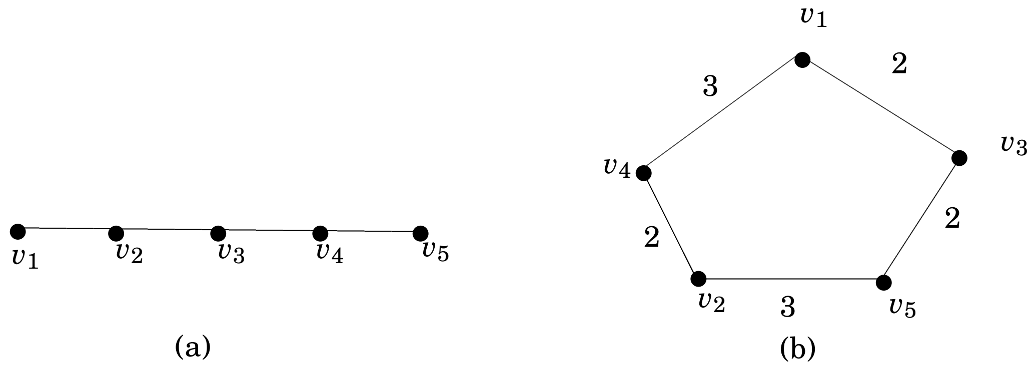


Figure 1

A graph $G = (V,E)$ is (i,j)-distance graph, if there exits a graph H such that G is isomorphic to (i,j)-distance graph of H. For example the graph in Figure 2(a) is $(2,3)$-distance graph of the graph as shown in Figure 2(b).

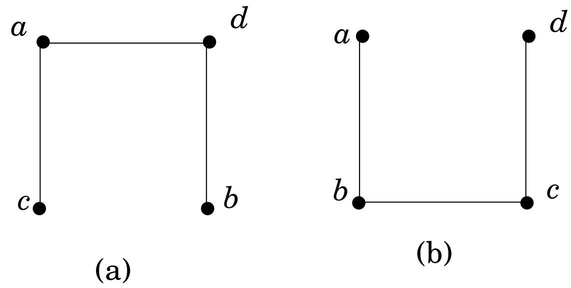


Figure 2

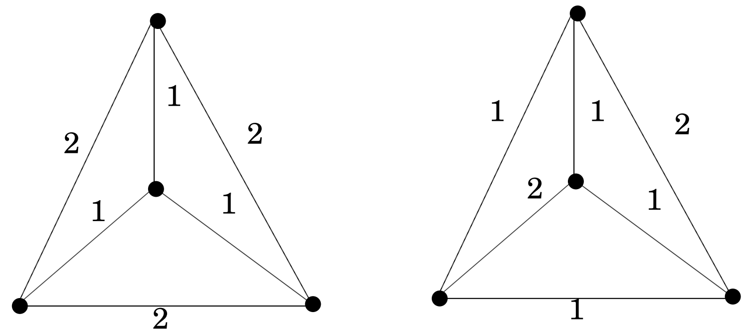
From the definition of (i,j)-Distance Graph of a Graph, we have the following:

Theorem 1. *Path on 5 vertices is not (i,j)-distance graph of any graph.*

A graph $G = (V,E)$ is (i,j)-connected if the (i,j)-distance graph $G^{(i,j)}$ is connected. The (i,j)-clique denoted by $\omega_{(i,j)}(G)$ is the maximum order a clique in $G^{(i,j)}$.

Given a connected graph G and two positive integers i and j, there may exist more than one type of labeling of edges with labels i and j such that G is the (i,j)-distance graph of two non-isomorphic graphs, say H_1 and H_2 such that the label on the edges of G corresponds to the distance between the end vertices in H_1 and H_2 respectively.

For example consider two edge labeling of K_4 by 1 and 2 as shown in Figure 3(a) and 3(b).



(a) (b)

Figure 3

Each of the above edge labeled graph is $(1,2)$-distance graph, for the graph in Figure 3(a) is $(1,2)$-distance graph of $K_{1,3}$, where as the graph in Figure 3(b) is $(1,2)$-distance graph of C_4.

We now give a characterization of paths and cycles which are (i,j)-distance graphs.

Theorem 2. (1) *The path on n vertices, P_n is (i,j)-distance graph if and $(i,j)=1$ and $i+j=n+1$.*

(2) *The cycle on n vertices C_n is (i,j)-distance graph if and only if $i+j=n$ and $(i,j)=1$.*

In [4], P.S.K. Reddy et al. defined the k-distance bipartite graphs as: A graph $G=(V,E)$ is said to be k-distance bipartite (or D_k-bipartite) if its vertex set can be partitioned into two D_k independent sets. If the diameter of G is $<k$, then G is distance k-bipartite and so if G is not distance k-bipartite then diameter of G is at least k.

Let $G=(V,E)$ be a graph. Given any integer $k>0$, we can associate a graph $G^{(k)}$ as follows: The D_K-graph of G, denoted by $G^{(k)}$ is the graph on same vertex set V and two vertices u and v are adjacent if and only if distance between them is equal to k. Clearly, a graph is D_k-bipartite if and only if $G^{(k)}$ is bipartite.

3. Strongly (i,j)-Bipartite graphs

Let $G=(V,E)$ be a graph. For any two distinct integers $i,j\geq 1$, a set of vertices S is said to be an (i,j)-*independent* set or simply an (i,j)-set if no two vertices in S are at a distance i or j. Further a graph G is said to be *Strongly (i,j)-bipartite* if its vertex set can be partitioned in to two (i,j)-independent sets.

Remark. (1) If either i or j is grater than $d(G)$, the diameter of G, then for any vertex v there will no vertex which is at a distance i and so every D_j independent set is (i,j)-set. Hence we assume that $1\leq i\neq j\leq d(G)$.

(2) If a graph G is strongly (i,j)-bipartite then G is strongly (j,i)-bipartite. Hence we always assume that $i > j$.

(3) Let $i > j > k$. If G is both strongly (i,j)-bipartite and strongly (j,k)-bipartite then G need not be strongly (i,k)-bipartite.

(4) If a graph G is strongly (i,j)-bipartite then then G is both D_i bipartite and D_j-bipartite.

For example, path on 5 vertices is not strongly $(2,3)$-bipartite. The characterization of strongly (i,j)-bipartite graphs is an interesting problem. In this section, we list the possible values of i and j for a fixed value of n for which path on n vertices is strongly (i,j)-bipartite. We also give some possible values of i and j for which path on n vertices is not strongly (i,j)-bipartite. Since path on n vertices, P_n has diameter $n-2$, throughout this section let i and j are integers such that $1 \leq i \neq j \leq n-2$.

Finally for a fixed integer n, we list possible values of i and j for which path on n vertices is not strongly (i,j)-bipartite.

Theorem 3. *For a fixed value of n, path on n vertices P_n is not strongly (i,j)-bipartite if,*

(i) *$i + j$ is odd*

(ii) *i is a even multiple of j.*

We now list the possible values of i and j for a fixed value of n for which cycle on n vertices is strongly (i,j)-bipartite. We also give some possible values of i and j for which cycle on n vertices is not strongly (i,j)-bipartite. Since cycle on n vertices, C_n has diameter $\lceil n/2 \rceil$, throughout this section let i and j are integers such that $1 \leq i \neq j \leq \lceil n/2 \rceil$.

Theorem 4. *For any odd integer n, C_n the cycle on n vertices is not strongly (i, j) bipartite for any i and j with $1 \leq i \neq j \leq \lceil n/2 \rceil$.*

Proof.

Case (i): Suppose that i and n are relatively prime then, the set of points which are at distance i in C_n, C_n^i is C_n and since C_n^i is a subgraph of $C_n^{(i,j)}$ it follows that $C_n^{(i,j}$ and hence C_n is not strongly (i,j) bipartite.

Case (ii): Suppose that $n/i = d_1$ and $n/j = d_2$ with both $d_1 > 1$ and $d_2 > 1$. Since n is odd both i and d_1 are odd. □

Remark 5. If both i and j are grater than $d(G)$, the diameter of the graph G, then $G_{(i,j)}$ is trivial graph and if exactly one of them say $i$is grater than the diameter of the graph or if $i = j$, then $G_{(i,j)}$ is nothing but $G_(i)$, the D_i-graph of G. Hence we assume that $1 \leq i < j \leq d(G)$.

Theorem 6. *Let $G = (V,E)$ be a graph of diameter k, and i,j be two integers such that $2 \leq i < j \leq k$.Then, G is Strongly (i,j)-bipartite if and only if $G^(i,j)$ is bipartite.*

Remark 7. If both i and j are odd, then path on n vertices is strongly (i,j)-bipartite. Hence let us assume that at least one of i or j. is even.

The following result gives some possible values of i and j for which a graph G is strongly (i,j)-distance bipartite.

Theorem 8. *Path on n vertices is strongly (i,j)-bipartite if*

(i) $i+j=n-1$

(ii) $i+j=n$ *and both* i *and* j *are even.*

(iii) $2i+j<n$

(iv) i *is an odd multiple of* j.

4. Two Distance Graph of a Graph

Let $G=(V,E)$ be a graph. For any two vertices u,v distance between u and v denoted by $d(u,v)$ is the length of the shortest path between u and v. The 2-distance graph of G, denoted by $D_2(G)$ is the graph on the same vertex set V and two vertices $u,v\in V$ are adjacent if $d(u,v)=2$. Equivalently, the 2-distance graph $D_2(G)$ of G is the graph on the same vertex set V and two vertices u and v are adjacent if and only if $(u,v)\notin E$ u and v have a common neighbor. For any vertex $v\in V$, the open neighborhood of v denoted by $N(v)$ is the set of all vertices adjacent to v. Let $N(V)$ denote the set of all open neighborhoods $N(v)$of v, $v\in V$. Then $D_2(G)$ can be considered as a subgraph $mathcalH$of intersection graph of open neighborhoods of vertices of G, where $\mathscr{H}=(N(V),\mathscr{E})$, where $\mathscr{E}=\{(N(u),N(v)):N(u)\cap N(v)\neq\phi,\ uv\notin E\}$. Note that for any graph G, $D_2(G)$ is a subgraph of $\overline{G}$. The following result gives a characterization of D_2 graphs which is analogous to the the characterization of open neighborhood graphs.

Theorem 9. *A graph $G=(V,E)$ of order p with vertex set $V=v_1,v_2,\dots,v_p$ is a D_2 graph if, and only if its there exists p complete subgraphs $K_1,K_2,\dots,K_p$ indexed such that*

(i) $E(G)=\bigcup\limits_{i=1}^{p}E(K_i)$;

(ii) $v_i\notin K_i$;

(iii) $v_j\in K_i$ *if, and only if* $v_i\in K_j$ *and*

(iv) *if* $v_j\in K_i$ *then* $v_iv_j\notin E(G)$.

Proof. Suppose that $G=D_2(H)$ for some graph $H=(V,E')$. For any vertex $v_i\in V$, consider the subset $N_{d_2}(v_i)=\{v_j\in N_H(v_i)$ such that $v_jv_k\notin E(H)$ for any $v_k\in N_H(v_i)\}S$ of $N_H(v_i)$. Then the subgraph induced by $N_{d_2}(v_i)$ forms a complete subgraph in $D_2(G)$ since they are at a distance 2 in H. Denote this complete subgraph by K_i for each $v_i\in V$. Now consider the these cliques K_i of G, $1\le i\le p$. Then (ii) follows from the fact that $v_i\notin N_{d_2}(v_i)$ for any $v_i\in V$.

Next suppose that (i) does not hold. That is there exists an edge $v_jv_k\in E(G)$ which is not covered by any of the K_i, $1\le i\le p$. Since $D_2(G)=H$, it follows that there exists a vertex v_s such that v_sv_i, v_sv_j are in $E(G)$ and $h_ih_j\notin E(G)$. This implies that $v_iv_j\in K_s$, a contradiction to our assumption. This proves (ii).

Condition (iii) follows from the fact that $u\in N_{d_2}(v)$ if, and only if, $v\in N_{d_2}(u)$ for any two vertices u and v in H and that $G=D_2(H)\subset\overline{H}$, where $\overline{H}$ denotes the complement of H.

Conversely, suppose that conditions (i) to (iii) holds for G. Let H be the graph with vertex set V where $v_iv_j \in E(H)$ if and only if $v_i \in K_j$ in G and $v_iv_j \notin E(G)$. We shall now prove that $G \cong D_2(H)$. □

5. Conclusion

The branch of graph theory in which the study of distance graphs lie is called extremal graph theory. The ability to construct a graph from a few rules or a small amount of information is a very powerful tool. In the terms of a distance graph we will use the elements of a distances set called distances to generate the edges of the graph. In this paper, we have defined the (i,j)-distance graph of a graph and presented several characterizations based on this notion. Certain classes of graphs which can be constructed fairly easily with (i,j)-distance graphs will be shown in a separate paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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