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Research Article

Gaussian Tetranacci Numbers

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Abstract. In this paper, we define Gaussian Tetranacci sequence. Moreover, we give generating function, Binet-like formula, sum formulas and matrix representation of Gaussian tetranacci numbers.

Keywords. Tetranacci numbers; Gaussian Tetranacci numbers

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1. Introduction

The Tetranacci sequence was defined by Feinberg in [3]. Then, some new properties and theorems were obtained from this definition in [13], [8], [9], [12] and [1]. In [14], Waddill defined new Tetranacci sequences by choosing arbitrary integers as initial conditions. Weisstein determined prime Tetranacci numbers by choosing prime numbers for initial conditions in [10].

In this study, we define Gaussian Tetranacci numbers by using Gauss integers as initial conditions. In [4] Gürel used Gauss integers in Tribonacci sequence and thus defined Gaussian Tribonacci numbers. On the other hand, Gauss integers have extremely been used in Fibonacci sequence studies, some of which are [2], [5], [11] and [7]. Indeed, the idea of using complex numbers in Fibonacci sequence belongs to Horadam [6].

The Tetranacci sequence is the sequence of integers M_n defined by the initial values $M_0 = M_1 = 0$, $M_2 = M_3 = 1$ and the recurrence relation [3],

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} \quad \text{for all } n \geq 4 \tag{1.1}$$

The first few values of M_n are $0,0,1,1,2,4,8,15,\ldots$.

2. Gaussian Tetranacci Numbers

Gaussian Tetranacci numbers are defined by

$$GM_n = GM_{n-1} + GM_{n-2} + GM_{n-3} + GM_{n-4} \quad (n \geq 4) \tag{2.1}$$

for the initial conditions

$$GM_0 = GM_1 = 0, GM_2 = 1 \text{ and } GM_3 = 1 + i.$$

We note that

$$GM_n = M_n + iM_{n-1}. \tag{2.2}$$

The first few values of Gaussian Tetranacci numbers are given the following table.

n	0	1	2	3	4	5	6	7	8
GM_n	0	0	1	$1+i$	$2+i$	$4+2i$	$8+4i$	$15+8i$	...

Theorem 2.1. *The generating function of Gaussian Tetranacci numbers,*

$$g(x) = \sum_{n=0}^{\infty} GM_n \cdot x^n = \frac{x^2 + i \cdot x^3}{1 - x - x^2 - x^3 - x^4}. \tag{2.3}$$

Proof. Let

$$g(x) = GM_0 + GM_1 \cdot x + GM_2 \cdot x^2 + GM_3 \cdot x^3 + \cdots + GM_n \cdot x^n = \sum_{n=0}^{\infty} GM_n \cdot x^n$$

be generating function of Gaussian Tetranacci numbers. Then we have

$$g(x) \cdot x = GM_0 \cdot x + GM_1 \cdot x^2 + GM_2 \cdot x^3 + GM_3 \cdot x^4 + \cdots + GM_{n-1} \cdot x^n + \cdots$$

$$g(x) \cdot x^2 = GM_0 \cdot x^2 + GM_1 \cdot x^3 + GM_2 \cdot x^4 + GM_3 \cdot x^5 + \cdots + GM_{n-2} \cdot x^n + \cdots$$

$$g(x) \cdot x^3 = GM_0 \cdot x^3 + GM_1 \cdot x^4 + GM_2 \cdot x^5 + GM_3 \cdot x^6 + \cdots + GM_{n-3} \cdot x^n + \cdots$$

and

$$g(x) \cdot x^4 = GM_0 \cdot x^4 + GM_1 \cdot x^5 + GM_2 \cdot x^6 + GM_3 \cdot x^7 + \cdots + GM_{n-4} \cdot x^n + \cdots$$

So, we obtain

$$\begin{aligned}
&g(x) - g(x) \cdot x - g(x) \cdot x^2 - g(x) \cdot x^3 - g(x) \cdot x^4 \\
&\quad = GM_0 + (GM_1 - GM_0) \cdot x + (GM_2 - GM_1 - GM_0) \cdot x^2 \\
&\qquad + (GM_3 - GM_2 - GM_1 - GM_0) \cdot x^3 \\
&\qquad + (GM_4 - GM_3 - GM_2 - GM_1 - GM_0) \cdot x^4 \\
&\qquad + (GM_5 - GM_4 - GM_3 - GM_2 - GM_1 - GM_0) \cdot x^5 + \cdots \\
&\qquad + (GM_n - GM_{n-1} - GM_{n-2} - GM_{n-3} - GM_{n-4}) \cdot x^n + \cdots
\end{aligned}$$

$$g(x)\cdot(1-x-x^2-x^3-x^4)=x^2+i\cdot x^3$$

or

$$g(x)=\frac{x^2+i\cdot x^3}{1-x-x^2-x^3-x^4}.$$

The roots of the equation $t^4-t^3-t^2-t-1=0$ are

$$\alpha=-7.6379\times 10^{-2}-0.8147i\,,$$

$$\beta=-7.6379\times 10^{-2}+0.8147i\,,$$

$$\gamma=-0.7748\,,$$

$$\theta=1.9276\,.$$

We note that, $\alpha+\beta+\gamma+\theta=1$ and $\alpha\cdot\beta\cdot\gamma\cdot\theta=-1$. As well known, the Binet-like formula of Tetranacci numbers is

$$M_n=\frac{\alpha^{n+3}}{(\alpha-\beta)\cdot(\alpha-\gamma)\cdot(\alpha-\theta)}+\frac{\beta^{n+3}}{(\beta-\alpha)\cdot(\beta-\gamma)\cdot(\beta-\theta)}+\frac{\gamma^{n+3}}{(\gamma-\alpha)\cdot(\gamma-\beta)\cdot(\gamma-\theta)}+\frac{\theta^{n+3}}{(\theta-\alpha)\cdot(\theta-\beta)\cdot(\theta-\gamma)}. \tag{2.4}$$

□

Now, we give the Binet-like formula for Gaussian Tetranacci numbers.

Theorem 2.2. *The Binet formula for the Gaussian Tetranacci numbers is*

$$\begin{aligned}GM_n=&\left[\frac{\alpha^{n+3}}{(\alpha-\beta)\cdot(\alpha-\gamma)\cdot(\alpha-\theta)}+\frac{\beta^{n+3}}{(\beta-\alpha)\cdot(\beta-\gamma)\cdot(\beta-\theta)}\right.\\&\left.+\frac{\gamma^{n+3}}{(\gamma-\alpha)\cdot(\gamma-\beta)\cdot(\gamma-\theta)}+\frac{\theta^{n+3}}{(\theta-\alpha)\cdot(\theta-\beta)\cdot(\theta-\gamma)}\right]\\&+i\cdot\left[\frac{\alpha^{n+3}}{(\alpha-\beta)\cdot(\alpha-\gamma)\cdot(\alpha-\theta)}+\frac{\beta^{n+3}}{(\beta-\alpha)\cdot(\beta-\gamma)\cdot(\beta-\theta)}\right.\\&\left.+\frac{\gamma^{n+3}}{(\gamma-\alpha)\cdot(\gamma-\beta)\cdot(\gamma-\theta)}+\frac{\theta^{n+3}}{(\theta-\alpha)\cdot(\theta-\beta)\cdot(\theta-\gamma)}\right].\end{aligned} \tag{2.5}$$

Proof. Considering eq. (2.2) the proof is easily seen. □

3. Equations

Theorem 3.1. *Sum of Gaussian Tetranacci numbers;*

$$\sum_{k=1}^{n} GM_k=\frac{1}{3}[GM_{n+4}-GM_{n+2}-2GM_{n+1}-(1+i)]. \tag{3.1}$$

Proof. Using the recurrence relation,

$$GM_k=GM_{k-1}+GM_{k-2}+GM_{k-3}+GM_{k-4},\quad k\geq 4,$$

we have

$$\begin{aligned}
GM_0 &= GM_4 - GM_3 - GM_2 - GM_1\\
GM_1 &= GM_5 - GM_4 - GM_3 - GM_2\\
GM_2 &= GM_6 - GM_5 - GM_4 - GM_3\\
GM_3 &= GM_7 - GM_6 - GM_5 - GM_4\\
\vdots\ \ &\ \ \vdots\ \ \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
GM_{k-3} &= GM_{k+1} - GM_k - GM_{k-1} - GM_{k-2}\\
GM_{k-2} &= GM_{k+2} - GM_{k+1} - GM_k - GM_{k-1}\\
GM_{k-1} &= GM_{k+3} - GM_{k+2} - GM_{k+1} - GM_k\\
GM_k &= GM_{k+4} - GM_{k+3} - GM_{k+2} - GM_{k+1}
\end{aligned}$$

If the equations are added side by side, we obtain

$$\sum_{k=1}^{n} GM_k = \frac{1}{3}[GM_{k+4} - GM_{k+2} - 2\cdot GM_{k+1} - (1+i)].$$ □

Theorem 3.2.

$$\sum_{k=0}^{n} GM_k \cdot GM_{k+1} = M_n\cdot(M_{n+1} - M_{n-1}) + i\cdot(M_n^2 + M_{n+1}\cdot M_{n-1}). \tag{3.2}$$

where M_n denotes nth Tetranacci number.

Proof. If the equation $GM_n = M_n + i\cdot M_{n-1}$ is used, then we write

$$\begin{aligned}
GM_n\cdot GM_{n+1} &= (M_n + i\cdot M_{n-1})\cdot(M_{n+1} + i\cdot M_n)\\
&= M_n\cdot M_{n+1} + i\cdot M_n\cdot M_n + i\cdot M_{n-1}\cdot M_{n+1} - M_n\cdot M_{n-1}\\
&= M_n\cdot M_{n+1} + (GM_{n+1} - M_{n+1})\cdot M_n + (GM_n - M_n)\cdot M_{n+1} - M_n\cdot M_{n-1}\\
&= M_n\cdot M_{n+1} + GM_{n+1}\cdot M_n - M_{n+1}\cdot M_n + GM_n\cdot M_{n+1} - M_n\cdot M_{n+1} - M_n\cdot M_{n-1}\\
&= M_n\cdot(M_{n+1} + i\cdot M_n - M_{n-1}) + i\cdot M_{n+1}\cdot M_{n-1}\\
&= M_n\cdot M_{n+1} + i\cdot M_n^2 - M_n\cdot M_{n-1} + i\cdot M_{n+1}\cdot M_{n-1}\\
&= M_n\cdot(M_{n+1} - M_{n-1}) + i\cdot(M_n^2 + M_{n+1}\cdot M_{n-1}).
\end{aligned}$$ □

Theorem 3.3. *For $n \geq 1$,*

$$\sum_{k=0}^{n} GM_k^2 = (M_n^2 - M_{n-1}^2) + 2\cdot i\cdot M_{n-1}\cdot M_n. \tag{3.3}$$

Proof. If the equation $GM_n = M_n + i\cdot M_{n-1}$ is used, then we write

$$\begin{aligned}
GM_n\cdot GM_n &= (M_n + i\cdot M_{n-1})\cdot(M_n + i\cdot M_{n-1})\\
&= M_n\cdot M_n + i\cdot M_n\cdot M_{n-1} + i\cdot M_{n-1}\cdot M_n - M_{n-1}\cdot M_{n-1}
\end{aligned}$$

$$= M_n^2 + 2 \cdot i \cdot M_n \cdot M_{n-1} - M_{n-1}^2$$
$$= M_n^2 + 2 \cdot i \cdot M_n.(GM_n - M_n) - M_{n-1}^2$$
$$= M_n^2 + 2 \cdot M_n \cdot GM_n - 2 \cdot M_n^2 - M_{n-1}^2$$
$$= M_n^2 + 2 \cdot M_n \cdot (M_n + i \cdot M_{n-1}) - 2 \cdot M_n^2 - M_{n-1}^2$$
$$= M_n^2 + 2 \cdot M_n^2 + 2 \cdot i \cdot M_n \cdot M_{n-1} - 2 \cdot M_n^2 - M_{n-1}^2$$
$$= M_n^2 - M_{n-1}^2 + 2 \cdot i \cdot M_{n-1} \cdot M_n.$$ □

Theorem 3.4. *For $n \geq 1$,*

$$\sum_{k=1}^{n} GM_{2k+1} = \frac{1}{3} \cdot [GM_{2n+2} + GM_{2n+1} + 2 \cdot GM_{2n} + GM_{2n-2} + i - 2]. \tag{3.4}$$

Proof. The following equations are written by using eq. (2.1).

$$GM_k = GM_{k-1} + GM_{k-2} + GM_{k-3} + GM_{k-4} \qquad (k \geq 4).$$
$$GM_4 = GM_3 + GM_2 + GM_1 + GM_0$$
$$GM_6 = GM_5 + GM_4 + GM_3 + GM_2$$
$$GM_8 = GM_7 + GM_6 + GM_5 + GM_4$$
$$GM_{10} = GM_9 + GM_8 + GM_7 + GM_6$$
$$\vdots$$
$$GM_{2n+2} = GM_{2n+1} + GM_{2n} + GM_{2n-1} + GM_{2n-2}.$$

If the equations are regulated;

$$GM_3 = GM_4 - GM_2 - GM_1 - GM_0$$
$$GM_5 = GM_6 - GM_4 - GM_3 - GM_2$$
$$GM_7 = GM_8 - GM_6 - GM_5 - GM_4$$
$$GM_9 = GM_1 0 - GM_8 - GM_7 - GM_6$$
$$\vdots$$
$$GM_{2n+1} = GM_{2n+2} - GM_{2n} - GM_{2n-1} - GM_{2n-2}.$$

From here,

$$\sum_{k=1}^{n} GM_{2k+1} = GM_{2n+2} - GM_2 - \sum_{k=1}^{2n-1} GM_k$$
$$= GM_{2n+2} - GM_2 - \frac{1}{3} \cdot [GM_{2n+3} - GM_{2n+1} - 2 \cdot GM_{2n} - (1+i)]$$
$$= \frac{1}{3} \cdot [3 \cdot GM_{2n+2} - 3 - GM_{2n+3} + GM_{2n+1} + 2 \cdot GM_{2n} - (1+i)]$$

$$= \frac{1}{3}\cdot[2\cdot GM_{2n+2} - 3 + GM_{2n+2} + GM_{2n+1} + GM_{2n} - GM_{2n+3} + GM_{2n} - (1+i)]$$
$$= \frac{1}{3}\cdot[GM_{2n+2} + GM_{2n+2} - 3 - GM_{2n-1} + GM_{2n} + (1+i)]$$
$$= \frac{1}{3}\cdot[GM_{2n+2} + GM_{2n+1} + 2\cdot GM_{2n} + GM_{2n-2} + i - 2]. \qquad \square$$

Theorem 3.5. *For* $n \geq 1$,

$$\sum_{k=1}^{n} GM_{2k} = \frac{1}{3}[2\cdot GM_{2n+1} + GM_{2n-1} - GM_{2n-2} + 1 - 2i]. \tag{3.5}$$

Proof.

$$\sum_{k=1}^{2n+1} GM_k = GM_1 + GM_2 + GM_3 + \cdots + GM_{2n+1},$$
$$\sum_{k=1}^{n} GM_{2k+1} = GM_1 + GM_3 + GM_5 + \cdots + GM_{2n+1},$$
$$\sum_{k=1}^{2n+1} GM_k - \sum_{k=1}^{n} GM_{2k+1} = GM_2 + GM_4 + GM_6 + \cdots + GM_{2n}$$
$$= \frac{1}{3}\cdot[GM_{2n+5} - GM_{2n+3} - 2\cdot GM_{2n+2} - (1+i)]$$
$$+ \frac{1}{3}\cdot[GM_{2n+2} + GM_{2n+1} + 2\cdot GM_{2n} + GM_{2n-2} + i - 2]$$
$$= \frac{1}{3}\cdot[GM_{2n+4} - 2\cdot GM_{2n+2} - 2\cdot GM_{2n} - GM_{2n-2} + 1 - 2\cdot i]$$
$$= \frac{1}{3}\cdot[GM_{2n+3} + GM_{2n+2} + GM_{2n+1} + GM_{2n} - 2\cdot GM_{2n+2} - 2\cdot GM_{2n} - GM_{2n-2} + 1 - 2\cdot i]$$
$$= \frac{1}{3}\cdot[2\cdot GM_{2n+1} + GM_{2n-1} - GM_{2n-2} + 1 - 2\cdot i].$$

In this section,we give two new generating matrices for Gaussian Tetranacci numbers. Then we get an explicit formula for the sums.

Let θ_4, R_4 and $E_{4,n}$ matrices are defined as follows. θ is the analogue of the Qmatrix in [4].

$$\theta_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 1+i & 1 & 0 & 0 \\ 1 & 0 & 0 & i \\ 0 & 0 & i & 1-i \\ 0 & i & 1-i & -1 \end{bmatrix}$$

and

$$E_{4,n} = \begin{bmatrix} GM_{n+3} & GM_{n+2} & GM_{n+1} & GM_n \\ GM_{n+2} & GM_{n+1} & GM_n & GM_{n-1} \\ GM_{n+1} & GM_n & GM_{n-1} & GM_{n-2} \\ GM_n & GM_{n-1} & GM_{n-2} & GM_{n-3} \end{bmatrix}.$$ □

Theorem 3.6. *For $n \geq 3$,*

$$\theta_4^n \cdot R_4 = E_{4,n}. \tag{3.6}$$

Proof. (Induction on n) If $n = 1$, then $\theta_4 \cdot R_4 = E_{4,1}$. Suppose that the equation holds for $n-1$, which means $\theta_4^{n-1} \cdot R_4 = E_{4,n-1}$. Then we show that the equation holds for n.

$$\begin{aligned} \theta_4^n \cdot R_4 &= \theta_4 \cdot \theta_4^{n-1} \cdot R_4 \\ &= \theta_4 \cdot E_{4,n-1} \\ &= E_{4,n}. \end{aligned}$$

Thus the proof is complete. □

Theorem 3.7. *For $n \geq 1$,*

$$\theta_4^n \cdot \begin{bmatrix} 2+i \\ 1+i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} GM_{n+4} \\ GM_{n+3} \\ GM_{n+2} \\ GM_{n+1} \end{bmatrix}. \tag{3.7}$$

Proof. (Induction on n) If $n = 1$, then

$$\theta_4^1 \cdot \begin{bmatrix} 2+i \\ 1+i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+2i \\ 2+i \\ 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} GM_5 \\ GM_4 \\ GM_3 \\ GM_2 \end{bmatrix}.$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for n.

$$\begin{aligned} \theta_4^n \cdot \begin{bmatrix} 2+i \\ 1+i \\ 1 \\ 0 \end{bmatrix} &= \theta_4 \cdot \theta_4^{n-1} \cdot \begin{bmatrix} 2+i \\ 1+i \\ 1 \\ 0 \end{bmatrix} \\ &= \theta_4 \cdot \begin{bmatrix} GM_{n+3} \\ GM_{n+2} \\ GM_{n+1} \\ GM_n \end{bmatrix} \\ &= \begin{bmatrix} GM_{n+4} \\ GM_{n+3} \\ GM_{n+2} \\ GM_{n+1} \end{bmatrix}. \end{aligned}$$

Thus the proof is complete. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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