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Research Article

Monotone Iterative Technique for Nonlinear Impulsive Conformable Fractional Differential Equations With Delay

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Abstract. In this paper, we investigate the existence of solutions for boundary value problems of nonlinear impulsive conformable fractional differential equations with delay. By establishing the associate Green's function and a comparison result for the linear impulsive problem, we obtain that the lower and upper solutions converge to the extremal solutions via the monotone iterative technique. An example is also presented in the last section.

Keywords. Boundary value problem; Conformable fractional derivative; Impulsive differential equation; Monotone iterative technique

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1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. For a detailed account of applications and recent results on initial and boundary value problems of fractional differential equations, we refer the reader to a series of books and papers ([5–11,14,17,19,23–29]) and references cited therein.

Many definitions for the fractional derivative are available. Most of these definitions use an integral form. The most popular definitions are:

(i) *Riemann-Liouville Definition*: If n is a positive integer and $\alpha \in [n-1,n)$, the derivative of f is given by

$$(D_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds.$$

(ii) *Caputo Definition*: For $\alpha \in [n-1,n)$ the derivative of f is

$$({}^C D_a^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds.$$

Among the inconsistencies of the existing fractional derivatives are: Most of the fractional derivatives, except Riemann-Liouville-type derivatives, do not satisfy $D_a^\alpha(1) = 0$ if α is not a natural number; all fractional derivatives do not obey the Product Rule for two functions, the Quotient Rule for two functions, the Chain Rule, do not have a corresponding Rolle's Theorem, and a corresponding Mean Value Theorem etc.

To overcome some of these and other difficulties, Khalil *et al.* [18], came up with an interesting idea that extends the familiar limit definition of the derivative of a function, *the conformable fractional derivative*. This new theory is improved by Abdeljawad [2].

As a consequence of this new definition, the authors in [18], showed that the conformable fractional derivative, obeys the Product rule, Quotient rule and has results similar to the Rolle's Theorem and the Mean Value Theorem in classical calculus. For recent results on conformable fractional derivatives we refer the reader to [1,3,4,12,15].

In this paper, we consider the following boundary value problem for impulsive conformable fractional differential equation with delay:

$$\begin{cases} {}_{t_k}D^\alpha x(t) = f(t,x(t),x(\theta(t))), & t \in J := [0,T], t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\dots,m, \\ x(0) = \lambda x(T), \end{cases} \tag{1.1}$$

where ${}_aD^\alpha$ denotes the conformable fractional derivative of order $0 < \alpha \le 1$ starting from $a \in \{t_0,\dots,t_m\}$, $t_0 = 0 < t_1 < \dots < t_m < t_{m+1} = T$, $f \in C(J \times \mathbb{R}^2,\mathbb{R})$, $\theta \in C(J,J)$, $\theta(t) \le t$, $I_k \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\lambda \in \mathbb{R}^+$. Not that if $\lambda = 1$ then (1.1) is reduced to the periodic boundary value problem.

In the year 2016 [13], the authors studied the periodic boundary value problems for impulsive conformable fractional integro-differential equation of the form

$$\begin{cases} {}_{t_k}D^\alpha x(t) = f(t,x(t),(Fx)(t),(Sx)(t)), & t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\dots,m, \\ x(0) = x(T), \end{cases} \tag{1.2}$$

where $(Fx)(t) = \int_0^t l(t,s)x(s)ds$ and $(Sx)(t) = \int_0^T h(t,s)x(s)ds$. By using the method of lower and upper solutions in reversed order coupled with the monotone iterative technique, they formulated the existence of solutions for impulsive problem (1.2).

It is well known that the method of upper and lower solutions coupled with its associated monotone iteration scheme is an interesting and powerful mechanism that offers theoretical as well constructive existence results for nonlinear problems in a closed set, generated by the lower and upper solutions; see, for instance, [16, 20–22, 30]. By mean of two new maximal principles and new definitions of lower and upper solutions, the monotone iterative technique will be applied in our investigation of the impulsive problem (1.1).

The rest of the paper is organized as follows: In Section 2 we recall some definitions and results from conformable fractional calculus. In Section 3 we define the lower and upper solutions, obtain the Green's functions and prove two new maximum principles. The existence of a unique solution for linear problem is proved in this section. The existence results of problem (1.1) via monotone iterative technique are contained in Section 4, while an example illustrating the main result is presented in Section 5.

2. Conformable Fractional Calculus

In this section, we recall some definitions, notations and results which will be used in our main results.

Definition 2.1 ([2]). The conformable fractional derivative starting from a point a of a function $f:[a,\infty)\to\mathbb{R}$ of order $0<\alpha\le 1$ is defined by

$$ {}_aD^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}, \tag{2.1}$$

provided that the limit exists.

If f is differentiable then ${}_aD^{\alpha}f(t)=(t-a)^{1-\alpha}f'(t)$. In addition, if the conformable fractional derivative of f of order α exists on $[a,\infty)$, then we say that f is α-differentiable on $[a,\infty)$.

The following lemma has been stated in the paper [13].

Lemma 2.2. *Let $\alpha\in(0,1]$, $k_1,k_2,p,\lambda\in\mathbb{R}$ and functions f,g be α-differentiable on $[a,\infty)$. Then:*

(i) ${}_aD^{\alpha}(k_1f+k_2g)=k_1{}_aD^{\alpha}(f)+k_2{}_aD^{\alpha}(g)$;

(ii) ${}_aD^{\alpha}(t-a)^p=p(t-a)^{p-\alpha}$;

(iii) ${}_aD^{\alpha}\lambda=0$ *for all constant functions* $f(t)=\lambda$;

(iv) ${}_aD^{\alpha}(fg)=f{}_aD^{\alpha}g+g{}_aD^{\alpha}f$;

(v) ${}_aD^{\alpha}\left(\frac{f}{g}\right)=\frac{g{}_aD^{\alpha}f-f{}_aD^{\alpha}g}{g^2}$ *for all functions* $g(t)\neq 0$.

Definition 2.3 ([2]). Let $\alpha\in(0,1]$. The conformable fractional integral starting from a point a of a function $f:[a,\infty)\to\mathbb{R}$ of order α is defined as

$$ {}_aI^{\alpha}f(t)=\int_a^t(s-a)^{\alpha-1}f(s)ds. \tag{2.2}$$

Remark 2.4. If $a=0$, the definitions of the conformable fractional derivative and integral above will be reduced to the results in [18].

3. Auxiliary Impulsive Results

Let $J^- = J\setminus\{t_1,t_2,\dots,t_m\}$, $J_0=[t_0,t_1]$, $J_k=(t_k,t_{k+1}]$ for $k=1,2,\dots,m$ be sub-intervals of J and the set $PC(J,\mathbb{R})=\{x:J\to\mathbb{R}: x(t)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-)=x(t_k)$, $k=1,2,\dots,m\}$. Let $E=PC(J,\mathbb{R})$, then E is a Banach space with the norm $\|x\|=\sup_{t\in J}|x(t)|$. A function $x\in E$ is called a solution of the impulsive boundary value problem (1.1) if it satisfies (1.1).

Definition 3.1. A function $\mu_0\in E$ is called a lower solution of boundary value problem (1.1) if there exist $M>0$ and $N,L_k\ge 0$, $k=1,2,\dots,m$, such that

$$\begin{cases} {}_{t_k}D^\alpha \mu_0(t)\le f(t,\mu_0(t),\mu_0(\theta(t)))-a(t), & t\in J^-,\\ \Delta\mu_0(t_k)\le I_k(\mu_0(t_k))-\delta_k, & k=1,2,\dots,m. \end{cases} \tag{3.1}$$

Analogously, a function $\nu_0\in E$ is called an upper solution of boundary value problem (1.1) if the following inequalities

$$\begin{cases} {}_{t_k}D^\alpha \nu_0(t)\ge f(t,\nu_0(t),\nu_0(\theta(t)))+b(t)), & t\in J^-,\\ \Delta\nu_0(t_k)\ge I_k(\nu_0(t_k))+\eta_k, & k=1,2,\dots,m, \end{cases} \tag{3.2}$$

hold, where

$$a(t)=\begin{cases} 0, & \mu_0(0)\le\lambda\mu_0(T),\\ \frac{(t-t_k)^{1-\alpha}+Mt+N\theta(t)}{\lambda T}[\mu_0(0)-\lambda\mu_0(T)], & \mu_0(0)>\lambda\mu_0(T), \end{cases}$$

$$\delta_k=\begin{cases} 0, & \mu_0(0)\le\lambda\mu_0(T)\\ \frac{L_kt_k}{\lambda T}[\mu_0(0)-\lambda\mu_0(T)], & \mu_0(0)>\lambda\mu_0(T), \end{cases}$$

and

$$b(t)=\begin{cases} 0, & \nu_0(0)\ge\lambda\nu_0(T),\\ \frac{(t-t_k)^{1-\alpha}+Mt+N\theta(t)}{\lambda T}[\nu_0(0)-\lambda\nu_0(T)], & \nu_0(0)<\lambda\nu_0(T), \end{cases}$$

$$\eta_k=\begin{cases} 0, & \nu_0(0)\ge\lambda\nu_0(T),\\ \frac{L_kt_k}{\lambda T}[\nu_0(0)-\lambda\nu_0(T)], & \nu_0(0)<\lambda\nu_0(T). \end{cases}$$

In our analysis, we use the following notations. For $a,b\in\{0,1,2,\dots,m\}$ with $a\le b$,

$$\phi(a,b)=\prod_{i=a}^{b} e^{-\frac{M}{\alpha}(t_{i+1}-t_i)^\alpha}(1-L_{i+1}), \tag{3.3}$$

where $\prod_{i=b+1}^{b}(\cdot)=1$. Let $f=f_i(t)$ for $t\in J_i$, $i=0,1,2,\dots,m$. The impulsive integral notation is defined as

$$\int_a^b f(s)\widehat{d}s=\int_a^{t_p} f_{p-1}(s)ds+\int_{t_p}^{t_{p+1}} f_p(s)ds+\cdots+\int_{t_q}^{b} f_q(s)ds, \quad a,b\in J, \tag{3.4}$$

where $a\le t_p<\cdots<t_q\le b$.

For clearing the new notations, we consider an example.

Example 3.2. For $J=[0,5]$, $t_k=k$, $k=1,2,3,4$, two notations can be expressed as

$$\phi(2,4)=\prod_{i=2}^{4} e^{-\frac{M}{\alpha}(t_{i+1}-t_i)^\alpha}(1-L_{i+1})=e^{-\frac{M}{\alpha}(t_3-t_2)^\alpha}(1-L_3)\cdot e^{-\frac{M}{\alpha}(t_4-t_3)^\alpha}(1-L_4)\cdot e^{-\frac{M}{\alpha}(t_5-t_4)^\alpha}(1-L_5),$$

$$\int_{0.5}^{3.5} f(s)\widehat{d}s = \int_{0.5}^{1} f_0(s)ds + \int_{1}^{2} f_1(s)ds + \int_{2}^{3} f_2(s)ds + \int_{3}^{3.5} f_3(s)ds.$$

It is easy to prove the following property.

Property 3.3. Let $a \le c \le b \le d$ be nonnegative integers. The following relations hold:

(i) $\phi(a,c)\phi(c+1,b) = \phi(a,b)$.

(ii) $\phi(a,b)\phi(c,d) = \phi(a,d)\phi(c,b)$.

Now, we consider the following boundary value problem of a linear impulsive conformable fractional differential equation with delay subject to boundary condition as:

$$\begin{cases} {}_{t_k}D^{\alpha}x(t) = -Mx(t) - Nx(\theta(t)) + v(t), \quad 0 < \alpha \le 1,\ t \in J^-, \\ \Delta x(t_k) = -L_k x(t_k) + \gamma_k, \quad k = 1,2,\dots,m, \\ x(0) = \lambda x(T), \end{cases} \tag{3.5}$$

where $M > 0$, $N \ge 0$, $L_k \ge 0$, $\gamma_k, \lambda \in \mathbb{R}$, $k = 1,2,\dots,m$ with $\lambda\phi(0,m-1) \ne e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}$ are given constants and a function $v \in E$.

Lemma 3.4. *The problem* (3.5) *is equivalent to the following impulsive integral equation*

$$x(t) = \int_0^T G_1(t,s)P(s)\widehat{d}s + \sum_{j=1}^{m} G_2(h,j),\ t \in J_h; \quad h = 0,1,2,\dots,m, \tag{3.6}$$

where $P(t) = -Nx(\theta(t)) + v(t)$,

$$G_1(t,s) = \begin{cases} \dfrac{(s-t_l)^{\alpha-1}\phi(l,h-1)e^{\frac{M}{\alpha}(s-t_l)^{\alpha}}e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}e^{-\frac{M}{\alpha}(t-t_h)^{\alpha}}}{e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}-\lambda\phi(0,m-1)}, & 0 \le s < t \le T, \\ \dfrac{\lambda(s-t_l)^{\alpha-1}\phi(0,h-1)\phi(l,m-1)e^{\frac{M}{\alpha}(s-t_l)^{\alpha}}e^{-\frac{M}{\alpha}(t-t_h)^{\alpha}}}{e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}-\lambda\phi(0,m-1)}, & 0 \le t \le s \le T, \end{cases} \tag{3.7}$$

and

$$G_2(h,j) = \begin{cases} \dfrac{\gamma_j\phi(j,h-1)e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}e^{-\frac{M}{\alpha}(t-t_h)^{\alpha}}}{e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}-\lambda\phi(0,m-1)}, & 0 \le j < h \le T, \\ \dfrac{\lambda\gamma_j\phi(0,h-1)\phi(j,m-1)e^{-\frac{M}{\alpha}(t-t_h)^{\alpha}}}{e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}-\lambda\phi(0,m-1)}, & 0 \le h \le j \le T, \end{cases} \tag{3.8}$$

with $t_l = \max\{t_k;\ k = 0,1,\dots,m \text{ and } t_k \le s\}$.

Proof. Let $x(t)$ be a solution of the problem (3.5). Multiplying by $e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}$ both sides of the first equation of problem (3.5) and applying Lemma 2.2(iv), for $t \in J_0$,

$$e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}\,{}_{t_k}D^{\alpha}x(t) + e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}Mx(t) = {}_{t_k}D^{\alpha}\left[e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}x(t)\right],$$

we have

$${}_{t_k}D^{\alpha}\left[e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}x(t)\right] = e^{M\frac{(t-t_k)^{\alpha}}{\alpha}}P(t). \tag{3.9}$$

Using the conformable fractional integral of order α to both sides of (3.9) for $t \in J_0$, we get

$$x(t) = e^{-\frac{M}{\alpha}(t-t_0)^{\alpha}}\left[x(0) + \int_{t_0}^{t}(s-t_0)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_0)^{\alpha}}P(s)ds\right].$$

The conformable fractional integration of order α from t_1 to t for $t \in J_1$, of (3.9) yields

$$x(t) = e^{-\frac{M}{\alpha}(t-t_1)^\alpha}\left[x(t_1^+) + \int_{t_1}^{t}(s-t_1)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_1)^\alpha}P(s)ds\right].$$

From $x(t_1^+) = x(t_1) - L_1x(t_1) + \gamma_1 = (1-L_1)x(t_1) + \gamma_1$ and

$$x(t_1) = e^{-\frac{M}{\alpha}(t_1-t_0)^\alpha}\left[x(0) + \int_{t_0}^{t_1}(s-t_0)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_0)^\alpha}P(s)ds\right],$$

by using (3.3), we deduce that

$$\begin{aligned}
x(t) &= e^{-\frac{M}{\alpha}(t-t_1)^\alpha}\left[(1-L_1)x(t_1) + \gamma_1 + \int_{t_1}^{t}(s-t_1)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_1)^\alpha}P(s)ds\right] \\
&= e^{-\frac{M}{\alpha}(t-t_1)^\alpha}\Bigg[(1-L_1)e^{-\frac{M}{\alpha}(t_1-t_0)^\alpha}\left(x(0) + \int_{t_0}^{t_1}(s-t_0)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_0)^\alpha}P(s)ds\right) \\
&\qquad + \int_{t_1}^{t}(s-t_1)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_1)^\alpha}P(s)ds + \gamma_1\Bigg] \\
&= e^{-\frac{M}{\alpha}(t-t_1)^\alpha}\Bigg[\phi(0,0)\left(x(0) + \int_{t_0}^{t_1}(s-t_0)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_0)^\alpha}P(s)ds\right) \\
&\qquad + \int_{t_1}^{t}(s-t_1)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_1)^\alpha}P(s)ds + \gamma_1\Bigg].
\end{aligned}$$

Repeating the above process, for $t \in J_h$, we have

$$\begin{aligned}
x(t) = e^{-\frac{M}{\alpha}(t-t_h)^\alpha}\Bigg(&\phi(0,h-1)x(0) + \sum_{j=0}^{h-1}\phi(j,h-1)\int_{t_j}^{t_{j+1}}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}P(s)ds \\
&+ \int_{t_h}^{t}(s-t_h)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_h)^\alpha}P(s)ds + \sum_{j=1}^{h}\gamma_j\phi(j,h-1)\Bigg).
\end{aligned} \tag{3.10}$$

Substituting $t = T$ in (3.10), we get

$$\begin{aligned}
x(T) = e^{-\frac{M}{\alpha}(T-t_m)^\alpha}\Bigg(&\phi(0,m-1)x(0) + \sum_{j=0}^{m}\phi(j,m-1)\int_{t_j}^{t_{j+1}}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}P(s)ds \\
&+ \sum_{j=1}^{m}\gamma_j\phi(j,m-1)\Bigg).
\end{aligned}$$

From the boundary condition $x(0) = \lambda x(T)$, we obtain a constant

$$\begin{aligned}
x(0) = \frac{\lambda}{e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)}\Bigg(&\sum_{j=0}^{m}\phi(j,m-1)\int_{t_j}^{t_{j+1}}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}P(s)ds \\
&+ \sum_{j=1}^{m}\gamma_j\phi(j,m-1)\Bigg).
\end{aligned} \tag{3.11}$$

Putting (3.11) into (3.10), one has

$$\begin{aligned}
x(t) = e^{-\frac{M}{\alpha}(t-t_h)^\alpha}\Bigg[&\frac{\lambda\phi(0,h-1)}{e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)}\Bigg(\sum_{j=0}^{m}\phi(j,m-1) \\
&\times \int_{t_j}^{t_{j+1}}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}P(s)ds + \sum_{j=1}^{m}\gamma_j\phi(j,m-1)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=0}^{h-1} \phi(j,h-1) \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds \\
&\quad + \int_{t_h}^{t} (s-t_h)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_h)^\alpha} P(s)ds + \sum_{j=1}^{h} \gamma_j \phi(j,h-1) \Bigg] \\
&= \frac{\lambda e^{-\frac{M}{\alpha}(t-t_h)^\alpha}}{e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)} \Bigg[\sum_{j=0}^{m} \phi(0,h-1)\phi(j,m-1) \\
&\quad \times \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds + \sum_{j=1}^{m} \gamma_j \phi(0,h-1)\phi(j,m-1) \\
&\quad - \sum_{j=0}^{h-1} \phi(0,m-1)\phi(j,h-1) \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds \\
&\quad - \phi(0,m-1) \int_{t_h}^{t} (s-t_h)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_h)^\alpha} P(s)ds - \sum_{j=1}^{h} \gamma_j \phi(0,m-1)\phi(j,h-1) \Bigg] \\
&\quad + \frac{e^{\frac{M}{\alpha}(T-t_m)^\alpha}}{\lambda} \Bigg(\sum_{j=0}^{h-1} \phi(j,h-1) \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds \\
&\quad + \int_{t_h}^{t} (s-t_h)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_h)^\alpha} P(s)ds + \sum_{j=1}^{h} \gamma_j \phi(j,h-1) \Bigg) \Bigg].
\end{aligned}
$$

Applying Property 3.3 (i) and (ii), we have

$$
\begin{aligned}
x(t) &= \frac{\lambda e^{-\frac{M}{\alpha}(t-t_h)^\alpha}}{e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)} \Bigg[\sum_{j=h+1}^{m} \phi(0,h-1)\phi(j,m-1) \\
&\quad \times \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds + \sum_{j=h+1}^{m} \gamma_j \phi(0,h-1)\phi(j,m-1) \\
&\quad + \phi(0,m-1) \int_{t}^{t_{h+1}} (s-t_h)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_h)^\alpha} P(s)ds \Bigg] \\
&\quad + \frac{e^{\frac{M}{\alpha}(T-t_m)^\alpha}}{\lambda} \Bigg(\sum_{j=0}^{h-1} \phi(j,h-1) \int_{t_j}^{t_{j+1}} (s-t_j)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_j)^\alpha} P(s)ds \\
&\quad + \int_{t_h}^{t} (s-t_h)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_h)^\alpha} P(s)ds + \sum_{j=1}^{h} \gamma_j \phi(j,h-1) \Bigg) \Bigg] \\
&= \frac{\lambda e^{-\frac{M}{\alpha}(t-t_h)^\alpha}}{e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)} \Bigg[\int_{t}^{T} (s-t_l)^{\alpha-1} \phi(0,h-1)\phi(l,m-1) e^{\frac{M}{\alpha}(s-t_l)^\alpha} P(s)\widehat{d}s \\
&\quad + \frac{e^{\frac{M}{\alpha}(T-t_m)^\alpha}}{\lambda} \int_{0}^{t} (s-t_l)^{\alpha-1} \phi(l,h-1) e^{\frac{M}{\alpha}(s-t_l)^\alpha} P(s)\widehat{d}s \\
&\quad + \sum_{j=h+1}^{m} \gamma_j \phi(0,h-1)\phi(j,m-1) + \frac{e^{\frac{M}{\alpha}(T-t_m)^\alpha}}{\lambda} \sum_{j=1}^{h} \gamma_j \phi(j,h-1) \Bigg) \Bigg].
\end{aligned}
$$

Hence, we get the integral equation (3.6).

Conversely, it can easily be shown that the integral equation (3.6) satisfies the impulsive problem (3.5). The proof is completed. □

Next, we give two new maximum principles.

Lemma 3.5. *Let $0<\alpha\le 1$. Assume that $x\in E$ satisfies*

$$\begin{cases} {}_{t_k}D^{\alpha}u(t)\le -Mu(t)-Nu(\theta(t)), & t\in J^-,\\ \Delta u(t_k)\le -L_k u(t_k), & k=1,2,\dots,m,\\ u(0)\le \lambda u(T), \end{cases} \tag{3.12}$$

where $M>0$, $N\ge 0$, $0<\lambda\le e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}$, $0\le L_k\le 1$, $k=1,2,\dots,m$ are given constants. In addition suppose that

$$\frac{N}{\phi(0,m-1)}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}(s-t_i)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_i)^{\alpha}}ds\le 1. \tag{3.13}$$

Then $u(t)\le 0$ for all $t\in J$.

Proof. Suppose, to the contrary, that $u(t)>0$ for some $t\in J$. Then, the analysis can be separated in to two cases:

(i) There exists a point $t^*\in J$, such that $u(t^*)>0$ and $u(t)\ge 0$ for all $t\in J$.

(ii) There exist two points $t^*,t_*\in J$, such that $u(t^*)>0$ and $u(t_*)<0$.

Case (i): Setting $v(t)=e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}u(t)$ for $t\in J_k$, $k=0,1,2,\dots$ then we have

$$\begin{cases} {}_{t_k}D^{\alpha}v(t)\le -Ne^{\frac{M}{\alpha}[(t-t_k)^{\alpha}-(\theta(t)-t_k)^{\alpha}]}v(\theta(t)), & t\in J^-,\\ \Delta v(t_k)\le -L_k v(t_k), & k=1,2,\dots,m,\\ v(0)\le \lambda e^{-\frac{M}{\alpha}(T-t_m)^{\alpha}}v(T). \end{cases} \tag{3.14}$$

Observe that, $v(t)$ and $u(t)$ have the same sign. The first inequality of problem (3.14) implies that ${}_{t_k}D^{\alpha}v(t)\le 0$ and $\Delta v(t_k)\le 0$ for $k=1,2,\dots,m$. Therefore, $v(t)$ is nonincreasing in J. Hence, we have $v(0)\ge v(t^*)>0$. If $\lambda=e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}$, then $v(0)\le v(T)$, and consequently $v(0)=v(T)$, which implies that $v(t)\equiv$ costant, for all $t\in J$. Therefore, $u(t)\equiv 0$, a contradiction. If $0<\lambda<e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}$, then

$$\frac{e^{\frac{M}{\alpha}(T-t_m)^{\alpha}}}{\lambda}v(0)\le v(T)\le v(0),$$

which is a contradiction.

Case (ii): Let $\inf\{u(t):t\in J\}=-b$. Then, we assume that $b>0$ and also there exists a point $t_*\in J_i$, $i\in\{0,1,\dots,m\}$, such that $u(t_*)=-b$ or $u(t_i^+)=-b$. Now, we only consider the case $u(t_*)=-b$. For the case $u(t_i^+)=-b$, the proof is similar. It is easy to see that

$${}_{t_k}D^{\alpha}\left(e^{\frac{M}{\alpha}(t-t_k)^{\alpha}}u(t)\right)\le bNe^{\frac{M}{\alpha}(t-t_k)^{\alpha}}. \tag{3.15}$$

First, we claim that $u(T)\le 0$. Otherwise, if $u(T)>0$, then by (3.12) and (3.15) we have

$$\begin{aligned} u(T)\le e^{-\frac{M}{\alpha}(T-t_m)^{\alpha}}\Bigg(&\phi(i+1,m-1)(1-L_{i+1})u(t_{i+1}^-)\\ &+bN\sum_{l=i+1}^{m}\phi(l,m-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^{\alpha}}ds\Bigg). \end{aligned} \tag{3.16}$$

Since ${}_{t_i}D^{\alpha}f(t)=(t-t_i)^{1-\alpha}f'(t)$ and (3.12), we obtain

$$u(t_{i+1}^-)\le e^{-\frac{M}{\alpha}(t_{i+1}-t_i)^{\alpha}}\left(e^{\frac{M}{\alpha}(t_*-t_i)^{\alpha}}u(t_*)+bN\int_{t_*}^{t_{i+1}}(s-t_i)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_i)^{\alpha}}ds\right). \tag{3.17}$$

From (3.16) and (3.17), we get

$$u(T) \le e^{-\frac{M}{\alpha}(T-t_m)^\alpha}\bigg(\phi(i,m-1)e^{\frac{M}{\alpha}(t_*-t_i)^\alpha}u(t_*)+bN\phi(i,m-1)\int_{t_*}^{t_{i+1}}(s-t_i)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_i)^\alpha}ds$$
$$+bN\sum_{l=i+1}^{m}\phi(l,m-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds\bigg), \tag{3.18}$$

which leads to

$$0<u(T) \le e^{-\frac{M}{\alpha}(T-t_m)^\alpha}\bigg(-b\phi(i,m-1)e^{\frac{M}{\alpha}(t_*-t_i)^\alpha}+bN\phi(i,m-1)\int_{t_*}^{t_{i+1}}(s-t_i)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_i)^\alpha}ds$$
$$+bN\sum_{l=i+1}^{m}\phi(l,m-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds\bigg).$$

Thus

$$\phi(i,m-1)e^{\frac{M}{\alpha}(t_*-t_i)^\alpha}<N\phi(i,m-1)\int_{t_*}^{t_{i+1}}(s-t_i)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_i)^\alpha}ds$$
$$+N\sum_{l=i+1}^{m}\phi(l,m-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds.$$

Therefore, we have

$$1<\frac{N}{\phi(0,m-1)}\sum_{l=0}^{m}\phi(l,m-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds,$$

a contradiction, and so $u(T)\le 0$. From $u(0)\le \lambda u(T)$, we get $u(0)\le 0$. Thus there exist $\bar{t}$ such that $u(\bar{t})\le 0$ and $\bar{t}<t^*$.

Suppose that $t^*\in J_j$ and $u(\bar{t})=\inf\{u(t): t\in[0,t^*)\}=-c\le 0$, such that $\bar{t}\in J_h$ for some $j,h\in\{0,1,\dots,m\}$. It is easy to see that $h\le j$. As in (3.18), we have

$$u(t^*) \le e^{-\frac{M}{\alpha}(t^*-t_j)^\alpha}\bigg(\phi(h,j-1)e^{\frac{M}{\alpha}(\bar{t}-t_h)^\alpha}u(\bar{t})+cN\phi(h,j-1)\int_{\bar{t}}^{t_{h+1}}(s-t_h)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_h)^\alpha}ds$$
$$+cN\sum_{l=h+1}^{j-1}\phi(l,j-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds+cN\int_{t_j}^{t^*}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}ds\bigg).$$

Thus

$$u(t^*) \le e^{-\frac{M}{\alpha}(t^*-t_j)^\alpha}\bigg(-c\phi(h,j-1)e^{\frac{M}{\alpha}(\bar{t}-t_h)^\alpha}+cN\phi(h,j-1)\int_{\bar{t}}^{t_{h+1}}(s-t_h)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_h)^\alpha}ds$$
$$+cN\sum_{l=h+1}^{j-1}\phi(l,j-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds+cN\int_{t_j}^{t^*}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}ds\bigg).$$

If $c=0$, then we get $u(t^*)\le 0$ a contradiction. If $c>0$, then we obtain that

$$0<u(t^*) \le e^{-\frac{M}{\alpha}(t^*-t_j)^\alpha}\bigg(-c\phi(h,j-1)e^{\frac{M}{\alpha}(\bar{t}-t_h)^\alpha}+cN\phi(h,j-1)\int_{\bar{t}}^{t_{h+1}}(s-t_h)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_h)^\alpha}ds$$
$$+cN\sum_{l=h+1}^{j-1}\phi(l,j-1)\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds+cN\int_{t_j}^{t^*}(s-t_j)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_j)^\alpha}ds\bigg),$$

which yields

$$1<\frac{N}{\phi(0,m-1)}\sum_{l=0}^{m}\int_{t_l}^{t_{l+1}}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}ds,$$

a contradiction, and so $u(t)\le 0$. This is a contradiction to (3.13). The proof is completed. □

Lemma 3.6. *Let* $0 < \alpha \le 1$. *Assume that* $x \in E$ *satisfies*

$$\begin{cases} {}_{t_k}D^\alpha u(t) \le -Mu(t) - Nu(\theta(t)) - \frac{(t-t_k)^{1-\alpha}+Mt+N\theta(t)}{\lambda T}[u(0)-\lambda u(T)],\ t \in J^-, \\ \Delta u(t_k) \le -L_k u(t_k) - \frac{L_k t_k}{\lambda T}[u(0)-\lambda u(T)], \quad k = 1,2,\dots,m, \\ u(0) > \lambda u(T), \end{cases} \tag{3.19}$$

where $0 < \lambda \le e^{\frac{M}{\alpha}(T-t_m)^\alpha}$, $M > 0$, N, $L_k \ge 0$, $k = 1,2,\dots,m$ *are given constants satisfying* (3.13). *Then* $u(t) \le 0$ *for all* $t \in J$.

Proof. Setting $v(t) = u(t) + \frac{t}{\lambda T}[u(0)-\lambda u(T)]$ then we have $v \ge u$, and for all $t \ne t_k$, $t \in J$,

$$\begin{aligned} &{}_{t_k}D^\alpha v(t) + Mv(t) + Nv(\theta(t)) \\ &= {}_{t_k}D^\alpha u(t) + Mu(t) + Nu(\theta(t)) + \frac{(t-t_k)^{1-\alpha}+Mt+N\theta(t)}{\lambda T}[u(0)-\lambda u(T)] \\ &\le 0. \end{aligned}$$

It is easy to verify that

$$\Delta v(t_k) = \Delta u(t_k) \le -L_k u(t_k) - \frac{L_k t_k}{\lambda T}[u(0)-\lambda u(T)] = -L_k v(t_k),$$

and $v(0) = u(0)$, $\lambda v(T) = u(0)$. Then we have $v(0) = \lambda v(T)$. By lemma 3.5, we obtain $v(t) \le 0$ for all $t \in J$, which implies that $u(t) \le 0$ for all $t \in J$. □

In view of Lemma 3.4, we define the operator $\mathcal{A} : E \to E$ by

$$\mathcal{A}x(t) = \int_0^T G_1(t,s)P(s)\widehat{d}s + \sum_{j=1}^m G_2(h,j), \tag{3.20}$$

where the Green's functions $G_1(t,s)$ and $G_2(t,s)$ are defined by (3.7) and (3.8), respectively. Next, we prove the existence of a unique solution for the linear problem (3.5). For convenience, we set a constant

$$\Lambda := \frac{\max\{\lambda, e^{\frac{M}{\alpha}(T-t_m)^\alpha}\}N\sum_{i=0}^m \left(e^{\frac{M}{\alpha}(t_{i+1}-t_i)^\alpha}-1\right)}{M\left|e^{\frac{M}{\alpha}(T-t_m)^\alpha}-\lambda\phi(0,m-1)\right|}.$$

Lemma 3.7. *Assume that* $\alpha \in (0,1]$, $M, \lambda > 0$, $N \ge 0$, $0 \le L_k \le 2$, $k = 1,2,\dots,m$ *and* $\lambda\phi(0,m-1) \ne e^{\frac{M}{\alpha}(T-t_m)^\alpha}$. *If*

$$\Lambda < 1, \tag{3.21}$$

then the boundary value problem (3.5) *has a unique solution on* J.

Proof. Case I. For $0 \le s < t \le T$, we see that

$$\left|(s-t_l)^{\alpha-1}\phi(l,h-1)e^{\frac{M}{\alpha}(s-t_l)^\alpha}e^{\frac{M}{\alpha}(T-t_m)^\alpha}e^{-\frac{M}{\alpha}(t-t_h)^\alpha}\right| \le (s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}e^{\frac{M}{\alpha}(T-t_m)^\alpha}.$$

Case II. For $0 \le t \le s \le T$, we have

$$\left|\lambda(s-t_l)^{\alpha-1}\phi(0,h-1)\phi(l,m-1)e^{\frac{M}{\alpha}(s-t_l)^\alpha}e^{-\frac{M}{\alpha}(t-t_h)^\alpha}\right| \le \lambda(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}.$$

From *Cases* I and II, it follows that

$$|G_1(t,s)| \le \frac{\max\left\{\lambda, e^{\frac{M}{\alpha}(T-t_m)^\alpha}\right\}(s-t_l)^{\alpha-1}e^{\frac{M}{\alpha}(s-t_l)^\alpha}}{\left|e^{\frac{M}{\alpha}(T-t_m)^\alpha}-\lambda\phi(0,m-1)\right|}.$$

Now, we transform the problem (3.5) into a fixed point problem $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined by (3.20). For any $x, y \in E$, we have

$$\begin{aligned}\|\mathcal{A}x - \mathcal{A}y\| &\le \|x-y\| N \int_0^T |G_1(t,s)| \widehat{d}s \\ &\le \|x-y\| \frac{N \max\left\{\lambda, e^{\frac{M}{\alpha}(T-t_m)^\alpha}\right\}}{\left|e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)\right|} \int_0^T (s-t_l)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_l)^\alpha} \widehat{d}s \\ &\le \Lambda \|x-y\|.\end{aligned}$$

As $\Lambda < 1$, $\mathcal{A}$ is a contraction. Therefore, by the Banach's contraction mapping principle, $\mathcal{A}$ has a fixed point which is the unique solution of problem (3.5). The proof is completed. □

4. Main Results

For functions $\mu_0, \nu_0 \in E$, we let

$$[\mu_0, \nu_0] = \{x \in E : \mu_0(t) \le x(t) \le \nu_0(t), t \in J\},$$

and we write $\mu_0 \le \nu_0$ if $\mu_0(t) \le \nu_0(t)$ for all $t \in J$.

Theorem 4.1. *Assume that the following conditions hold:*

(H_1) *the functions μ_0 and ν_0 are lower and upper solutions of boundary value problem* (1.1)*, respectively, such that $\mu_0(t) \le \nu_0(t)$ on J;*

(H_2) *the function $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ satisfies*

$$f(t,u,v) - f(t,\overline{u},\overline{v}) \ge -M(u-\overline{u}) - N(v-\overline{v}),$$

for $\mu_0(t) \le \overline{u}(t) \le u(t) \le \nu_0(t)$, $\mu_0(\theta(t)) \le \overline{v}(t) \le v(t) \le \nu_0(\theta(t))$, $t \in J$;

(H_3) *the functions $I_k \in C(\mathbb{R},\mathbb{R})$, $k = 1, \dots, m$ satisfy*

$$I_k(u(t_k)) - I_k(v(t_k)) \ge -L_k(u(t_k) - v(t_k)),$$

whenever $\mu_0(t_k) \le v(t_k) \le u(t_k) \le \nu_0(t_k)$, $L_k \ge 0$, $k = 1, 2, \dots, m$;

(H_4) *two inequalities* (3.13) *and* (3.21) *hold.*

Then there exist two monotone sequences $\{\mu_n\}$, $\{\nu_n\} \subset E$ such that $\lim_{n\to\infty} \mu_n(t) = x_*(t)$, $\lim_{n\to\infty} \nu_n(t) = x^*(t)$ uniformly on J and functions x_*, x^* are minimal and maximal solutions of problem (1.1)*, respectively, such that*

$$\mu_0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_n \le x_* \le x \le x^* \le \nu_n \le \cdots \le \nu_2 \le \nu_1 \le \nu_0,$$

on J, where x is any solution of the boundary value problem (1.1) *such that $\mu_0(t) \le x(t) \le \nu_0(t)$ on J.*

Proof. First, we consider the problem

$$\begin{aligned} {}_{t_k}D^\alpha p_n(t) &= f(t, p_{n-1}(t), p_{n-1}(\theta(t))) - M[p_n(t) - p_{n-1}(t)] \\ &\quad - N[p_n(\theta(t)) - p_{n-1}(\theta(t))], \quad t \in J^-, \\ \Delta p_n(t_k) &= I_k(p_{n-1}(t_k)) - [L_k p_n(t_k) - L_k p_{n-1}(t_k)], \quad k = 1, 2, \dots, m, \\ p_n(0) &= \lambda p_n(T), \end{aligned} \tag{4.1}$$

where $p_n = \mu_n$ or $p_n = \nu_n$, $n = 1, 2, \dots$. By Lemma 3.7, the iteration formula (4.1) has a unique solution. Next, we show that the sequences $\{\mu_n\}$ and $\{\nu_n\}$ are monotone sequences in two steps.

Step 1. We claim that $\mu_0 \le \mu_1$ and $\nu_1 \le \nu_0$. Let $u(t) = \mu_0 - \mu_1$, then from Definition 3.1 and (4.1), we have:

Case 1. $\mu_0(0) \le \lambda\mu_0(T)$. Then we have

$$\begin{aligned}
{}_{t_k}D^{\alpha}u(t) &= {}_{t_k}D^{\alpha}\mu_0(t) - {}_{t_k}D^{\alpha}\mu_1(t) \\
&\le f(t,\mu_0(t),\mu_0(\theta(t))) - f(t,\mu_0(t),\mu_0(\theta(t))) + M[\mu_1(t)-\mu_0(t)] + N[\mu_1(\theta(t))-\mu_0(\theta(t))] \\
&= -Mu(t) - Nu(\theta(t)), \quad t \in J^-, \\
\Delta u(t_k) &= \Delta\mu_0(t_k) - \Delta\mu_1(t_k) \\
&\le I_k\left(\mu_0(t_k)\right) - \left[I_k\left(\mu_0(t_k)\right) - L_k\mu_1(t_k) + L_k\mu_0(t_k)\right] \\
&= -L_k u(t_k), \quad k = 1,2,\dots,m,
\end{aligned}$$

and

$$\begin{aligned}
u(0) &= \mu_0(0) - \mu_1(0) \\
&\le \lambda\mu_0(T) - \lambda\mu_1(T) \\
&= \lambda u(T).
\end{aligned}$$

Using Lemma 3.5, we get that $u(t) \le 0$ for all $t \in J$, i.e., $\mu_0 \le \mu_1$. Similarly, we can prove that $\nu_1 \le \nu_0$.

Case 2. $\mu_0(0) > \lambda\mu_0(T)$. Then we have

$$\begin{aligned}
{}_{t_k}D^{\alpha}u(t) &= {}_{t_k}D^{\alpha}\mu_0(t) - {}_{t_k}D^{\alpha}\mu_1(t) \\
&\le f(t,\mu_0(t),\mu_0(\theta(t))) - \frac{(t-t_k)^{1-\alpha} + Mt + N\theta(t)}{\lambda T}\left[\mu_0(0) - \lambda\mu_0(T)\right] \\
&\quad - f(t,\mu_0(t),\mu_0(\theta(t))) + M[\mu_1(t)-\mu_0(t)] + N[\mu_1(\theta(t))-\mu_0(\theta(t))] \\
&= -Mu(t) - Nu(\theta(t)) - \frac{(t-t_k)^{1-\alpha} + Mt + N\theta(t)}{\lambda T}[u(0) - \lambda u(T)], \quad t \in J^-, \\
\Delta u(t_k) &= \Delta\mu_0(t_k) - \Delta\mu_1(t_k) \\
&\le I_k(\mu_0(t_k)) - \frac{L_k t_k}{\lambda T}[\mu_0(0) - \lambda\mu_0(T)] - \left[I_k(\mu_0(t_k)) - L_k\mu_1(t_k) + L_k\mu_0(t_k)\right] \\
&= -L_k u(t_k) - \frac{L_k t_k}{\lambda T}[u(0) - \lambda u(T)], \quad k = 1,2,\dots,m,
\end{aligned}$$

and

$$\begin{aligned}
u(0) &= \mu_0(0) - \mu_1(0) \\
&> \lambda\mu_0(T) - \lambda\mu_1(T) \\
&= \lambda u(T).
\end{aligned}$$

Using Lemma 3.6, again we have $u(t) \le 0$ for all $t \in J$, i.e., $\mu_0 \le \mu_1$. Similarly, we can prove that $\nu_1 \le \nu_0$.

Step 2. We show that $\mu_n \le \mu_{n+1}$, where $\mu_{n-1} \le \mu_n$ on J. Setting a function $u = \mu_n - \mu_{n+1}$, then for $t \in J$ and by (H_2), we obtain

$$\begin{aligned}
{}_{t_k}D^{\alpha}u(t) &= {}_{t_k}D^{\alpha}\mu_n(t) - {}_{t_k}D^{\alpha}\mu_{n+1}(t) \\
&= f(t,\mu_{n-1}(t),\mu_{n-1}(\theta(t))) - M[\mu_n(t)-\mu_{n-1}(t)] - N[\mu_n(\theta(t))-\mu_{n-1}(\theta(t))] \\
&\quad - f(t,\mu_n(t),\mu_n(\theta(t))) + M[\mu_{n+1}(t)-\mu_n(t)] + N[\mu_{n+1}(\theta(t))-\mu_n(\theta(t))] \\
&\le -Mu(t) - Nu(\theta(t)), \quad t \in J^-,
\end{aligned}$$

and by (H_3),

$$\begin{aligned}\Delta u(t_k) &= \Delta\mu_n(t_k) - \Delta\mu_{n+1}(t_k)\\ &= I_k(u_{n-1}(t_k)) - [L_k u_n(t_k) - L_k u_{n-1}(t_k)] - I_k(u_n(t_k)) + [L_k u_{n+1}(t_k) - L_k u_n(t_k)]\\ &\le -L_k u(t_k), \quad k = 1,2,\dots,m.\end{aligned}$$

It is easy to see that

$$u(0) = \mu_n(0) - \mu_{n+1}(0) = \lambda\mu_n(T) - \lambda\mu_{n+1}(T) = \lambda u(T).$$

Then, from Lemma 3.5, we get $u(t) \le 0$, which yields $\mu_n \le \mu_{n+1}$. Similarly, we can prove that $\nu_{n+1} \le \nu_n$ where $\nu_n \le \nu_{n-1}$ and $\mu_{n+1} \le \nu_{n+1}$, where $\mu_n \le \nu_n$ on J.

From *Steps* 1 and 2, we have that the two sequences $\{\mu_n\}$, $\{\nu_n\}$ satisfy the following inequalities

$$\mu_0 \le \mu_1 \le \dots \le \mu_n \le \dots \le \nu_n \le \dots \le \nu_1 \le \nu_0.$$

Therefore, there exist functions x_* and x^* on J, such that $\lim_{n\to\infty}\mu_n = x_*$ and $\lim_{n\to\infty}\nu_n = x^*$ uniformly on J. Clearly, x_*, x^* are solutions of boundary value problem (1.1).

Finally, we will show that x_*, x^* are minimal and maximal solutions of the problem (1.1). Let $x(t)$ be any solution of problem (1.1) for $t \in J$, such that $x \in [\mu_0, \nu_0]$. Then there exists a positive integer n such that $\mu_n(t) \le x(t) \le \nu_n(t)$ on J. Let $u = \mu_{n+1} - x$, then for $t \in J$, we have

$$\begin{aligned}{}_{t_k}D^\alpha u(t) &= {}_{t_k}D^\alpha \mu_{n+1}(t) - {}_{t_k}D^\alpha x(t)\\ &= f(t,\mu_n(t),\mu_n(\theta(t))) - M[\mu_{n+1}(t) - \mu_n(t)] - N[\mu_{n+1}(\theta(t)) - \mu_n(\theta(t))] - f(t,x(t),x(\theta(t)))\\ &\le -Mu(t) - Nu(\theta(t)), \quad t \in J^-,\\ \Delta u(t_k) &= \Delta\mu_{n+1}(t_k) - \Delta x(t_k)\\ &= I_k(\mu_n(t_k)) - (L_k\mu_{n+1}(t_k) - L_k\mu_n(t_k)) - I_k(x(t_k))\\ &\le -L_k u(t_k), \quad k = 1,2,\dots,m,\end{aligned}$$

and

$$u(0) = \mu_{n+1}(0) - x(0) = \lambda\mu_{n+1}(T) - \lambda x(T) \le \lambda u(T).$$

Therefore, using Lemma 3.5, we have $u(t) \le 0$, which leads to $\mu_{n+1} \le x$ on J. By similarly method, we can prove that $x \le \nu_{n+1}$ on J. From $\mu_0 \le x \le \nu_0$ on J, by mathematical induction, we get that $\mu_n \le x \le \nu_n$ on J for all $n \in \mathbb{N}$. Hence, by taking $n \to \infty$, we have $x_*(t) \le x(t) \le x^*(t)$ on J. The proof is complete. □

5. An Example

In this section, in order to illustrate our results, we consider an example.

Example 5.1. Consider the BVP

$$\begin{cases} {}_{t_k}D^{5/7}x(t) = -\frac{t}{5}x(t) + \frac{t}{7}\cos x\left(\frac{t}{2}\right) - \frac{t}{6}, & t \in [0,1] \setminus \left\{\frac{1}{3}\right\},\\ \Delta x\left(\frac{1}{3}\right) = -\frac{1}{3}x\left(\frac{1}{3}\right), & k = 1,\\ x(0) = 1.13x(1), \end{cases} \tag{5.1}$$

where $\alpha = 5/7$, $\theta = t/2$, $t_1 = 1/3$, $\lambda = 1.13$, $m = 1$.

Obviously, $\mu_0 = \begin{cases} -7, & t \in [0,1/3], \\ -6, & t \in (1/3,1], \end{cases}$ $\nu_0 = 0$ are lower and upper solutions for (5.1), respectively, and $\mu_0 \le \nu_0$. Let

$$f(t,u,v) = -\frac{t}{5}u + \frac{t}{7}\cos v - \frac{t}{6}.$$

Then

$$\begin{aligned} f(t,u,v) - f(t,\bar{u},\bar{v}) &= -\frac{t}{5}u + \frac{t}{7}\cos v + \frac{t}{5}\bar{u} - \frac{t}{7}\cos\bar{v} \\ &\ge -\frac{t}{5}(u-\bar{u}) - \frac{t}{7}(v-\bar{v}), \end{aligned}$$

for $\mu_0(t) \le \overline{u}(t) \le u(t) \le \nu_0(t)$, $\mu_0(\theta(t)) \le \overline{v}(t) \le v(t) \le \nu_0(\theta(t))$, $t \in J$. It is easy to see that

$$I_1(u) - I_1(v) = -\frac{1}{3}(u-v)$$

whenever $\mu_0(t_1) \le v(t_1) \le u(t_1) \le \nu_0(t_1)$.

Taking $M = 1/5$, $N = 1/7$, $L_1 = 1/3$, it follow that

$$\frac{N}{\phi(0,m-1)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} (s-t_i)^{\alpha-1} e^{\frac{M}{\alpha}(s-t_i)^\alpha} ds = 0.449772989 \le 1,$$

and

$$\Lambda := \frac{\max\{\lambda, e^{\frac{M}{\alpha}(T-t_m)^\alpha}\} N \sum_{i=0}^{m} \left(e^{\frac{M}{\alpha}(t_{i+1}-t_i)^\alpha} - 1\right)}{M\left|e^{\frac{M}{\alpha}(T-t_m)^\alpha} - \lambda\phi(0,m-1)\right|} = 0.8848227489 < 1.$$

Therefore, the conditions of Theorem 4.1 are satisfied, and therefore the BVP (5.1) has minimal and maximal solutions in the segment $[\mu_0,\nu_0]$.

6. Conclusions

In this paper, we investigated the existence of solutions for boundary value problems of nonlinear impulsive delay conformable fractional differential equations. The method of upper and lower solutions coupled with its associated monotone iteration scheme is an interesting and powerful mechanism that offers theoretical as well constructive existence results for nonlinear problems in a closed set, generated by the lower and upper solutions. By establishing the associate Green's function and a comparison result for the linear impulsive problem, we obtained that the lower and upper solutions converge to the extremal solutions via the monotone iterative technique. We illustrated the obtained results by an example.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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