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Research Article

Solvability, Unique Solvability, and Representation of Solutions for Rectangular Systems of Coupled Generalized Sylvester Matrix Equations

Thitison Nuchniyom and Pattrawut Chansangiam*

Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand
***Corresponding author:** pattrawut.ch@kmitl.ac.th

Abstract. We investigate a system of coupled linear matrix equations of the form

$$AXB + CYD = E,$$
$$CXD + AYB = F,$$

where A, B, C, D, E, F are rectangular complex matrices and X, Y are unknown complex matrices. We obtain several criterions for solvability and unique solvability of the system and its special cases. These criterions rely on Kronecker product, vector operator, Moore Penrose inverses, and ranks. Moreover, explicit formulas of solutions are presented.

Keywords. Linear matrix equation; Kronecker product; Vector operator; Moore-Penrose inverse

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1. Introduction

Linear matrix equations show up in several scientific fields such as, system and control theory, image processing, transportation problems, quantum mechanics (see e.g., [1, 4, 7, 17]). The current research on (systems of) linear matrix equations can be divided into three topics. The first one is to investigate necessary and sufficient condition for solvability and unique solvability of certain linear matrix equations with/without constraints, and then derive exact formulae for solutions (if possible). The second one is to obtain solutions that minimize least-squares errors (see e.g., [14, 21]). The last topic is to propose iterative schemes for solving linear matrix equations and deduce their convergence analysis (e.g., [9, 22]). This paper focuses on the first one.

One of the famous linear matrix equation, arising from stability analysis and optimal control, is the so called Lyapunov equation: $AX + XA^T = C$, here A, C are given square matrices, and X is an unknown square matrix. This equation is a special case of the Sylvester equation:

$$AX + XB = C. \tag{1.1}$$

In fact, the equation (1.1) has a unique solution for arbitrary matrix C if and only if A and $-B$ have no eigenvalues in common, and the solution can be represented in terms of Kronecker products and the vector operator (e.g., [5]). Let us take a look at a generalization of (1.1), namely the generalized Sylvester equation:

$$AXB + CXD = F. \tag{1.2}$$

Eq. (1.2) also includes the following equations as special cases:

$$AXB = F, \tag{1.3}$$

$$AXB^T + X = F, \tag{1.4}$$

$$AXA^T - X = F. \tag{1.5}$$

The equations (1.4) and (1.5) are known as the discrete-time Sylvester equation and the discrete-time Lyapunov equation, respectively. The equation (1.2) plays an important role in a generalized eigenvalue problem [8], numerical analysis for certain differential equations [10], and stability analysis of descriptor systems [6].

A general system of coupled linear matrix equations takes the form

$$\begin{aligned} A_1XA_2 + B_1YB_2 &= E, \\ C_1XC_2 + D_1YD_2 &= F, \end{aligned} \tag{1.6}$$

where A_i, B_i, C_i, D_i, E, F are given matrices for $i = 1,2$, and X, Y are unknown matrices. As a special case of (1.6), a system of coupled Sylvester matrix equations:

$$\begin{aligned} AX + YB &= E, \\ XB + AY &= F \end{aligned} \tag{1.7}$$

was investigated in [20]. More generally, the existence and uniqueness of solutions of (1.6) when every mentioned matrix is square and $C_iD_i = D_iC_i$ for all $i = 1,2$ was given in [13]. The following system was also discussed in [13]:

$$\begin{aligned} AXB + AYD &= E, \\ AXD + AYB &= F, \end{aligned} \tag{1.8}$$

here every mentioned matrix is square. In fact, this system is uniquely solvable if and only if A and $B \pm D$ are invertible. See more information on linear matrix equations in [2, 3, 11, 12, 16, 18, 19, 23] and references therein.

Our main task in the present work is to investigate a system of coupled generalized Sylvester matrix equations in the form

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F, \end{aligned} \tag{1.9}$$

where A, B, C, D, E, F are rectangular complex matrices and X, Y are unknown complex matrices. We apply Kronecker products and vector operator to reduce the system (1.9) to a simple vector-matrix equation. Then we obtain several necessary and sufficient conditions for solvability and unique solvability of the system (1.9) that hold for arbitrary E, F (see Section 3). These conditions rely on Kronecker products, vector operator, Moore-Penrose inverses, and ranks (see Section 2 for prerequisites on these topics). A discussion when the matrices E, F are fixed in the system (1.9) is also provided (see Section 4), and in this case we obtain explicit formulas of solutions. Cartain interesting special cases of (1.9) are also investigated. Note that when $C = 0$, the system (1.9) becomes the single equation (1.3). Our result also includes (1.7) and (1.8) as special cases.

2. Preliminaries

In this section, we provide fundamental tools for solving linear matrix equations. These tools include Kronecker products, the vector operator, Moore-Penrose inverses, and a block-matrix technique.

Denote by $M_{m,n}(\mathbb{C})$ the set of m-by-n complex matrices. We abbreviate $M_{n,n}(\mathbb{C})$ to $M_n(\mathbb{C})$.

2.1 Kronecker Products and the Vector Operator

Recall that the Kronecker product of two matrices $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$ is defined by

$$A \otimes B = [a_{ij}B] \in M_{mp,nq}(\mathbb{C}).$$

Lemma 1 (see e.g., [15]). *The Kronecker product satisfies the following properties (provided that every operation is well-defined):*

(i) *The map $(A,B) \mapsto A \otimes B$ is bilinear.*

(ii) *Compatability with transpose: $(A \otimes B)^T = A^T \otimes B^T$.*

(iii) *Mixed product property: $(A \otimes B)(C \otimes D) = AC \otimes BD$.*

(iv) *Compatability with ordinary inverse: $A \otimes B$ is invertible if and only if both A and B are invertible, in which case $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.*

(v) $\operatorname{rank}(A \otimes B) = (\operatorname{rank} A)(\operatorname{rank} B)$.

The vector operator is a column-stacking operator assigned to a matrix $A \in M_{m,n}(\mathbb{C})$ by

$$\operatorname{Vec} A = [a_{11}\, a_{21} \,\dots\, a_{m1}\, a_{12}\, a_{22} \,\dots\, a_{m2} \,\dots\, a_{1n}\, a_{2n} \,\dots\, a_{mn}]^T \in \mathbb{C}^{mn}.$$

This operator is clearly linear and bijective.

Lemma 2 (see e.g., [15]). *The vector operator can turn the usual matrix product to the Kronecker product as follows:*

$$\operatorname{Vec}(AXB) = (B^T \otimes A)\operatorname{Vec}X,$$

provided that every operation is well-defined.

2.2 Moore-Penrose Inverse and Linear Equations

Every matrix $A \in M_{m,n}(\mathbb{C})$ admits its Moore-Penrose inverse, which is the matrix $A^\dagger \in M_{n,m}(\mathbb{C})$ satisfying the following four conditions:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

If $A \in M_n(\mathbb{C})$ is invertible, then $A^\dagger = A^{-1}$. A simple and accurate way to compute the Moore-Penrose inverse is to use the singular value decomposition. Indeed, for any $A \in M_{m,n}(\mathbb{C})$ we can decompose $A = U\Sigma V^*$ where $U \in M_m(\mathbb{C})$ is unitary, $V \in M_n(\mathbb{C})$ is unitary and $\Sigma = [d_{ij}] \in M_{m,n}(\mathbb{C})$ is a rectangular diagonal matrix with nonnegative entries. Then $A^\dagger = V\Sigma^\dagger U^*$ where

$$\Sigma^\dagger = [d_{ij}^\dagger], \qquad d_{ij}^\dagger = \begin{cases} d_{ij}^{-1}, & d_{ij} \neq 0 \\ 0, & d_{ij} = 0. \end{cases}$$

Lemma 3 (see e.g., [15]). *Let $A \in M_{m,n}(\mathbb{C})$ and $B \in M_{p,q}(\mathbb{C})$. Then*

(i) $(A^\dagger)^T = (A^T)^\dagger$,

(ii) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

Lemma 4 (see e.g., [15]). *Let $A \in M_{m,n}(\mathbb{C})$ and $b \in \mathbb{C}^m$. The following statements are equivalent:*

(i) *The vector-matrix equation $Ax = b$ has a solution x.*

(ii) $\operatorname{rank}[A \mid b] = \operatorname{rank} A$.

(iii) $AA^\dagger b = b$.

In the above case, the general solution is given by $x = A^\dagger b + (I - A^\dagger A)q$ where $q \in \mathbb{C}^m$ is arbitrary.

Lemma 5 (see e.g., [15]). *Let $A \in M_{m,n}(\mathbb{C})$. The following statements are equivalent:*

(i) *The system $Ax = b$ is consistent for every $b \in \mathbb{C}^m$.*

(ii) *A has full row rank.*

(iii) $AA^\dagger = I_m$.

Lemma 6 (see e.g., [15]). *Let $A \in M_{m,n}(\mathbb{C})$. Assume that the system $Ax = b$ is consistent for every $b \in \mathbb{C}^m$. Then the following statements are equivalent:*

(i) *The system $Ax = b$ is uniquely solvable for every $b \in \mathbb{C}^m$.*

(ii) *A has full column rank.*

(iii) $A^\dagger A = I_n$.

2.3 A Block Matrix Technique

Recall the following fact; its proof is provided for benefits of audiences.

Lemma 7. *Let $A,B,C,D \in M_n(\mathbb{C})$ be such that A is invertible. Then $T \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible if and only if $D - CA^{-1}B$ is invertible.*

Proof. One can observe the following matrix decomposition:

$$\begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Taking determinants yields

$$\det \begin{bmatrix} I_n & 0 \\ -CA^{-1} & I_n \end{bmatrix} \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix},$$

and hence, $\det(T) = \det(A)\det(D - CA^{-1}B)$. Therefore, T is invertible if and only if $D - CA^{-1}B$ is invertible. □

3. Equivalent Conditions for Solvability and Unique Solvability for Arbitrary *E*, *F*

In this section, we investigate necessary and sufficient conditions for the rectangular system of coupled generalized Sylvester matrix equations (1.9) to be solvable and uniquely solvable for arbitrary E,F. Our main idea is to reduce the main system (1.9) to a simple vector-matrix equation by using Kronecker product, vector operator, and block-matrix algebra. Such vector-matrix equation can be treated by using Moore-Penrose inverses and rank argument.

Theorem 1. *Let $A,C \in M_{m,n}(\mathbb{C})$ and $B,D \in M_{p,q}(\mathbb{C})$. Denote*

$$H = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}, \; P = B^T \otimes A + D^T \otimes C, \; Q = B^T \otimes A - D^T \otimes C.$$

Then the following statements are equivalent:

(i) *The system* (1.9) *is solvable for arbitrary $E,F \in M_{m,q}(\mathbb{C})$.*

(ii) *The vector-matrix equation $Hx = b$ is solvable for arbitrary $b \in \mathbb{C}^{2mq}$.*

(iii) $\operatorname{rank} H = 2mq$.

(iv) $HH^{\dagger} = I_{2mq}$.

(v) *The equation $Px_1 = b_1$ is solvable for arbitrary $b_1 \in \mathbb{C}^{mq}$ and the equation $Qx_2 = b_2$ is solvable for arbitrary $b_2 \in \mathbb{C}^{mq}$.*

(vi) $\operatorname{rank} P = \operatorname{rank} Q = mq$.

(vii) $PP^{\dagger} = QQ^{\dagger} = I_{mq}$.

Proof. Taking the vector operator to (1.9) and then using Lemma 2, we get

$$(B^T \otimes A)\operatorname{Vec} X + (D^T \otimes C)\operatorname{Vec} Y = \operatorname{Vec} E,$$

$$(D^T \otimes C)\operatorname{Vec} X + (B^T \otimes A)\operatorname{Vec} Y = \operatorname{Vec} F.$$

For convenience, let us denote

$$x = \begin{bmatrix} \operatorname{Vec} X \\ \operatorname{Vec} Y \end{bmatrix}, \quad b = \begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix},$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{mq} & -I_{mq} \\ I_{mq} & I_{mq} \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{np} & I_{np} \\ -I_{np} & I_{np} \end{bmatrix}.$$

The system (1.9) is equivalent to the vector-matrix equation $Hx = b$ due to the injectivity of the vector operator. By Lemma 5, the conditions $(ii)-(iv)$ are equivalent. One can decompose H as follows:

$$\begin{aligned} H &= U \begin{bmatrix} B^T \otimes A + D^T \otimes C & 0 \\ 0 & B^T \otimes A - D^T \otimes C \end{bmatrix} V \\ &= U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} V. \end{aligned} \tag{3.1}$$

A direct computation reveals that

$$\begin{aligned} HH^\dagger &= U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} VV^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \\ &= \frac{1}{2} \begin{bmatrix} I_{mq} & -I_{mq} \\ I_{mq} & I_{mq} \end{bmatrix} \begin{bmatrix} PP^\dagger & 0 \\ 0 & QQ^\dagger \end{bmatrix} \begin{bmatrix} I_{mq} & I_{mq} \\ -I_{mq} & I_{mq} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} PP^\dagger + QQ^\dagger & PP^\dagger - QQ^\dagger \\ PP^\dagger - QQ^\dagger & PP^\dagger + QQ^\dagger \end{bmatrix}. \end{aligned}$$

Hence, the condition (iv) is equivalent to

$$PP^\dagger + QQ^\dagger = 2I_{mq} \text{ and } PP^\dagger - QQ^\dagger = 0,$$

which can be reduced to $PP^\dagger = QQ^\dagger = I_{mq}$. Lemma 5 tells us that the conditions $(v)-(vii)$ are equivalent. □

Theorem 2. *Let $A, C \in M_{m,n}(\mathbb{C})$ and $B, D \in M_{p,q}(\mathbb{C})$. Denote the matrices H, P, Q as in Theorem 1. Assume that the system* (1.9) *is solvable for arbitrary $E, F \in M_{m,q}(\mathbb{C})$. Then the following statements are equivalent:*

(i) *The system* (1.9) *is uniquely solvable for arbitrary $E, F \in M_{m,q}(\mathbb{C})$.*

(ii) *The vector-matrix equation $Hx = b$ is uniquely solvable for arbitrary $b \in \mathbb{C}^{2mq}$.*

(iii) $\operatorname{rank} H = 2np$.

(iv) $H^\dagger H = I_{2np}$.

(v) *The equation $Px_1 = b_1$ is uniquely solvable for arbitrary $b_1 \in \mathbb{C}^{mq}$ and the equation $Qx_2 = b_2$ is uniquely solvable for arbitrary $b_2 \in \mathbb{C}^{mq}$.*

(vi) $\operatorname{rank} P = \operatorname{rank} Q = np$.

(vii) $P^\dagger P = Q^\dagger Q = I_{np}$.

Proof. By applying the vector operator, the conditions (i) and (ii) can be seen to be equivalent. The conditions (ii)-(iv) are equivalent by Lemma 6. From the decomposition (3.1), we have

$$H^\dagger H = V^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* U \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} V$$

$$= \frac{1}{2}\begin{bmatrix} I_{np} & -I_{np} \\ I_{np} & I_{np} \end{bmatrix}\begin{bmatrix} P^\dagger P & 0 \\ 0 & Q^\dagger Q \end{bmatrix}\begin{bmatrix} I_{np} & I_{np} \\ -I_{np} & I_{np} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} P^\dagger P + Q^\dagger Q & P^\dagger P - Q^\dagger Q \\ P^\dagger P - Q^\dagger Q & P^\dagger P + Q^\dagger Q \end{bmatrix}.$$

Hence, the condition (iv) is equivalent to

$$P^\dagger P + Q^\dagger Q = 2I_{np} \text{ and } P^\dagger P - Q^\dagger Q = 0,$$

which can be reduced to $P^\dagger P = Q^\dagger Q = I_{np}$. Lemma 6 tells us that the conditions $(v)-(vii)$ are equivalent. □

4. Equivalent Conditions for Solvability and Unique Solvability for Fixed *E*, *F*, and Representation of Solutions

In this section, we investigate existence and uniqueness of solution for the system (1.9) and its interesting special cases. Exact formula of the unique solution is explicitly presented.

Theorem 3. *Let $A, C \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Denote the matrices P, Q as in Theorem* 1. *Consider the coupled generalized Sylvester matrix equations*

$$\begin{aligned} AXB + CYD &= E, \\ CXD + AYB &= F. \end{aligned} \tag{4.1}$$

Then the following statements are equivalent:

(i) *The system* (4.1) *has a solution.*

(ii) $\operatorname{rank} P + \operatorname{rank} Q = \operatorname{rank}\begin{bmatrix} B^T \otimes A & D^T \otimes C & \operatorname{Vec} E \\ D^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{bmatrix}$.

(iii) $\left(2I_{mq} - (PP^\dagger + QQ^\dagger)\right)\operatorname{Vec} E = (PP^\dagger - QQ^\dagger)\operatorname{Vec} F$, *and*
$\left(2I_{mq} - (PP^\dagger + QQ^\dagger)\right)\operatorname{Vec} F = (PP^\dagger - QQ^\dagger)\operatorname{Vec} E$.

In the above case, the general solution is given by

$$\operatorname{Vec} X = \frac{1}{2}\left[(P^\dagger + Q^\dagger)\operatorname{Vec} E + (P^\dagger - Q^\dagger)\operatorname{Vec} F + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_1 - (P^\dagger P - Q^\dagger Q)q_2\right],$$

$$\operatorname{Vec} Y = \frac{1}{2}\left[(P^\dagger - Q^\dagger)\operatorname{Vec} E + (P^\dagger + Q^\dagger)\operatorname{Vec} F - (P^\dagger P - Q^\dagger Q)q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_2\right]$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary.

Proof. Denote the matrix H and the vector b as in Theorem 1. Then the system (4.1) is equivalent to the vector-matrix equation $Hx = b$. From the decomposition (3.1), since U and V are invertible, we have

$$\operatorname{rank} H = \operatorname{rank}\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = \operatorname{rank} P + \operatorname{rank} Q.$$

By Lemma 4, the conditions (i) and (ii) are equivalent. Lemma 4 also implies that the system (4.1) is consistent if and only if $HH^\dagger b = b$. One can compute

$$HH^\dagger b = \frac{1}{2}\begin{bmatrix} PP^\dagger + QQ^\dagger & PP^\dagger - QQ^\dagger \\ PP^\dagger - QQ^\dagger & PP^\dagger + QQ^\dagger \end{bmatrix}\begin{bmatrix} \operatorname{Vec} E \\ \operatorname{Vec} F \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}(PP^\dagger + QQ^\dagger)\mathrm{Vec}E + (PP^\dagger - QQ^\dagger)\mathrm{Vec}F \\ (PP^\dagger - QQ^\dagger)\mathrm{Vec}E + (PP^\dagger + QQ^\dagger)\mathrm{Vec}F\end{bmatrix}.$$

It follows that the conditions (i) and (iii) are equivalent.

If the solution x exists, Lemma 4 ensures that it must be in the form

$$x = H^\dagger b + (I_{2np} - H^\dagger H)q\ ,$$

where $q \in \mathbb{C}^{2np}$ is arbitrary. Setting $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ when $q_1, q_2 \in \mathbb{C}^{np}$, we have

$$\begin{aligned} x &= V^* \begin{bmatrix} P^\dagger & 0 \\ 0 & Q^\dagger \end{bmatrix} U^* \begin{bmatrix} \mathrm{Vec}E \\ \mathrm{Vec}F \end{bmatrix} + \left(\begin{bmatrix} I_{np} & 0 \\ 0 & I_{np} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} P^\dagger P + Q^\dagger Q & P^\dagger P - Q^\dagger Q \\ P^\dagger P - Q^\dagger Q & P^\dagger P + Q^\dagger Q \end{bmatrix} \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}(P^\dagger + Q^\dagger)\mathrm{Vec}E + (P^\dagger - Q^\dagger)\mathrm{Vec}F \\ (P^\dagger - Q^\dagger)\mathrm{Vec}E + (P^\dagger + Q^\dagger)\mathrm{Vec}F\end{bmatrix} + \frac{1}{2}\begin{bmatrix} \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ -(P^\dagger P - Q^\dagger Q) q_1 + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_2 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}(P^\dagger + Q^\dagger)\mathrm{Vec}E + (P^\dagger - Q^\dagger)\mathrm{Vec}F + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_1 - (P^\dagger P - Q^\dagger Q) q_2 \\ (P^\dagger - Q^\dagger)\mathrm{Vec}E + (P^\dagger + Q^\dagger)\mathrm{Vec}F - (P^\dagger P - Q^\dagger Q) q_1 + \left(2I_{np} - (P^\dagger P + Q^\dagger Q)\right) q_2 \end{bmatrix}. \end{aligned}$$

Therefore, the general (vector) solution of system (4.1) is given by

$$\mathrm{Vec}X = \frac{1}{2}[(P^\dagger + Q^\dagger)\mathrm{Vec}E + (P^\dagger - Q^\dagger)\mathrm{Vec}F + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_1 - (P^\dagger P - Q^\dagger Q)q_2],$$

$$\mathrm{Vec}Y = \frac{1}{2}[(P^\dagger - Q^\dagger)\mathrm{Vec}E + (P^\dagger + Q^\dagger)\mathrm{Vec}F - (P^\dagger P - Q^\dagger Q)q_1 + (2I_{np} - (P^\dagger P + Q^\dagger Q))q_2]. \square$$

Theorem 4. *Let $A, B, C, D, E, F \in M_n(\mathbb{C})$. Denote $P = B^T \otimes A + D^T \otimes C$ and $Q = B^T \otimes A - D^T \otimes C$. Then the system* (4.1) *has a unique solution if and only if P and Q are invertible. In this case, the unique solution is given by*

$$\mathrm{Vec}X = \frac{1}{2}[P^{-1}(\mathrm{Vec}E + \mathrm{Vec}F) + Q^{-1}(\mathrm{Vec}E - \mathrm{Vec}F)],$$

$$\mathrm{Vec}Y = \frac{1}{2}[P^{-1}(\mathrm{Vec}E + \mathrm{Vec}F) - Q^{-1}(\mathrm{Vec}E - \mathrm{Vec}F)].$$

Proof. As in the proof Theorem 1, the system (4.1) is equivalent to the vector-matrix equation $Hx = b$. The decomposition (3.1) says that the system (4.1) has a unique solution if and only if both P and Q are invertible. To obtain the unique solution of (4.1), we substitute $P^\dagger = P^{-1}$ and $Q^\dagger = Q^{-1}$ into the general solution of Theorem 3, and we thus obtain

$$\begin{aligned} \mathrm{Vec}X &= \frac{1}{2}[(P^{-1} + Q^{-1})\mathrm{Vec}E + (P^{-1} - Q^{-1})\mathrm{Vec}F + (2I_{n^2} - (P^{-1}P + Q^{-1}Q))q_1 - (P^{-1}P - Q^{-1}Q)q_2] \\ &= \frac{1}{2}[P^{-1}(\mathrm{Vec}E + \mathrm{Vec}F) + Q^{-1}(\mathrm{Vec}E - \mathrm{Vec}F)]. \end{aligned}$$

Similarly, we get the above formula of $\mathrm{Vec}Y$. $\square$

Theorem 5. *Let $A, C \in M_n(\mathbb{C})$, $B, D \in M_p(\mathbb{C})$ and $E, F \in M_{n,p}(\mathbb{C})$ be such that A and B are invertible. Then the system* (4.1) *has a unique solution if and only if $B^T \otimes A - \left((DB^{-1}D)^T \otimes CA^{-1}C\right)$ is invertible.*

Proof. From the proof of Theorem 3, we see that the system (4.1) has a unique solution if and only if the matrix

$$H = \begin{bmatrix} B^T \otimes A & D^T \otimes C \\ D^T \otimes C & B^T \otimes A \end{bmatrix}$$

is invertible. Since A and B are invertible, so is $B^T \otimes A$ by Lemma 1. Then Lemma 7 implies that the invertibility of H is equivalent to that of

$$(B^T \otimes A) - (D^T \otimes C)(B^T \otimes A)^{-1}(D^T \otimes C),$$

which can be reduced to that of $B^T \otimes A - \big((DB^{-1}D)^T \otimes CA^{-1}C\big)$ by Lemma 1. □

Theorem 6. *Let $A \in M_{m,n}(\mathbb{C})$, $B, D \in M_{p,q}(\mathbb{C})$ and $E, F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations*

$$\begin{aligned} AXB + AYD &= E, \\ AXD + AYB &= F. \end{aligned} \tag{4.2}$$

Then the following statements are equivalent:

(i) *The system* (4.2) *has a solution.*

(ii) $(\operatorname{rank} A)(\operatorname{rank}(B+D) + \operatorname{rank}(B-D)) = \operatorname{rank}\begin{bmatrix} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E \\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F \end{bmatrix}$.

(iii) *The following two conditions hold:*

$$AA^\dagger\big[(E+F)(B+D)^\dagger(B+D) + (E-F)(B-D)^\dagger(B-D)\big] = 2E, \tag{4.3}$$

$$AA^\dagger\big[(E+F)(B+D)^\dagger(B+D) - (E-F)(B-D)^\dagger(B-D)\big] = 2F. \tag{4.4}$$

In the above case, the general solution is given by

$$\begin{aligned} X =& Q_1 + \frac{1}{2}\big[A^\dagger\{(E+F)(B+D)^\dagger + (E-F)(B-D)^\dagger\} \\ &- A^\dagger A\{(Q_1+Q_2)(B+D)(B+D)^\dagger + (Q_1-Q_2)(B-D)(B-D)^\dagger\}\big], \end{aligned} \tag{4.5}$$

$$\begin{aligned} Y =& Q_2 + \frac{1}{2}\big[A^\dagger\{(E+F)(B+D)^\dagger - (E-F)(B-D)^\dagger\} \\ &- A^\dagger A\{(Q_1+Q_2)(B+D)(B+D)^\dagger - (Q_1-Q_2)(B-D)(B-D)^\dagger\}\big], \end{aligned} \tag{4.6}$$

where $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. Denote $R = (B+D)^T \otimes A$ and $S = (B-D)^T \otimes A$. In the viewpoint of Theorem 3 when $A = C$, the existence of a solution of the system (4.2) is equivalent to any of the following conditions:

(ii)′ $\operatorname{rank} R + \operatorname{rank} S = \operatorname{rank}\begin{bmatrix} B^T \otimes A & D^T \otimes A & \operatorname{Vec} E \\ D^T \otimes A & B^T \otimes A & \operatorname{Vec} F \end{bmatrix}$,

(iii)′ The following two conditions hold:

$$\big(2I_{mq} - (RR^\dagger + SS^\dagger)\big)\operatorname{Vec} E = (RR^\dagger - SS^\dagger)\operatorname{Vec} F, \tag{4.7}$$

$$\big(2I_{mq} - (RR^\dagger + SS^\dagger)\big)\operatorname{Vec} F = (RR^\dagger - SS^\dagger)\operatorname{Vec} E. \tag{4.8}$$

By Lemma 1, we have

$$\operatorname{rank} R = (\operatorname{rank} A)(\operatorname{rank}(B+D)), \quad \operatorname{rank} S = (\operatorname{rank} A)(\operatorname{rank}(B-D)).$$

Hence, the condition (ii)′ becomes the condition (ii). Now, we shall show that the equation (4.7) is reduced to (4.3). Indeed, by Lemma 3, we have

$$R^\dagger = ((B+D)^\dagger)^T \otimes A^\dagger, \quad S^\dagger = ((B-D)^\dagger)^T \otimes A^\dagger.$$

It follows that

$$RR^\dagger = \left((B+D)^\dagger(B+D)\right)^T \otimes AA^\dagger,$$
$$SS^\dagger = \left((B-D)^\dagger(B-D)\right)^T \otimes AA^\dagger.$$

Hence,

$$(RR^\dagger + SS^\dagger)\operatorname{Vec}E = \operatorname{Vec}\left[AA^\dagger E\{(B+D)^\dagger(B+D)+(B-D)^\dagger(B-D)\}\right],$$
$$(RR^\dagger - SS^\dagger)\operatorname{Vec}F = \operatorname{Vec}\left[AA^\dagger F\{(B+D)^\dagger(B+D)-(B-D)^\dagger(B-D)\}\right].$$

Now, the equation (4.7) becomes

$$2E - AA^\dagger E\left[(B+D)^\dagger(B+D)+(B-D)^\dagger(B-D)\right] = AA^\dagger F\left[(B+D)^\dagger(B+D)-(B-D)^\dagger(B-D)\right].$$

which is equivalent to (4.3). Similarly, (4.8) and (4.4) are equivalent.

To obtain a formula of the general solution, note that Theorem 3, Lemma 1 and Lemma 3 together imply that

$$\operatorname{Vec}X = \frac{1}{2}\left[K_1\operatorname{Vec}E + K_2\operatorname{Vec}F + K_3q_1 + K_4q_2\right],$$
$$\operatorname{Vec}Y = \frac{1}{2}\left[K_2\operatorname{Vec}E + K_1\operatorname{Vec}F + K_4q_1 + K_3q_2\right],$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary, and

$$K_1 = \left((B+D)^\dagger + (B-D)^\dagger\right)^T \otimes A^\dagger,$$
$$K_2 = \left((B+D)^\dagger - (B-D)^\dagger\right)^T \otimes A^\dagger,$$
$$K_3 = 2I_{np} - \left[\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A\right],$$
$$K_4 = -\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)^T \otimes A^\dagger A.$$

The bijectivity of Vec implies that $q_1 = \operatorname{Vec}Q_1$ and $q_2 = \operatorname{Vec}Q_2$ for some unique $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$, respectively. Then by Lemmas 2 and 3, we have

$$K_1\operatorname{Vec}E = \operatorname{Vec}\left[A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right)\right],$$
$$K_2\operatorname{Vec}F = \operatorname{Vec}\left[A^\dagger F\left((B+D)^\dagger - (B-D)^\dagger\right)\right],$$
$$K_3q_1 = \operatorname{Vec}\left[2Q_1 - A^\dagger AQ_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right)\right],$$
$$K_4q_2 = \operatorname{Vec}\left[-A^\dagger AQ_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\right].$$

Since the vector operator is linear and bijective, we obtain

$$\begin{aligned} X = \frac{1}{2}\Big[&A^\dagger E\left((B+D)^\dagger + (B-D)^\dagger\right) + A^\dagger F\left(B+D)^\dagger - (B-D)^\dagger\right) \\ &+ 2Q_1 - A^\dagger AQ_1\left((B+D)(B+D)^\dagger + (B-D)(B-D)^\dagger\right) \\ &- A^\dagger AQ_2\left((B+D)(B+D)^\dagger - (B-D)(B-D)^\dagger\right)\Big], \end{aligned}$$

which can be reformed to the desired formula (4.5). Similarly, we get the formula of Y as (4.6). □

Corollary 1. *Let $A,B,D,E,F \in M_n(\mathbb{C})$. The system* (4.2) *has a unique solution if and only if $A, B+D$ and $B-D$ are invertible. In this case, the unique solution is given by*

$$X = \frac{1}{2}A^{-1}\left[(E+F)(B+D)^{-1} + (E-F)(B-D)^{-1}\right],$$

$$Y = \frac{1}{2}A^{-1}\left[(E+F)(B+D)^{-1} - (E-F)(B-D)^{-1}\right].$$

Proof. In the viewpoint of Theorem 4 when $A = C$, the system (4.2) has a unique solution if and only if $(B+D)^T \otimes A$ and $(B-D)^T \otimes A$ are invertible. By Lemma 1, this condition is equivalent to the invertibility of A, $B+D$ and $B-D$. The formula of unique solution can be obtained by substituting the Moore-Penrose inverses of A, $B+D$, $B-D$ with their ordinary inverses in (4.5) and (4.6). □

Corollary 1 was obtained in [13, Theorem 4.11] under the restrict condition $AB = BA$.

Corollary 2. *Let $A \in M_n(\mathbb{C})$, $B,D \in M_p(\mathbb{C})$ and $E,F \in M_{n,p}(\mathbb{C})$ be such that A and B are invertible. Then the system* (4.2) *has a unique solution if and only if $B - DB^{-1}D$ is invertible.*

Proof. The idea of proof is similar to the proof of Theorem 5. □

Theorem 7. *Let $A,C \in M_{m,n}(\mathbb{C})$, $B \in M_{p,q}(\mathbb{C})$ and $E,F \in M_{m,q}(\mathbb{C})$. Consider the following coupled linear matrix equations*

$$\begin{aligned} AXB + CYB &= E, \\ CXB + AYB &= F. \end{aligned} \tag{4.9}$$

Then the following statements are equivalent:

(i) *The system* (4.9) *has a solution.*

(ii) $(\operatorname{rank} B)(\operatorname{rank}(A+C) + \operatorname{rank}(A-C)) = \operatorname{rank}\begin{bmatrix} B^T \otimes A & B^T \otimes C & \operatorname{Vec} E \\ B^T \otimes C & B^T \otimes A & \operatorname{Vec} F \end{bmatrix}$.

(iii) *The following two conditions hold:*

$$\left[(A+C)(A+C)^\dagger(E+F) + (A-C)(A-C)^\dagger(E+F)\right]B^\dagger B = 2E, \tag{4.10}$$

$$\left[(A+C)(A+C)^\dagger(E+F) - (A-C)(A-C)^\dagger(E+F)\right]B^\dagger B = 2F. \tag{4.11}$$

In the above case, the general solution is given by

$$\begin{aligned} X = Q_1 + \frac{1}{2}\Big[&\{(A+C)^\dagger(E+F) + (A-C)^\dagger(E-F)\}B^\dagger \\ &- \{(A+C)^\dagger(A+C)(Q_1+Q_2) + (A-C)^\dagger(A-C)(Q_1-Q_2)\}BB^\dagger\Big], \end{aligned} \tag{4.12}$$

$$\begin{aligned} Y = Q_2 + \frac{1}{2}\Big[&\{(A+C)^\dagger(E+F) - (A-C)^\dagger(E-F)\}B^\dagger \\ &- \{(A+C)^\dagger(A+C)(Q_1+Q_2) - (A-C)^\dagger(A-C)(Q_1-Q_2)\}BB^\dagger\Big], \end{aligned} \tag{4.13}$$

where $Q_1, Q_2 \in M_{n,p}(\mathbb{C})$ are arbitrary.

Proof. The idea of proof is similar to that of Theorem 6. In this case, the general solution is given by

$$\operatorname{Vec} X = \frac{1}{2}\left[J_1 \operatorname{Vec} E + J_2 \operatorname{Vec} F + J_3 q_1 + J_4 q_2\right],$$

$$\operatorname{Vec} Y = \frac{1}{2}\left[J_2 \operatorname{Vec} E + J_1 \operatorname{Vec} F + J_4 q_1 + J_3 q_2\right],$$

where $q_1, q_2 \in \mathbb{C}^{np}$ are arbitrary, and

$$J_1 = (B^\dagger)^T \otimes \{(A+C)^\dagger + (A-C)^\dagger\},$$

$$J_2 = (B^\dagger)^T \otimes \{(A+C)^\dagger - (A-C)^\dagger\},$$

$$J_3 = 2I_{np} - \left[(BB^\dagger)^T \otimes \{(A+C)^\dagger(A+C) + (A-C)^\dagger(A-C)\}\right],$$

$$J_4 = -(BB^\dagger)^T \otimes \{(A+C)^\dagger(A+C) - (A-C)^\dagger(A-C)\}.$$

We arrive at (4.12) and (4.13) by using properties of the vector operator. □

Corollary 3. *Let $A,B,C,E,F \in M_n(\mathbb{C})$. The system* (4.9) *has a unique solution if and only if $B, A+C$ and $A-C$ are invertible. In this case, the unique solution is given by*

$$X = \frac{1}{2}\left[(A+C)^{-1}(E+F) + (A-C)^{-1}(E-F)\right]B^{-1},$$

$$Y = \frac{1}{2}\left[(A+C)^{-1}(E+F) - (A-C)^{-1}(E-F)\right]B^{-1}.$$

Proof. The criterion for uniqueness of solution follows from Theorem 4 by setting $B = D$. The formula of the solutions X and Y can be derived from (4.12) and (4.13) by substituting the Moore-Penrose inverses of $B, A+C, A-C$ with their ordinary inverses. □

Corollary 3, under the restrict condition $AB = BA$, was obtained in [13, Theorem 4.10].

Corollary 4. *Let $A,C \in M_n(\mathbb{C})$, $B \in M_p(\mathbb{C})$ and $E,F \in M_{n,p}(\mathbb{C})$. If A and B are invertible, then the system* (4.9) *has a unique solution if and only if $A - CA^{-1}C$ is invertible.*

Proof. The proof is similar to that of Theorem 5. □

5. Conclusions

We investigate the following system of coupled generalized Sylvester matrix equations:

$$AXB + CYD = E,$$

$$CXD + AYB = F$$

where A, B, C, D, E, F are rectangular complex matrices and X, Y are unknown complex matrices. Equivalent conditions for solvability and unique solvability of the system that hold for arbitrary E, F and for given E, F are obtained in terms of Kronecker products, the vector operator, Moore-Penrose inverses, and ranks. Explicit formulas of solutions are also presented. Moreover, we discuss several special cases of this system.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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