Communications in Mathematics and Applications
Vol. 8, No. 1, pp. 69–86, 2017
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com

Research Article

Δ-Convergence and Uniform Distribution in Lacunary Sense

B. Aris and M. Küçükaslan*

Department of Mathematic, Mersin University, 33343 Mersin, Turkey
***Corresponding author:** mkucukaslan@mersin.edu.tr

Abstract. In this paper, by considering usual partition of $[0,\infty)$ Δ-convergence of non-negative real valued sequences is defined. It is shown that every convergent sequence is Δ-convergence but the converse is not true, in general. Besides, some basic properties of Δ-convergence as well as the second part of this paper by using any lacunary sequences as a partition of non-negative real numbers, lacunary uniform distribution is defined and some inclusion results between uniform distribution modulo 1 and lacunary uniform distribution has been given.

Keywords. Convergence of sequence; Statistically convergence; Uniformly distribution of sequence; Lacunary convergence

MSC. 40A05; 11K06; 11K31

Received: February 1, 2016 **Accepted:** June 11, 2017

1. Introduction

In 1916, formal definition of uniform distribution with modulo 1 (u.d. mod 1) was given by H. Weyl in ([15], [16]).

Let $\widetilde{x}=(x_n)$ be a sequence of non-negative real numbers and (a_n) be congruence (mod 1) of $\widetilde{x}$. If

$$\lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^{N} f(a_n)=\int_0^1 f(x)d(x)$$

holds, for every real valued function f defined on $[0,1]$, then $\widetilde{x}=(x_n)$ is called (u.d. mod 1).

Also, H. Weyl gave a theorem that known as Weyl Criteria: let m be a non-zero integer and $f(x) = e^{2\pi ix}$ be a function defined on $[0,1]$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{h=1}^{N} e(ma_n) = \int_0^1 e(mx)d(x) = 0,$$

where $e(ma_n) = e^{2\pi ima_n}$.

In 1953 as a generalization of (u.d. mod 1) the notation of uniformly distributed mod Δ was introduced by W.J. Le Veque in [13].

Let Δ be a subdivision of the interval $[0,\infty)$ such that $\Delta := \{z_0, z_1, \ldots\}$ where $0 = z_0 < z_1 < \ldots < z_n < \ldots$ and $\lim_{n\to\infty} z_n = \infty$.

For $z_{n-1} \le x < z_n$, integer part and fractional part of $x \in [0,\infty)$ are defined respectively as

$$[x]_\Delta := z_{n-1}, \quad \{x\}_\Delta := \frac{x-[x]_\Delta}{\delta(x)},$$

where $\delta(x) := z_n - z_{n-1}$. It is clear that $0 \le \{x\}_\Delta < 1$.

If $z_k = k$ for all $k \in \mathbb{N}$, then Δ is called the natural subdivision of $[0,\infty)$ and it is denoted by $\Delta^0 = \{1,2,3,\ldots\}$ and Δ-integer and Δ-fractional part reduces to $[x_n]_\Delta = [x_n]$ and $\{x_n\}_\Delta = \{x_n\}$ (see more [12]).

Let $\tilde{x} = (x_k)$ be a sequence of non-negative real numbers. If the sequence $\tilde{x}_\Delta = (\{x_k\}_\Delta)$ is uniformly distributed in $[0,1]$, then the sequence $\tilde{x} = (x_k)$ is said to be uniformly distributed modulo Δ (abbreviated u.d. mod Δ).

In 1960, the concept (u.d. mod Δ) was studied by J. Cigler, and given some results about (u.d. mod Δ) in [3].

Let $T = (t_{nk})$ be a positive Toeplitz matrix and $\lim_{n\to\infty}\sum_{k=1}^{\infty} t_{nk} = 1$ satisfied.

For a non-negative real valued sequence $\tilde{x} = (x_k)$ and characteristic function $\varphi_\alpha(\tilde{x})$ of $[0,\alpha)$ with $0 \le \alpha \le 1$,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} t_{nk}\varphi_\alpha(\{x_k\}) = z(\alpha)$$

holds. Where $z(\alpha) = \alpha$ and $z(0) = 0$, $z(1) = 1$.

If f is a continuous function defined on $[0,1]$, then

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} t_{nk} f(x_k) = \int_0^1 f(x)dz(x)$$

satisfied. In the same paper, J. Cigler also gave analogue of Weyl Criteria for a positive Toeplitz matrix. For q is a non-zero integers and if $f(x) = e^{2\pi iqx}$, then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} t_{nk} e^{2\pi iqx_k} = \int_0^1 e^{2\pi iqx}dz(x).$$

In literature, this notion was studied by H. Davenport and W.J. Le Veque [8], P. Erdös and H. Davenport [7], W.M. Schmidt [14], and R.E. Burkard [1, 2].

A detailed survey of the results on (u.d. mod 1) prior to 1936 can be found by J.F. Koksma in [11]. The period from 1936 to 1961 is covered in the survey article of J. Cigler and G. Helmberg [4]. Last four decades, there is no significant result about this notation. In this paper, we want again to draw attention to researchers this subject.

In section 2, Δ-convergence is defined. Especially, it is shown that every convergence sequence is Δ^0-convergence but the converse is not true. (Theorem 2.5, Remark 2.6).

The relation between (u.d. mod 1) and usual convergence of sequence will be given in section 4 with (Theorem 3.1). It is also shown that if dense subsequence of (x_n) is (u.d. mod 1), then $\tilde{x} = (x_n)$ is (u.d. mod 1) (viceversa) (Theorem 3.3).

In the last section of this paper, we consider a lacunary sequence of positive integers and give many inclusion results.

2. Δ-Convergence of Non-negative Sequences

Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers and $\Delta := \{0 = z_0 < z_1 < z_2 < \dots\}$ be a subdivision of the interval $[0,\infty)$. For all $n \in \mathbb{N}$, there exists a unique $n \in \mathbb{N}$ such that $z_{k-1} \le x_n < z_k$ holds. Δ-integer and Δ-fractional part of $\tilde{x} = (x_n)$ are

$$[x_n]_\Delta := z_{k-1}; \quad \{x_n\}_\Delta := \frac{x_n - z_{k-1}}{z_k - z_{k-1}},$$

respectively, so that $0 \le \{x_n\}_\Delta < 1$ holds for all $n \in \mathbb{N}$.

Let's say that during the study we consider usual partition of $[0,\infty)$ and we will use notations $[x_n]_\Delta = [x_n]$ and $\{x_n\}_\Delta = \{x_n\}$ instead of $[x_n]_{\Delta^0}$ and $\{x_n\}_{\Delta^0}$ for integer and fractional part.

Definition 2.1. A sequence $\tilde{x} = (x_n)$ of non-negative real numbers is said to be Δ-convergent if $\{x_n\}$ is convergent (in usual sense).

Theorem 2.2. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers and $l \in \mathbb{R}$. If (x_n) is monotone and convergent to l, then there exists an $n_0 \in \mathbb{N}$ such that $[x_n] = [l]$ for all $n \ge n_0$.*

Proof. Assume that for every $n \in \mathbb{N}$ there exists $k_n > n$ such that $[x_{k_n}] \ne [l]$. That is; for $n = 1$ there exists $k_1 > 1$ such that $[x_{k_1}] \ne [l]$. For k_1, there exists $k_2 > k_1$ such that $[x_{k_2}] \ne [l]$. If we continue this consecutively we find at least one monotone increasing sequence (k_n) such that $[x_{k_{n+1}}] \ne [l]$. Then, we have a subsequence $([x_{k_n}])$ of $([x_n])$ such that $[x_{k_n}] \ne [l]$ holds for all $n \in \mathbb{N}$. Thus, $[x_{k_n}]$ is not convergent to l. This is a contradiction to assumption on (x_n). □

Remark 2.3. The converse of Theorem 2.2 is not true, in general.

Let us consider $\tilde{x} = (x_n)$ as

$$x_n := \begin{cases} n_0 + t_1, & \text{if } n \text{ is even} \\ n_0 + t_2, & \text{if } n \text{ is odd.} \end{cases}$$

for $0 < t_1, t_2 < 1$. It is clear that (x_n) is not convergence. But $[x_n] = n_0$ for all $n \in \mathbb{N}$.

Remark 2.4. Monotonicity is not omitted in Theorem 2.2.

Let us consider $\tilde{x} = (x_n)$ as

$$x_n := \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ is even} \\ 1 + \frac{1}{n}, & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that the sequence (x_n) is convergent to 1. But Δ integer part of $\tilde{x} = (x_n)$ is

$$[x_n] := \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 2.5. *Let $\tilde{x} = (x_n)$ be a sequence of non negative real numbers. If $\tilde{x} = (x_n)$ is convergent, then $\tilde{x} = (x_n)$ is Δ-convergent.*

Proof. Assume that there exists $l \in \mathbb{R}$ such that $\tilde{x} = (x_n)$ is convergent to l. From Theorem 2.2, we know that there exists an $n_0 \in \mathbb{N}$ such that $[x_n] = [l]$ holds for all $n \geq n_0$. Therefore, we have

$$|\{x_n\} - \{l\}| = |(x_n - [x_n]) - (l - [l])| = |(x_n - l) - ([x_n] - [l])| \leq |x_n - l| + |[x_n] - [l]|.$$

So, if we take limit as $n \to \infty$, the right side of the above inequality tends to zero. Hence, we obtain desired result. □

Remark 2.6. The converse of Theorem 2.5 is not true, in general.

Let us consider $\tilde{x} = (x_n)$ as

$$x_n := \begin{cases} n + \frac{1}{n}, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

The sequence (x_n) is not convergent. It is clear that Δ-fractional part of $\tilde{x} = (x_n)$

$$\{x_n\} := \begin{cases} \frac{1}{n}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

is convergent to zero.

Corollary 2.7. *Let $\tilde{x} = (x_n)$ be a real valued sequence. If $[x_n]$ convergent to l_1 and $\{x_n\}$ convergent to l_2, then $\tilde{x} = (x_n)$ convergent to $l_1 + l_2$.*

Definition 2.8. Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence. It is said to be mutually constant sequence if there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, then x_n is a constant.

Theorem 2.9. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence such that $([x_n])_{n\in\mathbb{N}}$ be a mutually constant. If $(\{x_n\})_{n\in\mathbb{N}}$ is monotone, then the sequence (x_n) is convergent.*

Proof. Since $0 \leq \{x_n\} < 1$, then monotonicity of $(\{x_n\})$ is enough to be convergent. Hence, Corollary 2.7 gives the proof. □

Remark 2.10. The converse of Theorem 2.9 is not true, in general.

Let $t_0 \in (0,1)$ be an arbitrary point and consider monotone increasing (α_n) and monotone decreasing (β_n) such that $0 < \alpha_n < t_0 < \beta_n < 1$ holds for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = t_0$. Hence, let us consider $\tilde{x} = (x_n)$ as:

$$x_n := \begin{cases} n_0 + \alpha_n, & \text{if } n \text{ is even,} \\ n_0 + \beta_n, & \text{if } n \text{ is odd.} \end{cases}$$

It is clear that (x_n) convergent to $n_0 + t_0$ but, the fractional part of (x_n)

$$\{x_n\} := \begin{cases} \alpha_n, & \text{if } n \text{ is even,} \\ \beta_n, & \text{if } n \text{ is odd} \end{cases}$$

is not monotone sequence.

Definition 2.11. A real number $t \in [0,1)$ is said to be a Δ-accumulation point of $\tilde{x} = (x_n)$ if there exists a subsequence $(\{x_{n_k}\})_{k\in\mathbb{N}}$ of $(\{x_n\})_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} \{x_{n_k}\} = t$.

The set of Δ-accumulation points of $\tilde{x}$ is shown with $Y_{\tilde{x}}^{\Delta}$. Also, $Y_{\tilde{x}}$ is the set of accumulation points of $\tilde{x}$.

Theorem 2.12. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers. If $t \in Y_{\tilde{x}}$, then $\{t\} \in Y_{\tilde{x}}^{\Delta}$.*

Proof. Let $t \in Y_{\tilde{x}}$. There exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is convergent to t. Then, by the Theorem 2.5, $(\{x_{n_k}\})$ is convergent to $\{t\}$. Thus, the proof is obtained. □

Remark 2.13. The converse of Theorem 2.12 is not true, in general.

Let $(x_n) = \left(n + \frac{10}{3}\right)$. It is clear that $\left\{\frac{10}{3}\right\} \notin Y_{\tilde{x}}$. Because of $(\{x_n\}) = \left(\left\{\frac{10}{3}\right\}\right)$, every subsequence $(\{x_{n_k}\}) \subset (\{x_n\})$ is convergent to $\left\{\frac{10}{3}\right\}$. So, $\left\{\frac{10}{3}\right\} \in Y_{\tilde{x}}^{\Delta}$.

Corollary 2.14. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers. Then, $Y_{\tilde{x}}^{\Delta} \neq \emptyset$.*

Proof. From Bolzano-Weierstrass Theorem, we know that every bounded sequence has a limit point. Then, since $0 \leq \{x_n\} < 1$, then $Y_{\tilde{x}}^{\Delta} \neq \emptyset$. □

Definition 2.15. A sequence $\tilde{x} = (x_n)$ of non-negative real numbers is said to be Δ-Cauchy sequence if $(\{x_n\})$ is Cauchy sequence.

Theorem 2.16. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence. Then, $\tilde{x} = (x_n)$ is Δ-convergent if and only if $\tilde{x} = (x_n)$ is Δ-Cauchy sequence.*

Proof. Let $\tilde{x} = (x_n)$ be Δ-convergent. Then, $\{x_n\}$ is convergent to $\{\alpha\}$, $\alpha \in \mathbb{R}$. Therefore, we have

$$|\{x_n\} - \{x_m\}| = |\{x_n\} - \{\alpha\} + \{\alpha\} - \{x_m\}| \leq |\{x_n\} - \{\alpha\}| + |\{x_m\} - \{\alpha\}|$$

Since, $\{x_n\}$ is Cauchy sequence, then $\tilde{x} = (x_n)$ is Δ-Cauchy sequence. On the other hand, let $\tilde{x} = (x_n)$ be Δ-Cauchy sequence. Since every closed subset of complete metric space $\mathbb{R}$ is complete, then the sequence $(\{x_n\})_{n\in\mathbb{N}}$ is convergent and $\tilde{x} = (x_n)$ is Δ-convergent. □

Corollary 2.17. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence. If $\tilde{x} = (x_n)$ is convergent, then $\tilde{x} = (x_n)$ is Δ-Cauchy sequence.*

Proof. By the Theorem 2.5, since $\tilde{x} = (x_n)$ is convergent, then $\tilde{x} = (x_n)$ is Δ-convergent. Then, by the Theorem 2.16, $\tilde{x} = (x_n)$ is Δ-Cauchy sequence. □

Theorem 2.18. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence such that $[x_n]$ be mutually constant. Then, the sequence $(\{x_n\})$ is a Cauchy sequence if and only if the sequence $\tilde{x}$ is a Cauchy sequence.*

Proof. Let $(\{x_n\})$ be a Cauchy sequence. Then, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $|\{x_n\} - \{x_m\}| < \varepsilon$ holds for all $n, m \geq n_0$. Also, there exists $n_1 \in \mathbb{N}$ such that $[x_n] = [x_{n_1}]$ holds for all $n \geq n_1$. Let us choose $n^* = \max\{n_0, n_1\} \in \mathbb{N}$. So, we have following inequality for $n, m \geq n^*$

$$\begin{aligned} |x_n - x_m| &= |([x_n] + \{x_n\}) - ([x_m] + \{x_m\})| \\ &\leq |\{x_n\} - \{x_m\}| + |[x_n] - [x_m]| = |\{x_n\} - \{x_m\}|. \end{aligned}$$

Last inequality gives that (x_n) is a Cauchy sequence. For sufficiency of the proof, since (x_n) is a Cauchy sequence and $[x_n]$ is mutually constant, following inequality

$$|\{x_n\} - \{x_m\}| = |(x_n - [x_n]) - (x_m - [x_m])| \leq |x_n - x_m| + |[x_n] - [x_m]|$$

holds. This completed the proof. □

In following theorem a sufficient (not necessary) condition will be given for a sequence to be Δ-Cauchy.

Theorem 2.19. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers. If $\tilde{x}$ is Δ-Cauchy sequence, then $\lim_{n\to\infty} |\{x_{n+1}\} - \{x_n\}| = 0$ holds.*

Proof. Let $\tilde{x}$ be Δ-Cauchy sequence. We know that for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $|\{x_m\} - \{x_n\}| < \varepsilon$ holds for all $n, m \geq n_0$. If we take $m > n, m = n+1$, $\lim_{n\to\infty} |\{x_{n+1}\} - \{x_n\}| = 0$. Therefore, the proof is completed. □

Remark 2.20. The converse of Theorem 2.19 is not true, in general.

Let $(x_n) = \left(n + \frac{1}{n}\right)_n$. Because of the sequence (x_n) is not a Cauchy sequence in $\mathbb{Q}$, from the previous theorem, (x_n) is not Δ-Cauchy sequence. On the other hand, since for every $n \in \mathbb{N}$, $[x_n] = 2$ and $x_{n+1} - x_n = [x_{n+1}] + \{x_{n+1}\} - [x_n] - \{x_n\}$, then

$$\lim_{n\to\infty} \left| \left\{ \left(1 + \frac{1}{n+1}\right)^{n+1} \right\} - \left\{ (1 + \frac{1}{n})^n \right\} \right| = 0$$

holds.

3. Uniform Distribution and Some Properties

L. Kuipers [12] gave definition of uniform distribution (modulo 1) of a real valued sequence $\tilde{x} = (x_n)$.

Recall that the sequence $\tilde{x}$ is said to be uniformly distributed (modulo 1) if for every subinterval $[a,b)$ of $[0,1)$ we have

$$\lim_{N\to\infty} \frac{A([a,b),N,\{x_n\})}{N} = b-a, \tag{3.1}$$

where $A([a,b),N,\{x_n\})$ is a counting function by defined as the number of terms x_n, $1 \le n \le N$, for which $\{x_n\} \in [a,b)$.

This equality is equivalent to

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) = b-a. \tag{3.2}$$

Theorem 3.1. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence. If $\tilde{x} = (x_n)$ is (u.d. mod 1), then $\tilde{x} = (x_n)$ is not convergent.*

Proof. Assume that the sequence $\tilde{x} = (x_n)$ convergent to $L \in \mathbb{R}$. There are two case for L, (i) $L \in \mathbb{Z}$ or (ii) $L \notin \mathbb{Z}$.

If $L \in \mathbb{Z}$, then because of Theorem 2.5, the sequence $(\{x_n\})_{n\in\mathbb{N}}$ convergent to zero. In the case, for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\{x_n\} < \varepsilon$ holds for all $n \ge n_0$. Especially, if we take $0 < \varepsilon < a < b < 1$ and $[a,b) \subset [0,1)$, then we have

$$\lim_{N\to\infty} \frac{A([a,b),N,\{x_n\})}{N} \le \lim_{N\to\infty} \frac{n_0}{N} = 0.$$

This is a contradiction that

$$\lim_{N\to\infty} \frac{A([a,b),N,\{x_n\})}{N} = b-a.$$

Now, $L \notin \mathbb{Z}$, then $L = [L] + l_1$ such that $l_1 \in (0,1)$. From this we have $\{x_n\} \to l_1$ by Theorem 2.5. In the same way, for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|\{x_n\} - l_1| < \varepsilon$ holds for all $n \ge n_0$. We can consider subinterval $[a,b)$ of $[0,1)$ such that $l_1 + \varepsilon < a < b < 1$. Thus, the equality

$$\lim_{N\to\infty} \frac{A([a,b),N,\{x_n\})}{N} = \lim_{N\to\infty} \frac{n_0}{N} = 0$$

holds. This is also a contradiction to assumption on $\tilde{x} = (x_n)$. □

Remark 3.2. The converse of Theorem 3.1 is not true, in general.

Let $\tilde{x} = (x_n) = \left(n + \frac{1}{n}\right)$. It is clear that the sequence $\tilde{x}$ is not convergent. If we consider $[0,\frac{1}{2}) \subset [0,1)$, then

$$\lim_{N\to\infty} \frac{A\left([0,\frac{1}{2}),N,\{x_n\}\right)}{N} = \lim_{N\to\infty} \frac{N-2}{N} = 1 \ne \frac{1}{2}.$$

Hence, $\tilde{x}$ is not uniformly distributed sequence.

For a given sequence $\tilde{x}$, it is written $\{x_n : n \in \mathbb{N}\}$ to denote its range. If (x_{n_k}) is a subsequence of $\tilde{x}$ and $K = \{n_k : k \in \mathbb{N}\}$, we abbreviate (x_{n_k}) by $(\tilde{x})_K$.

The subsequence $(\tilde{x})_K$ is said to be a thin subsequence of $\tilde{x} = (x_n)$ if $\delta(K) = 0$, where

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \le n : k \in K\}|.$$

If $\delta(K) = 1$, then the subsequence $(\tilde{x})_K$ is called dense subsequence of $\tilde{x}$[10].

Theorem 3.3. *Let $\tilde{x} = (x_n)$ be a sequence of non negative real numbers. Then, a dense subsequence of $\tilde{x}$ is (u.d.mod 1) if and only if the sequence $\tilde{x}$ is (u.d.mod 1).*

Proof. Let (x_{n_k}) be an arbitrary dense subsequence of $\tilde{x} = (x_n)$ such that density of $K = \{n_k : k \in \mathbb{N}\}$ is one. Therefore, following equality

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) = \frac{1}{N}\left[\sum_{n_k \in K} \chi_{[a,b)}(\{x_{n_k}\}) + \sum_{n_k \notin K} \chi_{[a,b)}(\{x_{n_k}\})\right]. \tag{3.3}$$

holds for any $[a,b) \subset [0,1)$.

The second sum in the right side of above equality we have

$$\frac{1}{N}\sum_{n_k \notin K} \chi_{[a,b)}(\{x_{n_k}\}) = 0$$

because of $\delta(K^c) = 0$. If we take limit as $N \to \infty$ in (3.3), it is obtained that (x_n) is (u.d. mod 1) if and only if (x_{n_k}) is (u.d. mod 1). □

Definition 3.4. $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ are non-negative real valued sequences. They are called asymptotically equivalent if the set $A = \{n : x_n \neq y_n\}$ has zero asymptotic density. It is denoted by $\tilde{x} \asymp \tilde{y}$.

Definition 3.5. $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ are non-negative real valued sequences. They are called Δ-asymptotically equivalent if the density of the set $A_\Delta = \{n : \{x_n\} \neq \{y_n\}\}$ is zero. It is denoted by $\tilde{x} \overset{\Delta}{\asymp} \tilde{y}$.

Let us consider $\tilde{x} = (x_n)$ as $(x_n) = \left(n + \frac{1}{n}\right)$ and $\tilde{y} = (y_n)$ as

$$y_n := \begin{cases} n, & n \text{ is square,} \\ 2 + \frac{1}{n}, & \text{otherwise.} \end{cases}$$

It is clear that $\delta(\{n : \{x_n\} \neq \{y_n\}\}) = 0$ but $\delta(\{n : x_n \neq y_n\}) \neq 0$. So, sequences $\tilde{x}$ and $\tilde{y}$ are Δ-asymptotically equivalent but not asymptotically equivalent.

Now, let us $\tilde{x} = (x_n)$ as $(x_n) = \left(n - \frac{1}{n}\right)$ and $\tilde{y} = (y_n)$ as

$$y_n := \begin{cases} 1 + \frac{1}{n}, & n \text{ is square} \\ n - \frac{1}{n}, & \text{otherwise.} \end{cases}$$

It clear that $\tilde{x} \asymp \tilde{y}$. But, the sequences $(\{x_n\}) = \left(\frac{n-1}{n}\right)$ and

$$\{y_n\} := \begin{cases} \frac{1}{n}, & \text{n is square} \\ \frac{n-1}{n}, & \text{otherwise.} \end{cases}$$

are not asymptotically equivalent.

Thus, we understood that asymptotically equivalent and Δ-asymptotically equivalent are not comparable.

Theorem 3.6. *Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ be non-negative real valued sequences and $\tilde{x} \asymp \tilde{y}$. Then, (x_n) is (u.d.mod 1) if and only if (y_n) is (u.d. mod 1).*

Proof. For the necessity, let us assume that $\tilde{x} = (x_n)$ is (u.d. mod 1) sequence. By using (3.1), we have

$$\begin{aligned} b-a &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b)}(\{x_n\}) \\ &= \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n\in A\cap N} \chi_{[a,b)}(\{x_n\}) + \frac{1}{N} \sum_{n\notin A\cap N} \chi_{[a,b)}(\{x_n\}) \right) \\ &= 0 + \lim_{N\to\infty} \frac{1}{N} \sum_{n\notin A\cap N} \chi_{[a,b)}(\{y_n\}). \end{aligned}$$

From this equality, we say that $(y_n)_{n\in A^c}$ is (u.d. mod 1) as a dense subsequence of (y_n) because of $\delta(A^c) = 1$. Therefore, we obtain from Theorem 3.3 that (y_n) is (u.d. mod 1) sequence.

The sufficiency part can be proved by using same step. So, it is omitted here. □

Definition 3.7. A sequence $\tilde{x} = (x_n)$ of non-negative real numbers is said to be T-uniformly distributed with modulo 1 (T-u.d. mod 1) if for every pair a, b of real numbers with $0 \le a < b \le 1$ such that

$$\lim_{N\to\infty} \sum_{k=1}^{N} t_{nk} \chi_{A([a,b),n,\tilde{x})}(k) = b-a, \tag{3.4}$$

where $A([a,b),N,\tilde{x}) = |\{k \le N : \{x_k\} \in [a,b)\}|$ and $T = (t_{nk})$.

Let us note that, if we consider (t_{nk}) as

$$t_{nk} := \begin{cases} \frac{1}{n}, & k \le n, \\ 0, & k > n, \end{cases}$$

then (3.4) is coincided with (3.1) given by L. Kuipers [12].

Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers and $T = (t_{nk})$ be a non-negative regular summability matrix. The subsequence $(\tilde{x})_K$ said to be a T-thin subsequence of $\tilde{x} = (x_n)$ if $\delta_T(K) = 0$, where

$$\delta_T(K) := \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} \chi_K(k)$$

If $\delta_T(K) = 1$, then the subsequence $(\tilde{x})_K$ is called T-dense subsequence of $\tilde{x}$[5].

Definition 3.8. $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ are non-negative real valued sequences. They are called T-equivalent if T-density of the set $A = \{n : x_n \neq y_n\}$ is zero. It is denoted by $\tilde{x} \asymp \tilde{y}$ (w.r.t -T).

Theorem 3.9. *Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ be non-negative real valued sequences and $\tilde{x} \asymp \tilde{y}$ (w.r.t-T). Then, $\tilde{x} = (x_n)$ is (T-u.d.mod 1) if and only if $\tilde{y} = (y_n)$ is (T-u.d. mod 1).*

Proof. One can prove like the proof of the Theorem 3.6. Let $\tilde{x}=(x_n)$ be a (T-u.d.mod 1) sequence and assume that $\tilde{y}=(y_n)$ is not (T-u.d. mod 1). By using equation (3.4),

$$\begin{aligned} b-a &= \lim_{n\to\infty}\sum_{k=1}^{n} t_{nk}\chi_{[a,b)}(\{x_k\}) \\ &= \lim_{n\to\infty}\sum_{k=1,k\in A}^{n} t_{nk}\chi_{[a,b)}(\{x_k\}) + \lim_{n\to\infty}\sum_{k=1,k\notin A}^{n} t_{nk}\chi_{[a,b)}(\{x_k\}) \\ &= 0+\lim_{n\to\infty}\sum_{k=1,k\notin A}^{n} t_{nk}\chi_{[a,b)}(\{y_k\}). \end{aligned}$$

This is contradiction. Because, we know that if $\tilde{y}$ is not (u.d. mod 1), then the dense subsequence of $\tilde{y}$ is not (u.d. mod 1). Thus, we proved that the sequence $\tilde{y}=(y_n)$ is (T-u.d. mod 1). In the same way we can show sufficiency of the Theorem 3.9. □

Theorem 3.10. *Let $\tilde{x}=(x_n)$ be a sequence of non-negative real numbers, $T_1=(t^{(1)}_{nk})$ and $T_2=(t^{(2)}_{nk})$ be two regular matrices such that for every $\varepsilon>0$,*

$$\sum_{k=1}^{n}|t^{(1)}_{nk}-t^{(2)}_{nk}|<\varepsilon \tag{3.5}$$

holds. Then the sequence $\tilde{x}=(x_n)$ is (T_1 u.d. mod 1) if and only if the sequence $\tilde{x}=(x_n)$ is (T_2 u.d. mod 1).

Proof. For necessity of the theorem, we assume under the hypothesis $\tilde{x}=(x_n)$ is a (T_1 u.d. mod 1) sequence. Following inequality

$$\begin{aligned} \left|\sum_{k=1}^{n} t^{(2)}_{nk}\chi_{[a,b)}(\{x_k\})-(b-a)\right| &= \left|\sum_{k=1}^{n}(t^{(2)}_{nk}-t^{(1)}_{nk}+t^{(1)}_{nk})\chi_{[a,b)}(\{x_k\})-(b-a)\right| \\ &\le \left|\sum_{k=1}^{n}(t^{(2)}_{nk}-t^{(1)}_{nk})\chi_{[a,b)}(\{x_k\})\right| + \left|\sum_{k=1}^{n}(t^{(1)}_{nk}\chi_{[a,b)}(\{x_k\})-(b-a)\right| \\ &\le \sum_{k=1}^{n}|t^{(2)}_{nk}-t^{(1)}_{nk}| + \left|\sum_{k=1}^{n}(t^{(1)}_{nk}\chi_{[a,b)}(\{x_k\})-(b-a)\right| \end{aligned}$$

holds for an arbitrary subinterval $[a,b)$ of $[0,1)$. Therefore, assumption and (3.5) give the proof. Thus, $\tilde{x}=(x_n)$ is (T_2 u.d. mod 1) sequence.

The sufficiency of the theorem can be proved easily doing suitable changes. □

4. Uniform Distribution in Lacunary Sense

In this section, an arbitrary lacunary sequence considered as a partition of $[0,\infty)$ and some inclusion theorems will be given.

From [9], we know that $\theta:=(k_r)$, $r=1,2,\dots$, is a lacunary sequence of non-negative integers such that it is an increasing sequence such that $k_r-k_{r-1}\to\infty$, where $r\to\infty$.

The intervals determined by θ will be denoted by $I_r:=(k_{r-1},k_r]$ and $h_r:=k_r-k_{r-1}$ is length of the intervals I_r.

Now, let $\tilde{x} = (x_n)$ be a non-negative real valued sequence. For all $n \in \mathbb{N}$ there exists a unique $k \in \mathbb{N}$ such that $k_{r-1} \le x_n < k_r$ holds.

Let us define respectively θ-integer part and θ-fractional part of $\tilde{x} = (x_n)$ as follows:

$$[x]_\theta := k_{r-1} \quad \text{and} \quad \{x\}_\theta := \frac{x - k_{r-1}}{h_r}$$

so that $0 \le \{x\}_\theta < 1$.

Definition 4.1. A sequence $\tilde{x} = (x_n)$ of non-negative real numbers is said to be lacunary uniformly distributed modulo 1 (u.d. mod θ) if the sequence $(\{x_n\}_\theta)$ is uniformly distributed in $[0,1]$.

Lemma 4.2. *Let $\tilde{x} = (x_n)$ be non-negative real valued sequence. Then, the sequence $\tilde{x}$ is (u.d. mod 1) if and only if for every $\alpha \in [0,1)$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,\alpha)}(\{x_k\}) = \alpha$.*

Proof. Let $\tilde{x}$ be a (u.d. mod 1) sequence. By the equation (3.1), the following equality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,\alpha)}(\{x_k\}) = \alpha$$

satisfying for $a = 0$ and $b = \alpha$.

On the other hand, let the equation $\alpha \in [0,1]$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,\alpha)}(\{x_k\}) = \alpha$ holds for every $\alpha \in [0,1]$. Let $a, b \in [0,1)$ such that $b > a$. By the hypothesis, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,a)}(\{x_k\}) = a$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,b)}(\{x_k\}) = b.$$

By using the last two equations,

$$\begin{aligned} b - a &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\chi_{[0,b)} - \chi_{[0,a)})(\{x_k\}) \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b)}(\{x_k\}). \end{aligned}$$

It means that $\tilde{x}$ is (u.d. mod 1). □

Theorem 4.3. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers and $\theta = (k_r)_{r\in\mathbb{N}}$ be a lacunary sequence. If $\tilde{x} = (x_n)$ is (u.d. mod 1), then $\tilde{x} = (x_n)$ is (u.d. mod θ).*

Proof. Since (x_n) is (u.d. mod 1), then for an arbitrary $\alpha \in [0,1)$ we have from Lemma 4.2

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[0,\alpha)}(\{x_k\}) = \alpha. \tag{4.1}$$

Also, following equality

$$\frac{1}{h_r} \sum_{k_{r-1}+1}^{k_r} \chi_{(0,\alpha)}(\{x_k\}_\theta) = \frac{1}{h_r} \left[\sum_{k=1}^{k_r} \chi_{(0,\alpha)}(\{x_k\}_\theta) - \sum_{k=1}^{k_{r-1}} \chi_{(0,\alpha)}(\{x_k\}_\theta) \right]$$

$$= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{k=1}^{k_r}\chi_{(0,\alpha)}(\{x_k\}_\theta)\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}\chi_{(0,\alpha)}(\{x_k\}_\theta)\right)$$

holds.

On the other hand, from the equation (4.1), for every $\varepsilon > 0$

$$\alpha - \varepsilon < \frac{1}{k_r}\sum_{k=1}^{k_r}\chi_{(0,\alpha)}(\{x_k\}_\theta) < \alpha + \varepsilon \tag{4.2}$$

and

$$\alpha - \varepsilon < \frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}}\chi_{(0,\alpha)}(\{x_k\}_\theta) < \alpha + \varepsilon \tag{4.3}$$

hold.

By using (4.2) and (4.3), the following inequality

$$\frac{k_r}{h_r}\cdot\alpha - \frac{k_r}{h_r}\cdot\varepsilon - \frac{k_{r-1}}{h_r}\cdot\alpha + \frac{k_{r-1}}{h_r}\cdot\varepsilon < \frac{1}{h_r}\sum_{k_{r-1}+1}^{k_r}\chi_{(0,\alpha)}(\{x_k\}_\theta)$$
$$< \frac{k_r}{h_r}\cdot\alpha + \frac{k_r}{h_r}\cdot\varepsilon - \frac{k_{r-1}}{h_r}\cdot\alpha - \frac{k_{r-1}}{h_r}\cdot\varepsilon$$

satisfies. Therefore, for every $\varepsilon > 0$, following inequality

$$\alpha - \varepsilon < \frac{1}{h_r}\sum_{k_{r-1}+1}^{k_r}\chi_{(0,\alpha)}(\{x_k\}_\theta) < \alpha + \varepsilon,$$

holds. Hence, the last inequality gives that

$$\frac{1}{h_r}\sum_{k_{r-1}+1}^{k_r}\chi_{(0,\alpha)}(\{x_k\}_\theta) \to \alpha$$

when $r \to \infty$. So, desired result is obtained. □

Theorem 4.4. *Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ are non-negative real valued sequences and $\tilde{x} \asymp \tilde{y}$. $\tilde{x} = (x_n)$ is (u.d. mod θ) if and only if $\tilde{y} = (y_n)$ is (u.d. mod θ).*

Proof. For the necessity, we suppose that $\tilde{x} = (x_n)$ is (u.d. mod θ). For every $\alpha \in (0,1)$, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\chi_{(0,\alpha)}(\{x_k\}_\theta) = \alpha\,.$$

By hypothesis, we can take the set $B = \{k \in I_r : x_k \neq y_k\}$ with $\delta(B) = 0$. Now, we should show that $\tilde{y} = (y_n)$ (u.d. mod θ). For every $\alpha \in (0,1)$, by using the Theorem 3.3

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\chi_{(0,\alpha)}(\{y_k\}_\theta) = \lim_{r\to\infty}\frac{1}{h_r}\left(\sum_{k\in I_r,k\in B} + \sum_{k\in I_r,k\notin B}\right)\chi_{(0,\alpha)}(\{y_k\}_\theta)$$
$$= \left(\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r,k\in B} + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r,k\notin B}\right)\chi_{(0,\alpha)}(\{y_k\}_\theta)$$
$$= 0 + \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\chi_{(0,\alpha)}(\{x_k\}_\theta) = \alpha\,.$$

Conversely, in the same way we can show sufficiency of the Theorem 4.4. □

Lemma 4.5 ([9]). *If $b_1, b_2, \ldots, b_n$ are positive real numbers and if $a_1, a_2, \ldots, a_n$ are real numbers satisfying*

$$\frac{|a_1 + a_2 + \ldots + a_n|}{b_1 + b_2 + \ldots + b_n} > \varepsilon$$

then

$$\frac{|a_i|}{b_i} > \varepsilon$$

for some $1 \le i \le n$.

Now, let us give definition of lacunary refinement of a lacunary sequence $\theta = (k_r)$ (see [9]).

A lacunary refinement of $\theta = (k_r)$ is a lacunary sequence $\theta' = (k_r^{'})$ satisfying $(k_r) \subseteq (k_r^{'})$.

Theorem 4.6. *Let $\widetilde{x} = (x_n)$ be a non-negative real valued sequence and $\theta = (k_r)$ be a lacunary sequence and assume that θ' be a lacunary refinement of θ. If $\widetilde{x} = (x_n)$ is (u.d. mod θ), then $\widetilde{x} = (x_n)$ is (u.d. mod θ').*

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' such that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,v(r)} = k_r,$$

where $I'_{r,i} := (k'_{r,i-1}, k'_{r,i}]$.

Note that for all r, $v(r) \ge 1$ because of $\{k_r\} \subseteq \{k'_r\}$. Let $\{I_j^*\}_{j=1}^{\infty}$ be the sequence of abutting intervals $\{I'_{r,i}\}$ ordered by increasing right end points [9]. Then, we have for every $\alpha \in [0,1)$

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} \chi_{(0,\alpha)}(\{x_k\}_\theta) = \alpha.$$

So, for every $\varepsilon > 0$,

$$\alpha - \varepsilon < \frac{1}{h_r} \sum_{I_r} \chi_{(0,\alpha)}(\{x_k\}_\theta) < \alpha + \varepsilon$$

holds.

$$\frac{1}{h_j^*} \sum_{I_j^* \subset I_j} \chi_{(0,\alpha)}(\{x_k\}_\theta) = \frac{\left(\sum\limits_{I'_{r,1}} + \sum\limits_{I'_{r,2}} + \ldots + \sum\limits_{I'_{r,v(r)}}\right) \chi_{(0,\alpha)}(\{x_k\}_{\theta'})}{h'_{r,1} + h'_{r,2} + \ldots + h'_{r,v(r)}}. \tag{4.4}$$

In the equality (4.4) for every each $i = 1, 2, \ldots, v(r)$, by using Lemma 4.5,

$$\alpha - \varepsilon < \frac{\sum_{I'_{r,i}} \chi_{(0,\alpha)}(\{x_k\}_{\theta'})}{h'_{r,i}} < \alpha + \varepsilon.$$

As $r \to \infty$, we have

$$\lim_{r\to\infty} \frac{1}{h'_{r,i}} \sum_{I'_{r,i}} \chi_{(0,\alpha)}(\{x_k\}_{\theta'}) = \alpha.$$

So, we obtain desired result. □

Since usual partition of $[0,\infty)$ is a lacunary refinement of any $\theta = \{k_r\}$, then we can give following result:

Corollary 4.7. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative real numbers and $\theta = (k_r)$ be a lacunary sequence. If $\tilde{x} = (x_n)$ is (u.d.mod θ), then $\tilde{x} = (x_n)$ is (u.d. mod 1).*

Theorem 4.8. *Let $\tilde{x} = (x_n)$ be a non-negative real valued sequence and $\theta = (k_r)$ be a lacunary sequence. Assume that $\theta' = (k'_r)$ be a lacunary refinement of θ such that*

$$\lim_{r\to\infty} \frac{k_r - k_{r-1}}{k'_r - k'_{r-1}} = d(d \geq 1). \tag{4.5}$$

holds. If $\tilde{x} = (x_n)$ is (u.d. mod θ'), then $\tilde{x} = (x_n)$ is (u.d. mod θ).

Proof. Suppose each interval I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,v(r)} = k_r,$$

where $I'_{r,i} = (k'_{r,i-1}, k'_{r,i}]$. Because of $\{k_r\} \subseteq \{k'_r\}$, $v(r) \geq 1$ for all r. Let $\{I^*_j\}_{j=1}^{\infty}$ be the sequence of abutting intervals $\{I'_{r,i}\}$ ordered by increasing right end points. Take an arbitrary $\alpha \in (0,1)$, then we have

$$\begin{aligned}\frac{1}{k_r - k_{r-1}} \sum_{k_{r-1}+1}^{k_r} \chi_{(0,\alpha)}(\{x_k\}_{\theta'}) &= \frac{1}{k_r - k_{r-1}} \left(\sum_{k_{r-1}+1}^{k'_{r,1}} + \sum_{k'_{r,1}+1}^{k'_{r,2}} + \ldots + \sum_{k'_{r,v(r)-1}+1}^{k_r} \right) \chi_{(0,\alpha)}(\{x_k\}_{\theta'}) \\
&= \left(\frac{k'_{r,1} - k_{r-1}}{k_r - k_{r-1}} \cdot \frac{1}{k'_{r,1} - k_{r-1}} \sum_{k_{r-1}+1}^{k'_{r,1}} + \frac{k'_{r,2} - k'_{r,1}}{k_r - k_{r-1}} \cdot \frac{1}{k'_{r,2} - k'_{r,1}} \sum_{k'_{r,1}+1}^{k'_{r,2}} + \ldots \right. \\
&\quad \left. + \frac{k_r - k'_{r,v(r)-1}}{k_r - k_{r-1}} \frac{1}{k_r - k'_{r,v(r)-1}} \sum_{k'_{r,v(r)-1}+1}^{k_r} \right) \chi_{(0,\alpha)}(\{x_k\}_{\theta'}).\end{aligned}$$

On the other hand, for $i = 1,2,\ldots,v(r)$, we have

$$\lim_{r\to\infty} \frac{1}{k_r - k_{r-1}} \sum_{k_{r-1}+1}^{k_r} \chi_{(0,\alpha)}(\{x_k\}_{\theta}) = \alpha \cdot v(r) \frac{1}{v(r)} = \alpha$$

because $\tilde{x} = (x_n)$ is (u.d.mod θ'). Thus we obtain that $\tilde{x} = (x_n)$ is (u.d. mod θ). □

Definition 4.9. Let $\theta = (k_r)$ be a lacunary sequence. A sequence $\tilde{x} = (x_n)$ of non-negative real numbers is said to be B-lacunary uniformly distributed modulo 1 (B^θ-u.d. mod 1) if for every the real number α with $0 < \alpha < 1$, we have

$$\lim_{r\to\infty} \sum_{i\in I_r} b^\theta_{r,i} \chi_{A(\alpha,I_r,w)}(i) = \alpha, \tag{4.6}$$

where $A(\alpha,I_r,w) = |\{i \in I_r : \{x_i\}_\theta < \alpha\}|$ and the regular matrix $B^\theta = (b^\theta_{r,i})$ defined by

$$b^\theta_{r,i} := \begin{cases} (k_r - k_{r-1})^{-1}, & \text{for } k_{r-1} < i < k_r \\ 0, & \text{otherwise} \end{cases}$$

(see [9]).

Theorem 4.10. *Let $\tilde{x} = (x_n)$ be a sequence of non-negative numbers and $\theta = (k_r)$ be a lacunary sequence. Assume that $\liminf q_r > 1$. Then, $\tilde{x} = (x_n)$ is (u.d. mod 1) if and only if $\tilde{x} = (x_n)$ is (B^θ-u.d. mod 1).*

Proof. Suppose $\liminf q_r > 1$.

$$\begin{aligned}\sum_{i\in I_r} b^{\theta}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_\theta) &= \frac{1}{h_r}\left(\sum_{i=1}^{k_r}\chi_{(0,\alpha)}(\{x_i\}_\theta) - \sum_{i=1}^{k_{r-1}}\chi_{(0,\alpha)}(\{x_i\}_\theta)\right)\\ &= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}\chi_{(0,\alpha)}(\{x_i\}_\theta)\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}\chi_{(0,\alpha)}(\{x_i\}_\theta)\right)\\ &= a_r u_r + d_r v_r\,,\end{aligned}$$

where $u_r \to \alpha$, $v_r \to \alpha$,(a_r) and (d_r) are bounded sequences satisfying $a_r + d_r = 1$, for $r = 1,2,\ldots$, ($|a_r| \le \frac{F}{F-1}, |d_r| \le \frac{1}{F-1}, F = \inf q_r > 1$) we then observe

$$|a_r u_r + d_r v_r - \alpha| = |a_r(u_r-\alpha) + d_r(v_r-\alpha)| \le |a_r||u_r-\alpha| + |d_r||v_r-\alpha|$$

which converges to 0, and it follows that (x_n) is (B^θ-u.d. mod 1). □

Theorem 4.11. *Let $\widetilde{x} = (x_n)$ be a sequence of non-negative numbers and $\theta = (k_r)$ be a lacunary sequence. Let $\theta' = (k'_r)$ be lacunary refinement of θ and*

$$\lim_{r\to\infty}\frac{k_r - k_{r-1}}{k'_r - k'_{r-1}} = d \quad (d \ge 1). \tag{4.7}$$

If $\widetilde{x} = (x_n)$ is ($B^{\theta'}$ u.d. mod 1), then $\widetilde{x} = (x_n)$ is (B^θ u.d. mod 1).

Proof. Suppose each I_r of θ contains the points θ' so that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,v(r)} = k_r$$

where $I'_{r,i} = (k'_{r,i-1}, k'_{r,i}]$. So, $\{k_r\} \subseteq k'_r$, for all r, $v(r) \ge 1$. For an arbitrary $\alpha \in (0,1)$, by using (4.7), we have

$$\begin{aligned}\sum_{k_{r-1}+1}^{k_r} b^{\theta}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_\theta) &= \frac{1}{k_r-k_{r-1}}\sum_{k_{r-1}+1}^{k_r}\chi_{(0,\alpha)}(\{x_i\}_\theta)\\ &= \frac{1}{k_r-k_{r-1}}\left(\sum_{I'_{r,1}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'}) + \sum_{I'_{r,2}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'}) + \ldots + \sum_{I'_{r,v(r)}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)\\ &= \frac{k'_{r,1}-k_{r-1}}{k_r-k_{r-1}}\left(\frac{1}{k'_{r,1}-k_{r-1}}\sum_{I'_{r,1}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)\\ &\quad + \frac{k'_{r,2}-k'_{r,1}}{k_r-k_{r-1}}\left(\frac{1}{k'_{r,2}-k'_{r,1}}\sum_{I'_{r,2}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right) + \ldots\\ &\quad + \frac{k_r-k'_{r,v(r)-1}}{k_r-k_{r-1}}\left(\frac{1}{k_r-k'_{r,v(r)-1}}\sum_{I'_{r,v(r)}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)\\ &= \frac{k'_{r,1}-k_{r-1}}{k_r-k_{r-1}}\left(\sum_{I'_{r,1}} b^{\theta'}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)\end{aligned}$$

$$+\frac{k'_{r,2}-k'_{r,1}}{k_r-k_{r-1}}\left(\sum_{I'_{r,2}} b^{\theta'}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)+\dots$$

$$+\frac{k_r-k'_{r,v(r)-1}}{k_r-k_{r-1}}\left(\sum_{I'_{r,v(r)}} b^{\theta'}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})\right)$$

For each $i=1,2,\dots,v(r)$ and $r\to\infty$

$$\sum_{k_{r-1}+1}^{k_r} b^{\theta}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta})=v(r)\alpha\frac{1}{v(r)}=\alpha\,.$$

From last equality, we showed that $\widetilde{x}=(x_n)$ is (B^{θ} u.d. mod 1). □

Theorem 4.12. *Let $\widetilde{x}=(x_n)$ be a non-negative real valued sequence and $\theta=(k_r)$ be a lacunary sequence. Let $\theta'=(k'_r)$ be lacunary refinement of θ. If $\widetilde{x}=(x_n)$ is (B^{θ} u.d. mod 1), then $\widetilde{x}=(x_n)$ is ($B^{\theta'}$ u.d. mod 1).*

Proof. Suppose each I_r of θ contains the points θ' so that

$$k_{r-1}<k'_{r,1}<k'_{r,2}<\dots<k'_{r,v(r)}=k_r$$

where $I'_{r,i}=(k'_{r,i-1},k'_{r,i}]$. Let $\{I^*_j\}$ be the sequence of abutting intervals $\{I'_{r,i}\}$ ordered by increasing right end points. Because of $\widetilde{x}=(x_n)$ is (B^{θ} u.d. mod 1), for an arbitrary $\alpha\in[0,1)$ we have

$$\lim_{r\to\infty}\sum_{k_{r-1}+1}^{k_r} b^{\theta}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta})=\alpha.$$

By using Lemma 4.5,

$$\sum_{I^*_j\subset I_j} b^{\theta}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta})=\frac{1}{h_j{}^*}\sum_{I_j{}^*}\chi_{(0,\alpha)}(\{x_i\}_{\theta})=\frac{\left(\sum_{I'_{r,1}}+\sum_{I'_{r,2}}+\dots+\sum_{I'_{r,v(r)}}\right)\chi_{(0,\alpha)}(\{x_i\}_{\theta'})}{h'_{r,1}+h'_{r,2}+\dots+h'_{r,v(r)}}\,.$$

For every $i=1,2,\dots,v(r)$ and $\varepsilon>0$,

$$\alpha-\varepsilon<\frac{\sum_{I'_{r,i}}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})}{h'_{r,i}}<\alpha+\varepsilon\,.$$

We find last inequality. As $r\to\infty$

$$\sum_{I'_{r,i}} b^{\theta'}{}_{r,i}\chi_{(0,\alpha)}(\{x_i\}_{\theta'})=\alpha\,.$$

Hence, $\widetilde{x}=(x_n)$ is ($B^{\theta'}$ u.d. mod 1). □

5. Conclusion

In this paper, Δ-convergence has been defined and then main properties of Δ-convergence has been given by using usual partition of $[0,\infty)$. Mainly, relation between Δ-convergence and convergence has been given. In the second part, by using lacunary sequence, lacunary distribution has been defined and investigated. We hope that the results obtained in this paper can be applied to the concrete problems in analytic number theory. We strongly feel that the results which is obtained in this paper, can be re-write for any real valued sequence.

Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions that extend to the paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R.E. Burkard, *Zue Theorie der Glechverteilung modulo* Δ, Thesis, Univ. of Vienna (1967).

[2] R.E. Burkard, Gleichmasigen Glechverteilung Polynomiwerten modulo Δ, , *Sitzgsber. Österr. Akad. Wiss. Math. Naturw. K1., Abt. II*, **177** (1969), 395 – 406,.

[3] J. Cigler, Asymptotische Verteilun Reeler Zahlen mod 1, *Monatsh. Math.* **64** (1960), 201 – 225.

[4] J. Cigler and G. Helmberg, Neure Entwichlungen der Theorie der Gleichverteilung, *Jber. Deutsch. Math.-Verein.* **64** (1961), 1 – 50.

[5] J. Connor, On statistical limit points and the consistency of statistical convergence, *Jour. of Math. Anal. and Appl.* **197** (1996), 392 – 399.

[6] H. Davenport, Note on irregularities of distribution, *Mathematika* **3** (1956).

[7] H. Davenport and P. Erdös, Note on normal decimals, *Canad. J. Math.* (1952), 58 – 63.

[8] H. Davenport and W.J. Le Veque, Uniform distribution relative to a fixed sequence, *Michigan Math. J.* **10** (1963), 315 – 319.

[9] A.R. Freedman, J. Sember and M. Raphael Some Cesáro type summability spaces, *Proc. London Math. Soc.* **37** (3) (1978), 508 – 520.

[10] J.A. Fridy, On statistical convergence, *Analysis* **5** (1985), 301 – 313.

[11] J.F. Koksma, Diophantische Approximationen, *Erg. Math. Grenzgeb. Band* 4, Heft 4, Julius Springer, Berlin (1936).

[12] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley & Sons, New York (1974).

[13] W.J. Le Veque, On uniform distribution modulo a subdivision, *Pasific J. Math.* **3** (1953), 757 – 771.

[14] W. M. Schmidt, Disproof of some conjectures on diophantine approximations, *Studia Sci. Math. Soc.* **25** (1970), 608 – 614.

[15] H. Weyl, Über ein problem aus dem Geibete der Diophantischen approximationen, *Nachr. Ges. Wiss. Göttingen, Math.-Phys.* (1914), 234 – 244.

[16] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77** (1916), 313 – 352.