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Research Article

Connectivity of Suborbital Graphs for the Congruence Subgroups of the Extended Modular Group

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Abstract. Let $\widehat{\Gamma}$ be the extended modular group acting on $\widehat{\mathbb{Q}}$ by linear fractional transformation. In this paper, we investigate connectivity of the suborbital graphs $\widehat{\mathcal{F}}_{u,n}$ for the congruence subgroup $\widehat{\Gamma}_0(n)$ of $\widehat{\Gamma}$.

Keywords. Extended modular group; Congruence subgroups; Suborbital graphs; Connectivity

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1. Introduction

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, the set of all integer matrices of order 2 with the unit determinant and every matrix A is identified with its negative $-A$. With the matrix multiplication, Γ is actually the quotient of the group $\mathrm{SL}(2,\mathbb{Z})$ by its center $\{\pm I\}$. This quotient group is called the *modular group*. The group Γ can act on the upper half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by linear fractional transformation, that is,

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

where $z \in \mathbb{H}^2$ and $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. For convenience, we may omit the sign of matrices representing the elements of Γ and the related objects which we mention after.

In another manner, we can consider that the modular group is a group of linear fractional transformations on $\mathbb{H}^2$ determined by

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \right\},$$

and a group operation is composition of functions. The *extended modular group* $\widehat{\Gamma}$ is the group generated by the modular group Γ and the hyperbolic reflection $z \mapsto -\bar{z}$ across the imaginary axis. The explicit forms of elements of $\widehat{\Gamma}$ are demonstrated as follows,

$$\left.\begin{aligned} z &\mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1, \\ z &\mapsto \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=-1. \end{aligned}\right\} \tag{1.1}$$

In the forms of matrix representations, $\widehat{\Gamma}$ is generated by $\mathrm{PSL}(2,\mathbb{Z})$ and the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and acts on $\mathbb{H}^2$ by the action (1.1).

The action of $\widehat{\Gamma}$ on $\mathbb{H}^2$ can be extended to the set of rational numbers together with ∞, $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, where ∞ is represented by the fractions $\frac{1}{0} = \frac{-1}{0}$. Similarly, every element of $\mathbb{Q}$ is represented by the reduced fractions $\frac{x}{y} = \frac{-x}{-y}$. With these representations, the action (1.1) can be rewritten as follows,

$$z \mapsto \frac{ax+by}{cx+dy},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}$. The group Γ acts on $\widehat{\mathbb{Q}}$ by the same action. Further, it acts *transitively* on $\widehat{\mathbb{Q}}$, that is, for every $v \in \widehat{\mathbb{Q}}$ there exists an element $\gamma \in \Gamma$ such that $\gamma(\infty) = v$. This means that there is the only one orbit under the action of Γ. Exactly, $\widehat{\Gamma}$ acts transitively on $\widehat{\mathbb{Q}}$ since it contains Γ as a subgroup.

The suborbital graph is a directed graph arisen from the transitive group action. The concept of this graph was introduced by Sims in 1967 for finite permutation groups, see more details in [12]. Next, this idea was extended to the case of Γ by Jones, Singerman, and Wicks. They investigated and described many properties of suborbital graphs for Γ in [8]. After that there were many studies focusing on the suborbital graphs for the modular group and modular-group-like objects, see in [4, 7, 11] for examples, including the extended modular group which was studied in [9].

In [8], the authors let $\mathcal{G}_{u,n}$ denote the suborbital graph for Γ on $\widehat{\mathbb{Q}}$. They used the Γ-invariant equivalence relation induced by the congruence subgroup,

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \bmod n \right\},$$

of Γ to partition $\widehat{\mathbb{Q}}$ into many finite numbers of blocks, the equivalence classes. They restricted $\mathcal{G}_{u,n}$ on the block containing ∞, and denoted this subgraph by $\mathcal{F}_{u,n}$. Connectivity of $\mathcal{F}_{u,n}$ was studied there. Connectivity of suborbital graphs for various underline subgroups of Γ has been also investigated, see for example in [2, 3, 6, 10].

In the case of the extended modular group $\widehat{\Gamma}$, the notations of suborbital graphs are defined similar to the case of Γ. The authors used $\widehat{\mathcal{G}}_{u,n}$ to denote the suborbital graph for $\widehat{\Gamma}$ on $\widehat{\mathbb{Q}}$, and constructed the $\widehat{\Gamma}$-invariant equivalence relation on $\widehat{\mathbb{Q}}$ by using the congruence subgroup

$$\widehat{\Gamma}_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widehat{\Gamma} : c \equiv 0 \bmod n \right\}.$$

Certainly, $\widehat{\mathcal{F}}_{u,n}$ denotes the subgraph of $\widehat{\mathcal{G}}_{u,n}$ restricted on the block containing ∞. It contains $\mathcal{F}_{u,n}$ as a subgraph of the same vertex set. Some properties of the graphs in this case were studied following from the case of Γ. Nevertheless, connectivity of the graph was not yet examined.

Connectivity of an undirected graph is an easy concept. For the easiest explanation, if we pull one vertex of the connected graph, other vertices will be moved. The case of a directed graph is more complicated. There are many types of connectivity depending on directions. This research focuses on only weak and strong types. Connectivity of graphs in the studies mentioned above is actually *weakly connected* that we investigate graphs' connectivity when they are considered as undirected ones. For the strong type, the graph is said to be *strongly connected* if for any pair v and w of vertices, there is a directed path from v to w. Obviously, strong connectivity implies weak connectivity, so we say that the graph is *disconnected* or *not connected* if the graph does not satisfy the axiom of weak connectivity.

In this paper, we examine connectivity of the graph $\widehat{\mathcal{F}}_{u,n}$. We found that there are differences between the results of $\widehat{\mathcal{F}}_{u,n}$ and $\mathcal{F}_{u,n}$. The graph $\widehat{\mathcal{F}}_{u,n}$ is (weakly or strongly) connected in many more cases than the graph $\mathcal{F}_{u,n}$. More precisely, for $n \geq 5$, the graph $\mathcal{F}_{u,n}$ is not connected while $\widehat{\mathcal{F}}_{u,n}$ is (weakly or strongly) connected if $n = 5,7,9$ and $u \not\equiv \pm 1 \bmod n$. Moreover, we obtain that $\mathcal{F}_{u,4}$ is not strongly connected while $\widehat{\mathcal{F}}_{u,4}$ is strongly connected.

2. Preliminaries

This section summarizes several necessary basic backgrounds which can be found in [1, 8, 9]. We start this section with a general definition of a suborbital graph.

Let G be a group acting transitively on a nonempty set X. Then G can act on the Cartesian product $X \times X$ by

$$g(v,w) = (g(v), g(w)),$$

where $g \in G$ and $(v,w) \in X \times X$. The *suborbital graph* $\mathcal{G}(v,w)$ for G is a directed graph with the vertex set X and there is a directed edge from v_1 to v_2, which is traditionally denoted by $v_1 \to v_2$, if and only if the ordered pair (v_1, v_2) exists in the orbit $G(v,w)$. We can see that the graph $\mathcal{G}(w,v)$ is actually the graph $\mathcal{G}(v,w)$ but arrows are reversed. We say that $\mathcal{G}(w,v)$ and $\mathcal{G}(v,w)$ are *paired* suborbital graphs. If $\mathcal{G}(v,w) = \mathcal{G}(w,v)$, the graph is said to be *self-paired*.

Let $\alpha \in X$. Since G acts on X transitively, every element of X can be written in the form $g(\alpha)$ for some $g \in G$. Suppose that H is a subgroup of G containing the stabilizer G_α. Then the G-invariant equivalence relation $\sim$ on X induced by H is given by,

$$g(\alpha) \sim g'(\alpha) \text{ if and only if } g' \in gH.$$

An equivalence class is called a *block*, and the block containing $\beta \in X$ is denoted by $[\beta]$. If $\beta = g(\alpha)$, we see that $[\beta] = \{gh(\alpha) : h \in H\}$. Thus, the block $[\alpha]$ is actually the orbit

$H(\alpha) = \{h(\alpha) : h \in H\}$, so H acts transitively on $[\alpha]$. Let $\mathcal{P} = \{[g(\alpha)] : g \in G\}$. Since the relation $\sim$ is G-invariant, the action of G can be directly extended to $\mathcal{P}$. The induced action is certainly transitive.

Let us consider more about the subgroup H. For the cases $H = G_\alpha$ and $H = G$, the obtained relations are the identity and universal relations, respectively. The interesting case is the case $G_\alpha < H < G$ which provides the nontrivial G-invariant equivalence relation. If $v, w \in [\alpha]$, the graph $\mathcal{G}(v,w)$ restricted on the block $[\alpha]$ will be more complicated than the trivial graph. In this case, the obtained subgraph is, in fact, the suborbital graph for H on its orbit $H(\alpha) = [\alpha]$. We have known from the above paragraph that G permutes the blocks transitively. This implies that the restricted graphs on blocks in $\mathcal{P}$ are isomorphic. Of course, $\mathcal{G}(v,w)$ is the union of all those restricted graphs.

In the case of the modular group, G and X are replaced by Γ and $\widehat{\mathbb{Q}}$, respectively. Now, we have $v, w \in \widehat{\mathbb{Q}}$. Since Γ acts on $\widehat{\mathbb{Q}}$ transitively, we obtain that $(\infty, \frac{u}{n}) \in \Gamma(v,w)$ for some $\frac{u}{n} \in \widehat{\mathbb{Q}}$ with $n \geq 0$. Certainly, $\mathcal{G}(\infty, \frac{u}{n}) = \mathcal{G}(v,w)$. Then the graph is simply denoted by $\mathcal{G}_{u,n}$. If $\frac{u}{n} = \infty$, this is the trivial case of suborbital graphs. Thus, we assume that $n \geq 1$. In this case, the edges of the graph are the upper-semicircles connecting two rational numbers or the vertical half-lines in $\mathbb{H}^2$ joining the rational vertices on the real line to the ideal vertex ∞. The Γ-invariant equivalence relation is constructed by replacing G_α and H by Γ_∞ and $\Gamma_0(n)$, respectively. Here, the graph $\mathcal{F}_{u,n}$ is established to be the subgraph of $\mathcal{G}_{u,n}$ on $[\infty] = \{\frac{x}{y} \in \widehat{\mathbb{Q}} : y \equiv 0 \bmod n\}$. For the case $n = 1$, we have $\mathcal{G}_{1,1} = \mathcal{F}_{1,1}$. It is called the *Farey graph* and denoted by $\mathcal{F}$. In the same way, the suborbital graphs for the case of the extended modular group are established and the notations are determined likewise as described in the previous section. The first result provided below is a general property of a suborbital graph $\mathcal{G}$ for a group G. Other than vertices of the graph, G also acts transitively on edges of $\mathcal{G}$.

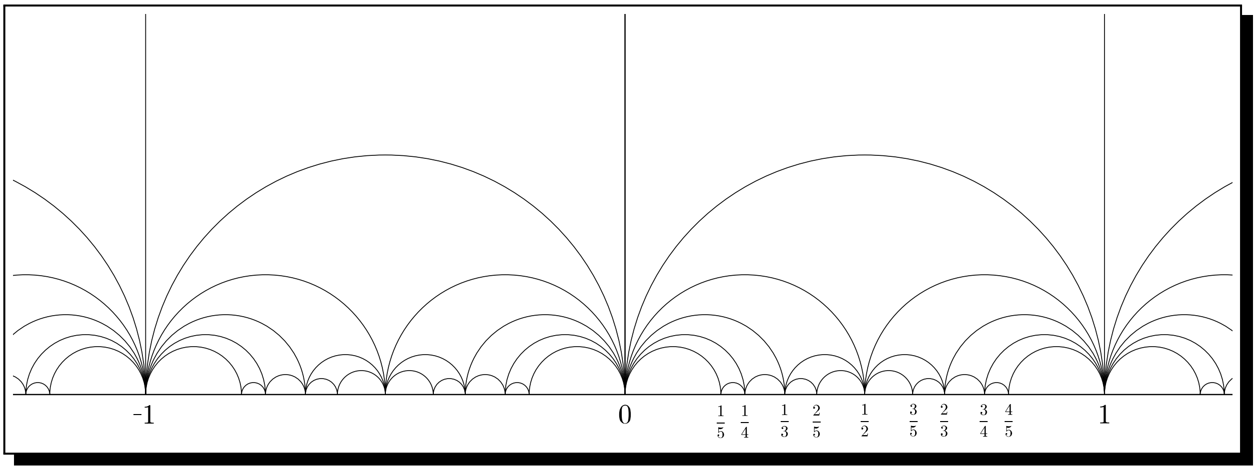


Figure 1. The Farey graph $\mathcal{F}$ embedded in $\mathbb{H}^2$

Proposition 1. *Let $\mathcal{G}$ be a suborbital graph for a group G. Then G acts on vertices and edges of $\mathcal{G}$ transitively.*

The following series of lemmas are fundamental properties of the graph $\mathcal{G}_{u,n}$ and $\widehat{\mathcal{G}}_{u,n}$.

Lemma 2. *$\mathcal{G}_{u,n} = \mathcal{G}_{u',n'}$ if and only if $n = n'$ and $u \equiv u' \bmod n$.*

Lemma 3. *$\mathcal{G}_{u,n}$ and $\mathcal{G}_{u',n}$ are paired if $-uu' \equiv 1 \bmod n$.*

Lemma 4. *$\mathcal{G}_{u,n}$ is self-paired if and only if $u^2 \equiv -1 \bmod n$.*

Lemma 5. *$\widehat{\mathcal{G}}_{u,n}$ and $\widehat{\mathcal{G}}_{u',n}$ are paired if $uu' \equiv \pm 1 \bmod n$.*

Lemma 6. *$\widehat{\mathcal{G}}_{u,n}$ is self-paired if and only if $u^2 \equiv \pm 1 \bmod n$.*

The next lemma provides that the Farey graph $\mathcal{F}$ can be embedded in the upper half-plane $\mathbb{H}^2$, in fact, in $\mathbb{H}^2$ union with its Euclidean boundary $\widehat{\mathbb{R}}$.

Lemma 7. *No edges of $\mathcal{F}$ cross in $\mathbb{H}^2$.*

The first fundamental result of the graph $\mathcal{F}_{u,n}$ is demonstrated in the next theorem, the edge conditions of the graph $\mathcal{F}_{u,n}$. It is obtained directly from the edge conditions for $\mathcal{G}_{u,n}$, see more details in [8, Theorem 3.2]. A bunch of results for $\mathcal{F}_{u,n}$ is also provided below.

Theorem 8. *There is an edge $\frac{r}{s} \to \frac{x}{y}$ in $\mathcal{F}_{u,n}$ if and only if it satisfies one of the following conditions,*

(1) *$x \equiv ur \bmod n$ and $ry - sx = n$,*

(2) *$x \equiv -ur \bmod n$ and $ry - sx = -n$.*

Lemma 9. *$\mathcal{F}_{u,n}$ is isomorphic to a subgraph of $\mathcal{F}$ by the isomorphism $v \mapsto nv$.*

Theorem 10. *$\mathcal{F}_{u,n}$ is weakly connected if and only if $n \le 4$.*

A *path* in a directed graph is a sequence of $m \ge 1$ different vertices $v_1, v_2, \dots, v_m$, allow $v_1 = v_m$, such that $v_1 \to v_2 \to \cdots \to v_m$ and some arrows may be reversed. The path in the first case is said to be *directed*. A *semi-infinite path* is similarly defined from an infinite sequence of vertices such that the first vertex exists. If $4 \le m < +\infty$ and $v_1 = v_m$, a (directed) path is called a (*directed*) *circuit*. A (directed) circuit of three vertices is called a (*directed*) *triangle*. We say that a graph is a *forest* if it contains no loops and circuits.

Theorem 11. *$\mathcal{F}_{u,n}$ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0 \bmod n$.*

Remark 12. $\mathcal{F}_{u,n}$ is a forest if and only if it contains no triangles, i.e., $u^2 \pm u + 1 \not\equiv 0 \bmod n$.

Similar to $\mathcal{F}_{u,n}$, the edge conditions for $\widehat{\mathcal{F}}_{u,n}$ is concluded from the edge conditions for the graph $\widehat{\mathcal{G}}_{u,n}$ which is described in [9, Theorem 2].

Theorem 13. *There is an edge $\frac{r}{s} \to \frac{x}{y}$ in $\widehat{\mathcal{F}}_{u,n}$ if and only if it satisfies one of the following conditions,*

(1) *$x \equiv ur \bmod n$ and $ry - sx = n$,*

(2) *$x \equiv -ur \bmod n$ and $ry - sx = -n$,*

(3) *$x \equiv -ur \bmod n$ and $ry - sx = n$,*

(4) $x \equiv ur \bmod n$ *and* $ry - sx = -n$.

We see that the first two conditions of Theorem 13 are the edge conditions for $\mathcal{F}_{u,n}$ and the others are for $\mathcal{F}_{-u,n}$, so we have the following remark immediately.

Remark 14. $\frac{r}{s} \to \frac{x}{y}$ is an edge in $\widehat{\mathcal{F}}_{u,n}$ if and only if it belongs to $\mathcal{F}_{u,n}$ or $\mathcal{F}_{-u,n}$. Then $\widehat{\mathcal{F}}_{u,n} = \widehat{\mathcal{F}}_{-u,n}$. Moreover, $\widehat{\mathcal{F}}_{u,n} = \mathcal{F}_{u,n}$ if and only if $1 \le n \le 2$, i.e., $\mathcal{F}_{u,n} = \mathcal{F}_{-u,n}$.

The series of obvious results below can be concluded by combining the previous remark together with some preceding results described above.

Corollary 15. $\widehat{\mathcal{F}}_{u,n} = \widehat{\mathcal{F}}_{u',n}$ *if and only if* $u \equiv u' \bmod n$ *or* $u \equiv -u' \bmod n$.

Corollary 16. $\mathcal{F}_{u,n}$ *and* $\widehat{\mathcal{F}}_{u,n}$ *are isomorphic to subgraphs of* $\mathcal{F}$ *by the isomorphism* $v \mapsto nv$.

Corollary 17. *No edges of* $\widehat{\mathcal{F}}_{u,n}$ *cross in* $\mathbb{H}^2$.

3. Connectivity of graphs

This section examines connectivity of graphs. We start with the strong type of connectivity of $\mathcal{F}_{u,n}$. By Theorem 10, the graph $\mathcal{F}_{u,n}$ is weakly connected if and only if $n \le 4$, so we can consider only for $n \le 4$. For $n = 4$, Lemma 4 and Remark 12 imply that $\mathcal{F}_{u,n}$ is a non-self-paired tree, so we can see easily that it cannot be strongly connected. In the remaining cases, the graph $\mathcal{F}_{u,n}$ is self-paired or contains directed triangles. The two types of connectivity are also equivalent after applying the following proposition.

Proposition 18. *Let* $\mathcal{G}$ *be a suborbital graph for a group* G*, and vertices* v *and* w *of the graph be joined together by some path. If* $\mathcal{G}$ *is self-paired or contains directed circuits, then there is a directed path from* v *to* w.

Proof. The self-paired case is obvious, so we prove only the remaining case. The proof can be obtained by verifying that a directed edge $a \to b$ can be replaced by some directed path of reverse direction. Suppose that $a_1 \to b_1 \to b_2 \cdots b_n \to a_1$ is a directed path of $\mathcal{G}$. We know from Proposition 1 that G acts transitively on edges of $\mathcal{G}$. Thus, there is $g \in G$ such that $g(a_1 \to b_1) = a \to b$. Then we obtains that $a \to b \to g(b_2) \cdots g(b_n) \to a$ is a directed circuit of $\mathcal{G}$ containing the edge $a \to b$. Hence we can replace the edge $a \to b$ by the reverse directed path $a \leftarrow g(b_n) \cdots g(b_2) \leftarrow b$. □

Theorem 19. $\mathcal{F}_{u,n}$ *is strongly connected if and only if* $n \le 3$.

Since $\mathcal{F}_{u,n}$ is a subgraph of $\widehat{\mathcal{F}}_{u,n}$ with the same vertex set, we obtain that $\widehat{\mathcal{F}}_{u,n}$ is also strongly connected for every $n \le 3$. By the similar reason applying on Theorem 10, we obtain the trivial result that $\widehat{\mathcal{F}}_{u,4}$ is weakly connected. It is also self-paired by Lemma 6, so Proposition 18 implies that $\widehat{\mathcal{F}}_{u,4}$ is strongly connected. We conclude all of these trivial results in the next proposition.

Proposition 20. $\widehat{\mathcal{F}}_{u,n}$ *is strongly connected for every* $n \le 4$.

The next proposition is another obvious result which can be concluded through the usage of $\mathcal{F}_{u,n}$. If $u \equiv \pm 1 \bmod n$, then $u^2 \equiv 1 \bmod n$. Thus, Lemma 3 implies that $\mathcal{F}_{u,n}$ and $\mathcal{F}_{-u,n}$ are paired suborbital graphs. Therefore, connectivity of $\widehat{\mathcal{F}}_{u,n}$ follows from that of $\mathcal{F}_{u,n}$.

Proposition 21. *If $n \geq 5$ and $u \equiv \pm 1 \bmod n$, the graph $\widehat{\mathcal{F}}_{u,n}$ is not connected.*

Next, we verify nontrivial cases which are benefited from the fact that $\widehat{\Gamma}_0(n)$ contains the translation $z \mapsto z+1$ and the reflection $z \mapsto -\bar{z}+1$ across the line $\mathrm{Re}(z) = \frac{1}{2}$. These transformations guarantee periodicity and symmetry of the graph $\widehat{\mathcal{F}}_{u,n}$. The translation implies that the graph is periodic with period 1 along the real axis, so we can consider $\widehat{\mathcal{F}}_{u,n}$ only in the strip $0 \leq \mathrm{Re}(z) \leq 1$. Together with the reflection, we can apply the narrower strip $0 \leq \mathrm{Re}(z) \leq \frac{1}{2}$ in some cases. In the next result, we prove disconnectedness of the graph where the two lemmas below are required.

Lemma 22. *Let $\frac{j}{k}$ be a fraction, need not be reduced, v and w be vertices of $\widehat{\mathcal{F}}_{u,n}$ such that $v < \frac{j}{k} < w$. If $k \mid n$, then v and w are not adjacent.*

Proof. We assume the contrary that v and w are adjacent. By Corollary 16, the vertices nv and nw are adjacent in $\mathcal{F}$, so the edge joining them crosses the edge $\frac{nj}{k} \to \infty$ of $\mathcal{F}$ in $\mathbb{H}^2$. This provides a contradiction to Lemma 7. □

Lemma 23. *Let $a,b,k \in \mathbb{Z}$ where $b \neq 0 \neq k$. Then the fraction $\frac{1+2abk}{4b^2k}$ is reduced.*

Proof. Since $\gcd(1+2abk, 2bk) = 1$, $\gcd(1+2abk, 4b^2k) = 1$. □

Theorem 24. *Let $n \neq 9$ be a non-prime integer greater than 5. Then the graph $\widehat{\mathcal{F}}_{u,n}$ is not connected.*

Proof. In the proof, we consider the graph on the strip $0 \leq \mathrm{Re}(z) \leq 1$. The case $u \equiv \pm 1 \bmod n$ is concluded in Proposition 21, so we can assume that $u \not\equiv \pm 1 \bmod n$. This implies that $n \neq 6$. By Corollary 15, we may suppose that $0 < \frac{u}{n} < 1$. The goal of this proof is to find a subset of $[0,1]$ separating some vertices of the graph from others.

In the case $n = 8$, we have $u = 3$ or $u = 5$. Since $3 \equiv -5 \bmod 8$, Corollary 15 implies that $\widehat{\mathcal{F}}_{3,8} = \widehat{\mathcal{F}}_{5,8}$. Thus, we can check only the case $u = 3$. By using Lemma 3, we can easily check that $\mathcal{F}_{3,8}$ and $\mathcal{F}_{-3,8}$ are paired. Then connectivity of $\widehat{\mathcal{F}}_{u,8}$ can be concluded directly from connectivity of $\mathcal{F}_{3,8}$. Since $\mathcal{F}_{3,8}$ is not connected, $\widehat{\mathcal{F}}_{u,8}$ is not connected.

For $n \geq 10$, we write $n = pm$ where p is the least prime factor of n. One can see that $m \geq 5$. We now partition the interval $(0,1]$ into m disjoint subintervals $(0, \frac{1}{m}], (\frac{1}{m}, \frac{2}{m}], (\frac{2}{m}, \frac{3}{m}], \ldots, (\frac{m-1}{m}, 1]$. Since $m \mid n$, Lemma 22 implies that vertices in each subinterval are not adjacent to rational vertices outside. By using Theorem 13, one can easily check that, there are at most 4 vertices of $\widehat{\mathcal{F}}_{u,n}$ in the interval $(0,1]$ adjacent to ∞. Since $m \geq 5$, there is at least one subinterval $(\frac{j}{m}, \frac{j+1}{m}], 0 \leq j \leq m-1$, not containing the vertices adjacent to ∞. We will show that the interval $(\frac{j}{m}, \frac{j+1}{m}]$ contains some vertices of $\widehat{\mathcal{F}}_{u,n}$, so that the strip $\frac{j}{m} < \mathrm{Re}(z) \leq \frac{j+1}{m}$ contains components of the graph. Therefore, the graph is not connected. To do this we replace variables a, b and

k in Lemma 23 by $2j+1, m$ and pk, respectively. Then we obtain the reduced proper fraction $\frac{1+2(2j+1)mpk}{4m^2pk} = \frac{1+2(2j+1)nk}{4mnk}$. It converges to $\frac{2j+1}{2m}$ as a sequence of index k. Since the interval $(\frac{j}{m}, \frac{j+1}{m}]$ is a neighborhood of $\frac{2j+1}{2m}$, the fraction $\frac{1+2(2j+1)nk}{4mnk}$ belongs to the interval $(\frac{j}{m}, \frac{j+1}{m}]$ for some large integer k. Certainly, $\frac{1+2(2j+1)nk}{4mnk}$ is a vertex of $\widehat{\mathcal{F}}_{u,n}$. The proof is now completed. □

The next example provides the different result from the case of the modular group. We have known by Theorem 10 that the graph $\mathcal{F}_{u,n}$ is not connected for every $n \geq 5$; however, it fails for the graph $\widehat{\mathcal{F}}_{u,n}$. We show that $\widehat{\mathcal{F}}_{2,5}$ is weakly connected. In fact, the graph is strongly connected because it is self-paired. Before demonstrating this example, we discuss the useful object, the Stern-Brocot tree. More properties of the tree can be found in [5, page 115-123].

The Stern-Brocot tree is an infinite tree whose vertex set is generated inductively by using the notion of mediants. Suppose that $\frac{r}{s}$ and $\frac{x}{y}$ are fractions, called *parents*. Their *mediant* is the fraction given by $\frac{r}{s} \oplus \frac{x}{y} := \frac{r+x}{s+y}$. We can see that the mediant of any two fractions is not unique, so we usually write the fractions in the lowest terms and assume that $s, y > 0$ to make the operation $\oplus$ well-defined. The concept is extended to the fraction $\frac{1}{0}$ representing ∞ to construct the Stern-Brocot tree. First, we start with the initial set $ST_0 = \mathcal{R}_0 = \{0 = \frac{0}{1}, \infty = \frac{1}{0}\}$. Then for every $n \geq 1$, $\mathcal{R}_n$ denotes the set of all mediants of successive fractions in ST_{n-1}, and define $ST_n = ST_{n-1} \cup \mathcal{R}_n$ written as the set of fractions arranged in increasing order. In this case, $\frac{1}{0}$ is assumed to be the greatest fraction. The union $\cup_{n\geq 0} ST_n$ is the vertex set of the tree. The following remark provides some facts of the Stern-Brocot tree.

Remark 25. (1) If $\frac{r}{s}$ and $\frac{x}{y}$ are consecutive fractions in ST_n with $s, y \geq 0$, then $ry - sx = -1$ and $\frac{r}{s} < \frac{r+x}{s+y} < \frac{x}{y}$. Moreover, they are reduced fractions.

(2) Every vertex of the Stern-Brocot tree appears only one time in some ST_n.

(3) $\cup_{n\geq 0} ST_n = \mathbb{Q}^+ \cup \{0, \infty\}$.

We see that the consecutive fractions $\frac{r}{s}$ and $\frac{x}{y}$ in ST_n satisfy the edge conditions of the Farey graph $\mathcal{F}$. Therefore, the fractions are adjacent in $\mathcal{F}$, so the tree is a subgraph of $\mathcal{F}$. We know by Corollary 16 that $\widehat{\mathcal{F}}_{u,n}$ is isomorphic to a subgraph of $\mathcal{F}$ by the isomorphism $\frac{r}{ns} \mapsto \frac{r}{s}$. This mapping sends all vertices of $\widehat{\mathcal{F}}_{u,n}$ in the interval $[0, +\infty)$ to be vertices of the Stern-Brocot tree. Obviously, its inverse is a function $\phi_n : \frac{r}{s} \mapsto \frac{r}{ns}$. We can see that ϕ_n preserves the operation $\oplus$, that is, $\phi_n(\frac{r}{s} \oplus \frac{x}{y}) = \phi_n(\frac{r}{s}) \oplus \phi_n(\frac{x}{y})$. This means that we can generate all vertices of $\widehat{\mathcal{F}}_{u,n}$ in the interval $[0, +\infty)$ by using the notion of mediants trough the function ϕ_n and the construction of the Stern-Brocot tree. Certainly, some images of vertices of the tree under ϕ_n are not reduced, so they cannot be vertices of $\widehat{\mathcal{F}}_{u,n}$. It is not necessary to write them in the lowest terms. Now, we are ready to provide the demonstration of $\widehat{\mathcal{F}}_{2,5}$.

Example 26. $\widehat{\mathcal{F}}_{2,5}$ is weakly connected.

Proof. We have known that the graph $\widehat{\mathcal{F}}_{2,5}$ is periodic along the real axis with period 1 and symmetric with respect to the line $\mathrm{Re}(z) = \frac{1}{2}$. Then the induced subgraph of $\widehat{\mathcal{F}}_{2,5}$ in the strip $0 \leq \mathrm{Re}(z) \leq \frac{1}{2}$ can represent induced subgraphs in other strips $\frac{j}{2} \leq \mathrm{Re}(z) \leq \frac{j+1}{2}$, $j \in \mathbb{Z}$. If the subgraph in the strip $0 \leq \mathrm{Re}(z) \leq \frac{1}{2}$ is weakly connected, then the subgraphs in other strips are

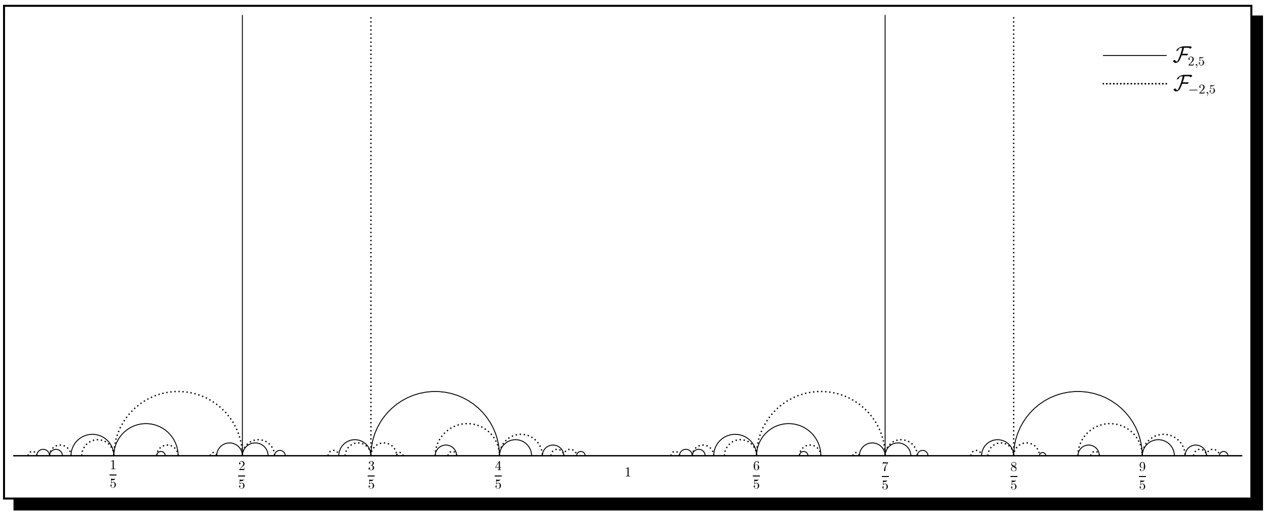


Figure 2. The graph $\widehat{\mathcal{F}}_{2,5}$

also weakly connected, and are joined to one another by the vertex ∞. This means that we can consider only the subgraph in the strip $0 \le \mathrm{Re}(z) \le \frac{1}{2}$ to verify that the graph $\widehat{\mathcal{F}}_{2,5}$ is weakly connected.

Now, let $\widehat{\mathcal{R}}_0 = \{\frac{0}{5}, \frac{1}{5}, \frac{2}{5}, \frac{5}{10}\}$. This set is the image of $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{5}{2}\}$ under ϕ_5. We see that $\frac{0}{1}$ and $\frac{1}{1}$ are successive terms in ST_1, $\frac{1}{1}$ and $\frac{2}{1}$ are successive terms in ST_2, $\frac{2}{1}$ and $\frac{5}{2}$ are successive terms in ST_4. Thus, we can generate all reduced fractions in $[0, \frac{5}{2}]$ by using mediants with the initial set $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{5}{2}\}$. Hence, all vertices of the graph $\widehat{\mathcal{F}}_{2,5}$ in $[0, \frac{1}{2}]$ are also generated by the same concept with the initial set $\widehat{\mathcal{R}}_0$.

We have known by Remark 25 that every pair of consecutive fractions $\frac{r}{s}$ and $\frac{x}{y}$ in ST_n satisfies the property $ry - sx = -1$. Then their images $\frac{r}{5s}$ and $\frac{x}{5y}$ satisfy the property $r(5y) - (5s)x = -5$. Hence, we can check only the remaining congruence conditions in Theorem 13 to show that every mediant which is a vertex of $\widehat{\mathcal{F}}_{2,5}$ can be joined with at least one of its parents.

Now, suppose that $\frac{r}{5s}$ and $\frac{x}{5y}$ are consecutive terms in $\cup_{i=0}^{n} \widehat{\mathcal{R}}_i$ for some $n \ge 0$. We will show that if the mediant $\frac{r+x}{5(s+y)}$ is a vertex of $\widehat{\mathcal{F}}_{2,5}$, it can be joined to $\frac{r}{5s}$ or $\frac{x}{5y}$. We consider all situations which can possibly occur. Start with the case that the numerator of one parent is divisible by 5. We may suppose that 5 divides r. Since numerators of every two successive fractions in $\widehat{\mathcal{R}}_0$ are not divisible by 5 in the same time, this situation also occurs for $\cup_{i=0}^{n} \widehat{\mathcal{R}}_i$ for every $n \ge 1$. Thus, 5 does not divide x, so $\frac{x}{5y}$ is a vertex of $\widehat{\mathcal{F}}_{2,5}$. Since 5 does not divide $r + x$ and $r + 2x$, the fractions $\frac{r+x}{5(s+y)}$ and $\frac{r+2x}{5(s+2y)}$ are reduced. Then they are vertices of $\widehat{\mathcal{F}}_{2,5}$. It is obvious that $r + 2x \equiv 2(r + x) \bmod 5$ and $r + 2x \equiv 2x \bmod 5$. Hence, Theorem 13 implies that the mediant $\frac{r+2x}{5(s+2y)}$ is adjacent to its parents $\frac{x}{5y}$ and $\frac{r+x}{5(s+y)}$. This shows that $\frac{r+x}{5(s+y)}$ can be joined to its parent $\frac{x}{5y}$ through the fraction $\frac{r+2x}{5(s+2y)}$. The case that x is divisible by 5 is similar. We will obtain that $\frac{r+x}{5(s+y)}$ is joined to $\frac{r}{5s}$.

For the other case, x and y are not divisible by 5, we obtain that if the mediant $\frac{r+x}{5(s+y)}$ is a vertex of $\widehat{\mathcal{F}}_{2,5}$, it is adjacent to at least one of its parents. The results are shown in Table 1.

The numerators of fractions are written in the least residue system modulo 5. If $r+x \equiv 0 \bmod 5$, the mediant $\frac{r+x}{5(s+y)}$ is not a vertex of the graph. If $r+x \not\equiv 0 \bmod 5$, the mediant $\frac{r+x}{5(s+y)}$ is a vertex of the graph and adjacent to the fractions $\frac{r}{s}$ or $\frac{x}{y}$ whose numerators are circled.

Table 1. Adjacent vertices of $\widehat{\mathcal{F}}_{2,5}$

r	x	$r+x$
①	①	2
①	2	3
1	③	4
1	4	0
②	②	4

r	x	$r+x$
2	3	0
②	4	1
③	③	1
3	④	2
④	④	3

We have proved that every time we obtain another mediant which is a vertex of $\widehat{\mathcal{F}}_{2,5}$ in the interval $[0, \frac{1}{2}]$, it can be joined to at least one of its parents. Combining this result with the fact that the vertices $\frac{1}{5}$ and $\frac{2}{5}$ of $\widehat{\mathcal{F}}_{2,5}$ in the initial set $\widehat{\mathcal{R}}_0$ are adjacent, so all vertices of $\widehat{\mathcal{F}}_{2,5}$ in $[0, \frac{1}{2}]$ belong to the same weakly connected component. Therefore, the graph $\widehat{\mathcal{F}}_{2,5}$ is weakly connected. □

The preceding example can be extended to a general case. After applying Corollary 15, we obtain that $\widehat{\mathcal{F}}_{u,5} = \widehat{\mathcal{F}}_{2,5}$, for every $u \not\equiv \pm 1 \bmod 5$. Thus, $\widehat{\mathcal{F}}_{u,5}$ is weakly connected for every $u \not\equiv \pm 1 \bmod 5$. For $n = 7$ and $n = 9$, connectivity of the graphs can be proved by using the similar arguments. We may verify only the case $u = 2$. For other $u \not\equiv \pm 1 \bmod n$, the graph $\widehat{\mathcal{F}}_{u,n}$ is either paired with or equal to $\widehat{\mathcal{F}}_{2,n}$. Combining the obtained results with Proposition 21, we obtain the theorem below.

Theorem 27. *For $n = 5, 7, 9$, $\widehat{\mathcal{F}}_{u,n}$ is weakly connected if and only if $u \not\equiv \pm 1 \bmod n$.*

Corollary 28. *For $n = 5, 7$, $\widehat{\mathcal{F}}_{u,n}$ is strongly connected if and only if $u \not\equiv \pm 1 \bmod n$.*

Proof. The forward is obtained directly by the theorem above. For the converse, we obtain that $\widehat{\mathcal{F}}_{u,5}$ and $\widehat{\mathcal{F}}_{u,7}$ are self-paired and contains directed triangles by Lemma 6 and Theorem 11, respectively. Thus, Proposition 18 and the previous theorem imply that the graphs are strongly connected. □

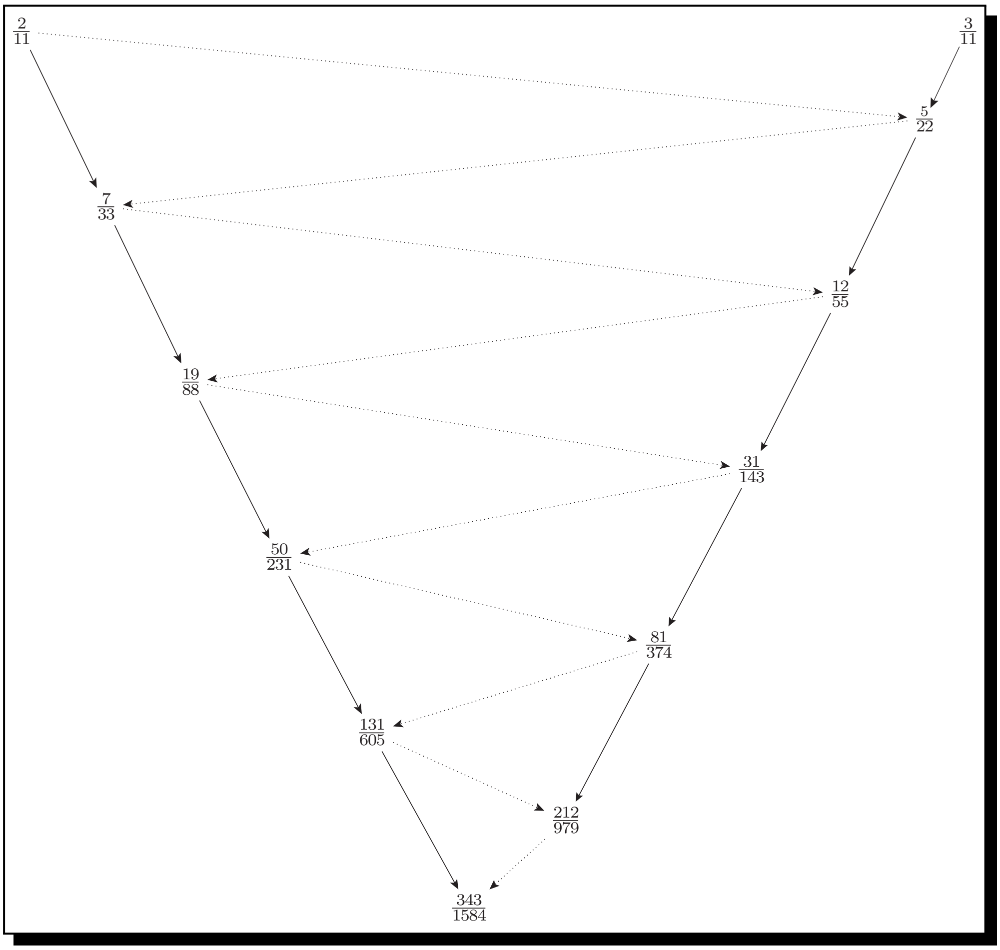
Now, we see that for a prime number p such that $5 \le p < 11$, the graph $\widehat{\mathcal{F}}_{u,p}$ is connected for every $u \not\equiv \pm 1 \bmod p$. However, it is not true for $p = 11$ and 13. The example of $\widehat{\mathcal{F}}_{2,11}$ below is also benefited from the notion of mediants similar to the previous example of $\widehat{\mathcal{F}}_{2,5}$. In the proof, we construct the Fibonacci sequence for the fraction using the concept of mediants instead of the usual addition on $\mathbb{R}$.

Example 29. $\widehat{\mathcal{F}}_{2,11}$ is not connected.

Proof. We know by Lemma 16 that $\widehat{\mathcal{F}}_{2,11}$ is isomorphic to a subgraph of the Farey graph $\mathcal{F}$ by the mapping $v \mapsto 11v$. Since the Farey graph has no edge-crossing in $\mathbb{H}^2$, all vertices of

$\widehat{\mathcal{F}}_{2,11}$ in the interval $(\frac{2}{11},\frac{5}{11})$ are not adjacent to every vertex outside the interval $[\frac{2}{11},\frac{5}{11}]$. Also, Theorem 13 implies that they are not adjacent to ∞. In particular, $\frac{3}{11}$ is not adjacent to every vertices outside the interval $(\frac{2}{11},\frac{4}{11}]$, and $\frac{4}{11}$ is not adjacent to every vertices outside the interval $[\frac{3}{11},\frac{5}{11})$. If there are non-vertices $\alpha \in (\frac{2}{11},\frac{3}{11})$ and $\beta \in (\frac{4}{11},\frac{5}{11})$ such that no edges of $\widehat{\mathcal{F}}_{2,11}$ cross the lines $\mathrm{Re}(z) = \alpha$ and $\mathrm{Re}(z) = \beta$, then the strip $\alpha \le \mathrm{Re}(z) \le \beta$ certainly separates some connected components of the graph from others.

To find the real number α, we consider the fractions $\frac{2}{11}$ and $\frac{3}{11}$, the images of $\frac{2}{1}$ and $\frac{3}{1}$ under the function $\phi_{11} : \frac{x}{y} \mapsto \frac{x}{11y}$. We see that $\frac{2}{1}$ and $\frac{3}{1}$ are successive terms in ST_3. Then all vertices of $\widehat{\mathcal{F}}_{2,11}$ in $[\frac{2}{11},\frac{3}{11}]$ can be generated by mediants with the initial values $\frac{2}{11}$ and $\frac{3}{11}$. First, we consider $\frac{5}{22}$, the mediant of $\frac{2}{11}$ and $\frac{3}{11}$. It is adjacent to its parent $\frac{3}{11}$, but not adjacent to $\frac{2}{11}$. Next, we consider the nonadjacent vertices $\frac{2}{11}$ and $\frac{5}{22}$. Their mediant is the fraction $\frac{7}{33}$ adjacent to $\frac{2}{11}$, but not adjacent to $\frac{5}{22}$. We then consider the next pair of nonadjacent vertices $\frac{7}{33}$ and $\frac{5}{22}$, and so on. The diagram below shows the first 10 steps of the method. The arrow heads aim to the mediants of two fractions at the arrow tails. The dotted lines mean that the mediants and their parents are not adjacent, and the others represent the edges of the graph. We see that $212 \equiv 3 \bmod 11$ and $343 \equiv 2 \bmod 11$. Then the situation is repeated with period 10. Finally, we obtain the sequence of vertices $\{a_m\} = \{\frac{3}{11},\frac{2}{11},\frac{5}{22},\dots\}$ such that $a_m \to a_{m+2}$ for every $m \ge 1$. We will show that it is convergent and its limit is the required value α.



Let $b_m = a_{2m}$ and $c_m = a_{2m-1}$. Then $\{11b_m\}$ and $\{11c_m\}$ are increasing and decreasing subsequences of $\{11a_m\}$, respectively. By boundedness of the sequences, the limits of $\{11b_m\}$ and $\{11c_m\}$ exist. If their limits are different, all reduced fractions between the limits will not be generated by using mediants. Thus, their limits must be equal, so it guarantee existence of the limit of the sequence $\{a_m\}$. Let $\alpha = \lim_{m\to+\infty} a_m$. It is easy to see that α belongs to (b_m, c_m) and (b_m, c_{m+1}) for every $m \geq 1$. We will show that $11\alpha \notin ST_n$ for every $n \geq 0$, so that α is not a vertex of $\widehat{\mathcal{F}}_{2,11}$. We verify this by using mathematical induction. Since $\alpha \in (\frac{2}{11}, \frac{3}{11})$ and $\frac{3}{1}$ first appears in ST_3, we have $11\alpha \notin ST_n$ for all $n \leq 3$. We now suppose that $11\alpha \notin ST_k$ for some $k > 3$. In the case that k is odd, we have $k = 2m+1$ for some $m \geq 2$. Further, we have $11b_m$ and $11c_m$ are consecutive terms in ST_k providing the mediant $11c_{m+1} \in ST_{k+1}$. Since $\alpha \in (b_m, c_{m+1})$ and $11\alpha \notin ST_k$, we obtain $11\alpha \notin ST_{k+1}$. In the case $k = 2m$ for some $m \geq 2$, we see that $11b_{m-1}$ and $11c_m$ are successive terms in ST_k providing the mediant $b_m \in ST_{k+1}$. Since $\alpha \in (b_m, c_m)$ and $11\alpha \notin ST_k$, α does not belong to ST_{k+1}. Thus, $11\alpha \notin ST_n$ for every $n \geq 0$. We have shown that α is not a vertex of $\widehat{\mathcal{F}}_{2,11}$, so it cannot be a vertex in a path joining $\frac{2}{11}$ and $\frac{3}{11}$. Next, we will show that there are not any edges of $\widehat{\mathcal{F}}_{2,11}$ crossing the line $\mathrm{Re}(z) = \alpha$.

Suppose that v and w are adjacent vertices of $\widehat{\mathcal{F}}_{2,11}$ such that $v < \alpha < w$. Since $\frac{2}{1}$ and $\frac{3}{1}$ are adjacent to ∞ in $\mathcal{F}$, to avoid edge-crossing of $\mathcal{F}$ in $\mathbb{H}^2$ the vertices v and w must belong to $[\frac{2}{11}, \frac{3}{11}]$. We see that $b_1 \to b_2 \to \cdots$ and $c_1 \to c_2 \to \cdots$ are semi-infinite paths in $\widehat{\mathcal{F}}_{2,11}$ converging to α on the left and right, respectively. Since there are not any edges of $\widehat{\mathcal{F}}_{2,11}$ crossing in $\mathbb{H}^2$, we must have $v = b_m$ and $w = c_{m'}$ for some $m, m' \in \mathbb{N}$. We know that b_m is not adjacent to c_m and c_{m+1}. Then $m+1 < m'$ or $m > m'$. In the case $m+1 < m'$, we see that $b_{m'-1} < c_{m'} < c_{m'-1}$. Since $11b_{m'-1}$ and $11c_{m'-1}$ are adjacent in $\mathcal{F}$ and $b_m < b_{m'-1}$, to avoid edge-crossing of $\mathcal{F}$ in $\mathbb{H}^2$ we obtain that b_m and $c_{m'}$ cannot be adjacent vertices of $\widehat{\mathcal{F}}_{2,11}$. In the case $m > m'$, we use the similar argument. We have $b_{m-1} < b_m < c_m$. Since $11b_{m-1}$ and $11c_m$ are adjacent in $\mathcal{F}$ and $c_m < c_{m'}$, the vertices b_m and $c_{m'}$ cannot be adjacent in $\widehat{\mathcal{F}}_{2,11}$. Thus, the adjacent vertices v and w do not exist. Hence no edges of $\widehat{\mathcal{F}}_{2,11}$ cross the line $\mathrm{Re}(z) = \alpha$.

Now, we obtain the required value $\alpha \in (\frac{2}{11}, \frac{3}{11})$. The value $\beta \in (\frac{4}{11}, \frac{5}{11})$ can be obtained similarly. Therefore, $\widehat{\mathcal{F}}_{2,11}$ is not connected. □

The following corollary is a consequence of Lemma 5, Corollary 15 and Proposition 21.

Corollary 30. *The graph* $\widehat{\mathcal{F}}_{u,11}$ *is not connected for every* $u \equiv \pm 1 \bmod 11$, $u \equiv \pm 2 \bmod 11$ *and* $u \equiv \pm 5 \bmod 11$.

Next, we provide another disconnected example without the usage of mediants. However, we need to use another notion, the Farey sequences. The *Farey sequence* $\mathcal{F}_m$ of order $m \geq 1$, is defined to be the set of all irreducible fractions $\frac{x}{y}$, with $|y| \leq m$, ordered increasingly.

Example 31. $\widehat{\mathcal{F}}_{3,13}$ is not connected.

Proof. In this case, we have $p = 13$ and $u = 3$. We can check that $\frac{3}{13}$ and $\frac{4}{13}$ are only two vertices of the graph in $[0, \frac{1}{2}]$ adjacent to ∞. It is easily seen that the interval $[\frac{1}{3}, \frac{1}{2}]$ contains $\frac{5}{13}$, a vertex of $\widehat{\mathcal{F}}_{3,13}$, but not containing $\frac{3}{13}$ and $\frac{4}{13}$. We need to verify that there are not any edges of the graph crossing the line $\mathrm{Re}(z) = \frac{1}{3}$ and the line $\mathrm{Re}(z) = \frac{1}{2}$, so that the strip $\frac{1}{3} \le \mathrm{Re}(z) \le \frac{1}{2}$ contains connected components of $\widehat{\mathcal{F}}_{3,13}$. Since there are not any vertices of the graph in the interval $[\frac{1}{3}, \frac{1}{2}]$ adjacent to ∞, the graph is not connected

We will show only the case of the line $\mathrm{Re}(z) = \frac{1}{3}$. The other case can be considered as a subcase of the proven case. By the sake of contradiction, we suppose that v and w are adjacent vertices of $\widehat{\mathcal{F}}_{3,13}$ such that $v < \frac{1}{3} < w$. Then Lemma 16 implies that $13v$ and $13w$ are adjacent vertices in $\mathcal{F}$. Certainly, $13v < \frac{13}{3} < 13w$, so $13v$ and $13w$ belong to Farey sequences $\mathcal{F}_m$ for some $m = 1$ or 2, see more in [8, Lemma 4.1]. If $m = 2$, we have $13v = 4$ and $13w = \frac{9}{2}$. Thus, $v = \frac{4}{13}$ and $w = \frac{9}{26}$. However, $9 \not\equiv \pm 3(4) \bmod 13$ and $4 \not\equiv \pm 3(9) \bmod 13$, so v and w are not adjacent in $\widehat{\mathcal{F}}_{3,13}$. In the case $m = 1$, we have $13v = 4$ and $13w = 5$, so $v = \frac{4}{13}$ and $w = \frac{5}{13}$. We see that $5 \not\equiv \pm 3(4) \bmod 13$ and $4 \not\equiv \pm 3(5) \bmod 13$. Thus, v and w are not adjacent in $\widehat{\mathcal{F}}_{3,13}$. The proof for the line $\mathrm{Re}(z) = \frac{1}{2}$ follows this last subcase. We now complete the proof of this example. □

Corollary 32. *The graph $\widehat{\mathcal{F}}_{u,13}$ is not connected for every $u \equiv \pm 1 \bmod 13$, $u \equiv \pm 3 \bmod 13$ and $u \equiv \pm 4 \bmod 13$.*

By the previous two examples of disconnectivity, we conjecture that for every prime number $p \ge 11$, the graph $\widehat{\mathcal{F}}_{u,p}$ is not connected.

4. Conclusion

There are differences between connectivity results of $\widehat{\mathcal{F}}_{u,n}$ and $\mathcal{F}_{u,n}$. The graph $\widehat{\mathcal{F}}_{u,n}$ is (weakly or strongly) connected in many more cases than $\mathcal{F}_{u,n}$. More precisely, the graph $\mathcal{F}_{u,n}$ is not connected for every $n \ge 5$ while $\widehat{\mathcal{F}}_{u,n}$ is (weakly or strongly) connected for $n = 5, 7, 9$ and $u \not\equiv \pm 1 \bmod n$. Further, $\mathcal{F}_{u,4}$ is not strongly connected while $\widehat{\mathcal{F}}_{u,4}$ is strongly connected.

Conjecture. For every prime number $p \ge 11$, the graph $\widehat{\mathcal{F}}_{u,p}$ is not connected.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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