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Research Article

Laplacian Minimum covering Randić Energy of a Graph

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Abstract. Randić energy was first defined in the article [6]. Using minimum covering set, we have introduced in this article Laplacian minimum covering Randić energy $LRE_C(G)$ of a graph G. This article contains computation of Laplacian minimum covering Randić energies for some standard graphs like star graph, complete graph, crown graph, complete bipartite graph and cocktail graph. At the end of this article upper and lower bounds for Laplacian minimum covering Randić energy are also presented.

Keywords. Minimum covering set; Minimum covering Randić matrix; Laplacian minimum covering Randić matrix; Laplacian minimum covering Randić eigenvalues; Laplacian minimum covering Randić energy

MSC. Primary 05C50, 05C69

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1. Introduction

Study on energy of graphs goes back to the year 1978, when Gutman [16] defined this while working with energies of conjugated hydrocarbon containing carbon atoms. All graphs considered in this article are assumed to be simple without loops and multiple edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph G with its eigenvalues $\rho_1, \rho_2, \rho_3, \cdots, \rho_n$ assumed in decreasing order. Since A is real symmetric, the eigenvalues of G are real numbers whose sum equal to zero. The sum of the absolute eigenvalues values of G is called the energy $E(G)$ of G. i.e., $E(G) = \sum_{i=1}^{n} |\rho_i|$.

Theories on the mathematical concepts of graph energy can be seen in the reviews [20], articles [9, 10, 17] and the references cited there in. For various upper and lower bounds for energy of a graph can be found in articles[22, 23] and it was observed that graph energy has chemical applications in the molecular orbital theory of conjugated molecules [15, 19].

Gutman and Zhou [21] defined the Laplacian energy of a graph G in the year 2006. Let G be a graph with n vertices and m edges. The Laplacian matrix of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order n whose elements are defined as

$$L_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i & \text{if } i = j, \end{cases}$$

where d_i is the degree of the vertex v_i. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Laplacian eigenvalues of G. Laplacian energy $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

The basic properties including various upper and lower bounds for Laplacian energy have been established in [2, 11, 14, 18, 27, 30–38] and it has found remarkable chemical applications, the molecular orbital theory of conjugated molecules [8].

1.1 Randić Energy

It was in the year 1975, Milan Randić invented a molecular structure descriptor called Randić index which is defined as [29]:

$$R(G) = \sum_{v_i v_j \in E(G)} \frac{1}{\sqrt{d_i d_j}}.$$

Motivated by this Bozkurt *et al.* [6] defined Randić matrix and Randić energy as follows.

Let G be graph of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set Randić matrix of G is a $n \times n$ symmetric matrix defined by $R(G) := (r_{ij})$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The characteristic equation of $R(G)$ is defined by $f_n(G, \rho) = \det(\rho I - R(G)) = 0$. The roots of this equation is called Randić eigenvalues of G. Since $R(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in decreasing order $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$. Randić energy of G is defined as $RE(G) := \sum_{i=1}^{n} |\rho_i|$.

Further studies on Randić energy can be seen in the articles [7, 12, 25] and the references cited therein.

1.2 Minimum Covering Energy

In the year 2012, Adiga *et al.* [1] introduced minimum covering energy of a graph, which depends on its particular minimum cover. A subset C of vertex set V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a minimum covering set. If C is a minimum covering set of a graph G then the minimum covering matrix of G is the $n \times n$ matrix defined by $A_C(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The minimum covering eigenvalues of the graph G are roots of the characteristic equation $f_n(G,\rho) = 0 = \det(\rho I - A_C(G)) = 0$, obtained from the matrix $A_C(G)$. Since $A_C(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in the order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The minimum covering energy of G is defined as $E_C(G) := \sum_{i=1}^{n} |\rho_i|$.

1.3 Minimum covering Randić Energy

Results on Randić energy and minimum covering energy of graph G motivates us to define minimum covering Randić energy. Consider a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. If C is a minimum covering set of a graph G then the minimum covering Randić matrix of G is the $n \times n$ matrix defined by $R_C(G) := (r_{ij})$, where

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G) \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $R_C(G)$ is defined by $f_n(G,\rho) = \det(\rho I - R_C(G))$. The minimum covering Randić eigenvalues of the graph G are the eigenvalues of $R_C(G)$. Since $R_C(G)$ is real and symmetric matrix so its eigenvalues are real numbers. We label the eigenvalues in order $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The minimum covering Randić energy of G is defined as $RE_C(G) := \sum_{i=1}^{n} |\rho_i|$.

In the year 2012, Adiga *et al.* [1] introduced minimum covering energy of a graph, which depends on its particular minimum cover. Motivated by this article, recently Rajesh Kanna and Jagadeesh in the article [28] introduced the concept of minimum covering Randić energy.

1.4 Laplacian Energy

Gutman and Zhou in article [21] introduced the Laplacian energy of a graph G in the year 2006.

Definition 1.1. Let G be a graph with n vertices and m edges. The *Laplacian matrix* of the graph G, denoted by $L = (L_{ij})$, is a square matrix of order $n \times n$ whose elements are defined as

$$L_{ij} = \begin{cases} -1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\ d_i & \text{if } i = j \end{cases}$$

where d_i is the degree of the vertex v_i.

Definition 1.2. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Laplacian eigenvalues of G. *Laplacian energy* $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$.

1.5 Laplacian Minimum covering Energy of A Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then $L_C(G) = D(G) - A_C(G)$ is called the Laplacian covering matrix of G. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of $L_C(G)$, arranged in non-increasing order. These eigenvalues are called Laplacian minimum covering eigenvalues of G. The *Laplacian minimum covering energy* of the graph G is defined as

$$LE_C(G) := \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where m is the number of edges of G and $\frac{2m}{n}$ is the average degree of G.

1.6 Laplacian Minimum Covering Randić Energy of A Graph

Let $D(G)$ be the diagonal matrix of vertex degrees of the graph G. Then $LR_C(G) = D(G) - R_C(G)$ is called the Laplacian covering matrix of G. Let $\rho_1, \rho_2, \rho_3, \ldots, \rho_n$ be the eigenvalues of $LR_C(G)$, arranged in non-increasing order. These eigenvalues are called Laplacian minimum covering Randić eigenvalues of G. The *Laplacian minimum covering Randić energy* of the graph G is defined as

$$LRE_C(G) := \sum_{i=1}^{n} \left| \rho_i - \frac{2m}{n} \right|,$$

where m is the number of edges of G and $\frac{2m}{n}$ is the average degree of G.

In this article, we are interested in studying mathematical aspects of the Laplacian minimum covering energy of a graph. The application of Laplacian minimum covering energy in other branches of science have to be investigated.

Example 1. (i) $C_1 = \{v_2, v_4, v_6\}$, (ii) $C_2 = \{v_2, v_4, v_5\}$ are the possible minimum covering sets for the Figure 1 as shown below.

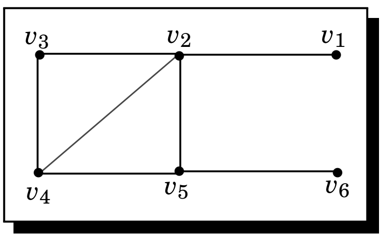

Figure 1

$$\text{(i)}\ R_{C_1}(G) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{12}} & 0 & 0 & \frac{1}{\sqrt{3}} & 1 \end{pmatrix}, \quad D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$LR_{C_1}(G)=D(G)-R_{C_1}(G)=\begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 3 & \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & 0 \\ 0 & \frac{-1}{\sqrt{8}} & 2 & \frac{-1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{6}} & 2 & \frac{-1}{3} & 0 \\ 0 & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{3} & 3 & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & \frac{-1}{\sqrt{3}} & 0 \end{pmatrix}.$$

Laplacian minimum covering Randić eigenvalues are $\rho_1 \approx -0.1172220$, $\rho_2 \approx 0.8641210$, $\rho_3 \approx 1.4559813$, $\rho_4 \approx 3.407766$, $\rho_5 \approx 3.0439018$, $\rho_6 \approx 2.3454518$.

Number of vertices $= 6$, number of edges $= 7$.

Therefore, average degree $= \frac{2m}{n} = \frac{2\times 7}{6} = \frac{7}{3}$.

Laplacian minimum covering Randić energy, $LRE_{C_1}(G) \approx 6.5942393$.

(ii) $R_{C_2}(G)=\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & 1 & \frac{1}{3} & 0 \\ 0 & \frac{1}{\sqrt{12}} & 0 & \frac{1}{3} & 1 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{12}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$, $D(G)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$,

$$LR_{C_2}(G)=D(G)-R_{C_2}(G)=\begin{pmatrix} 1 & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & 3 & \frac{-1}{\sqrt{8}} & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & 0 \\ 0 & \frac{-1}{\sqrt{8}} & 2 & \frac{-1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{12}} & \frac{-1}{\sqrt{6}} & 2 & \frac{-1}{3} & 0 \\ 0 & \frac{-1}{\sqrt{12}} & 0 & \frac{-1}{3} & 2 & \frac{-1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{12}} & 0 & 0 & \frac{-1}{\sqrt{3}} & 1 \end{pmatrix}.$$

Laplacian minimum covering Randić eigenvalues are $\rho_1 \approx 3.2591135$, $\rho_2 \approx 0.6640478$, $\rho_3 \approx 0.8892142$, $\rho_4 \approx 1.4512959$, $\rho_5 \approx 2.5725331$, $\rho_6 \approx 2.1637955$.
Number of vertices $= 6$, number of edges $= 7$.
Therefore, average degree $= \frac{2m}{n} = \frac{2\times 7}{6} = \frac{7}{3}$.
Laplacian minimum covering Randić energy, $LRE_{C_2}(G) \approx 5.3299599$.
Therefore, Laplacian minimum covering Randić energy depends on the covering set.

2. Main Results and Discussion

2.1 aplacian Minimum Covering Randić Energy of Some Standard Graphs

Theorem 2.1. *For $n \geq 2$, Laplacian minimum covering Randić energy of complete graph K_n is* $\frac{(n-2)^2+\sqrt{4n^2-8n+5}}{n-1}$.

Proof. Let K_n be a complete graph with vertex set $V = \{v_1, v_2, v_3, \dots, v_n\}$. The Laplacian minimum covering set for K_n is $C = \{v_1, v_2, v_3, \dots, v_{n-1}\}$. Then

$$R_C(K_n) = \begin{pmatrix} 1 & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & 1 & \frac{1}{n-1} & \dots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & 1 & \dots & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & 1 & \frac{1}{n-1} & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} & 1 & \frac{1}{n-1} \\ \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} & \frac{1}{n-1} & 0 \end{pmatrix}_{n \times n},$$

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & n-1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & n-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & n-1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & n-1 \end{pmatrix}_{n \times n},$$

$$LR_C(K_n) = D(K_n) - R_C(K_n) = \begin{pmatrix} n-2 & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & n-2 & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & n-2 & \dots & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & n-2 & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & n-2 & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & \frac{-1}{n-1} & n-1 \end{pmatrix}_{n \times n}.$$

Characteristic polynomial is

$$\frac{(-1)^n[(n-1)\rho - (n^2-3n+3)]^{n-2}[(n-1)\rho^2 - (2n^2-6n+5)\rho + (n^3-5n^2+8n-5)]}{(n-1)^{n-1}}.$$

Characteristic equation is

$$\frac{(-1)^n[(n-1)\rho - (n^2-3n+3)]^{n-2}[(n-1)\rho^2 - (2n^2-6n+5)\rho + (n^3-5n^2+8n-5)]}{(n-1)^{n-1}} = 0.$$

Laplacian minimum covering Randić spec

$$LR_c(K_n) = \begin{pmatrix} \frac{n^2-3n+3}{n-1} & \frac{(2n^2-6n+5)+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(2n^2-6n+5)-\sqrt{4n^2-8n+5}}{2(n-1)} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Number of vertices $= n$, number of edges $= nC_2 = \frac{n(n-1)}{2}$.

Therefore, average degree $= \frac{2m}{n} = \frac{2\frac{n(n-1)}{2}}{n} = n-1$.

Laplacian minimum covering Randić energy,

$$LRE_C(K_n) = \left|\frac{n^2-3n+3}{n-1} - (n-1)\right|(n-2) + \left|\frac{(2n^2-6n+5)+\sqrt{4n^2-8n+5}}{2(n-1)} - (n-1)\right|(1)$$

$$+\left|\frac{(2n^2-6n+5)-\sqrt{4n^2-8n+5}}{2(n-1)}-(n-1)\right|(1)$$
$$=\left|\frac{-n+2}{n-1}\right|(n-2)+\left|\frac{(3-2n)+\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1)+\left|\frac{(3-2n)-\sqrt{4n^2-8n+5}}{2(n-1)}\right|(1)$$
$$=\frac{(n-2)^2}{n-1}+\frac{\sqrt{4n^2-8n+5}}{n-1}$$
$$=\frac{(n-2)^2+\sqrt{4n^2-8n+5}}{n-1}.$$
□

Definition 2.1. Cocktail party graph is denoted by $K_{n\times 2}$, is a graph having the vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$ and the edge set $E=\{u_iu_j,v_iv_j : i\neq j\}\cup\{u_iv_j,v_iu_j : 1\leq i<j\leq n\}$.

Theorem 2.2. *Laplacian minimum covering Randić energy of cocktail party graph $K_{n\times 2}$ is* $\frac{(n-1)^2+\sqrt{4n^2-8n+5}}{n-1}$.

Proof. Consider cocktail party graph $K_{n\times 2}$ with vertex set $V=\bigcup_{i=1}^{n}\{u_i,v_i\}$. The Laplacian minimum covering set of cocktail party graph $K_{n\times 2}$ is $C=\bigcup_{i=1}^{n-1}\{u_i,v_i\}$. Then

$$R_C(K_{n\times 2})=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n\\
\hline
u_1 & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & \frac{1}{2n-2}\\
u_2 & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \dots & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \dots & \frac{1}{2n-2}\\
u_3 & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \dots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \dots & \frac{1}{2n-2}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
u_n & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & 0\\
\hline
v_1 & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & \frac{1}{2n-2}\\
v_2 & \frac{1}{2n-2} & 0 & \frac{1}{2n-2} & \dots & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \frac{1}{2n-2} & \dots & \frac{1}{2n-2}\\
v_3 & \frac{1}{2n-2} & \frac{1}{2n-2} & 0 & \dots & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & 1 & \dots & \frac{1}{2n-2}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
v_n & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & 0 & \frac{1}{2n-2} & \frac{1}{2n-2} & \frac{1}{2n-2} & \dots & 0
\end{array}\right),$$

$$D(K_{n\times 2})=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n\\
\hline
u_1 & 2n-2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
u_2 & 0 & 2n-2 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
u_3 & 0 & 0 & 2n-2 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
u_n & 0 & 0 & 0 & \dots & 2n-2 & 0 & 0 & 0 & \dots & 0\\
\hline
v_1 & 0 & 0 & 0 & \dots & 0 & 2n-2 & 0 & 0 & \dots & 0\\
v_2 & 0 & 0 & 0 & \dots & 0 & 0 & 2n-2 & 0 & \dots & 0\\
v_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 2n-2 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
v_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 2n-2
\end{array}\right),$$

$LR_C(K_{n\times 2}) = D(K_{n\times 2}) - R_C(K_{n\times 2})$

$$= \begin{pmatrix} & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \\ u_1 & 2n-3 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} \\ u_2 & \frac{-1}{2n-2} & 2n-3 & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} \\ u_3 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 2n-3 & \dots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \dots & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & 2n-2 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & 0 \\ v_1 & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} & 2n-3 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} \\ v_2 & \frac{-1}{2n-2} & 0 & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 2n-3 & \frac{-1}{2n-2} & \dots & \frac{-1}{2n-2} \\ v_3 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 0 & \dots & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & 2n-3 & \dots & \frac{-1}{2n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & 0 & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \frac{-1}{2n-2} & \dots & 2n-2 \end{pmatrix}.$$

Characteristic polynomial is,

$$\frac{[\rho-(2n-2)][\rho-(2n-3)]^{n-1}[(n-1)\rho^2-(4n^2-10n+7)\rho+(4n^3-16n^2+22n-11)]}{(n-1)^{n-1}}.$$

Characteristic equation is,

$$\frac{[\rho-(2n-2)][\rho-(2n-3)]^{n-1}[(n-1)\rho^2-(4n^2-10n+7)\rho+(4n^3-16n^2+22n-11)]}{(n-1)^{n-1}}=0.$$

Laplacian minimum covering Randić

$$\operatorname{spec}(K_{n\times 2}) = \begin{pmatrix} 2n-2 & 2n-3 & \frac{(4n^2-10n+7)+\sqrt{4n^2-8n+5}}{2(n-1)} & \frac{(4n^2-10n+7)-\sqrt{4n^2-8n+5}}{2(n-1)} \\ 1 & n-1 & 1 & 1 \end{pmatrix}.$$

Number of vertices $= 2n$, number of edges $= 2n(n-1)$.

Therefore, average degree $= \frac{2(2n)(n-1)}{2n} = 2(n-1)$.

Laplacian minimum covering Randić energy,

$$\begin{aligned} LRE_C(K_{n\times 2}) &= |(2n-2)-2(n-1)|(1)+|(2n-3)-2(n-1)|(n-1) \\ &\quad + \left|\frac{(4n^2-10n+7)+\sqrt{4n^2-8n+5}}{2(n-1)}-2(n-1)\right|(1) \\ &\quad + \left|\frac{(4n^2-10n+7)-\sqrt{4n^2-8n+5}}{2(n-1)}-2(n-1)\right|(1) \\ &= 0+\left|-1\right|(n-1)+\left|\frac{-2n+3+\sqrt{4n^2-8n+5}}{2n-2}\right|(1)+\left|\frac{-2n+3-\sqrt{4n^2-8n+5}}{2n-2}\right|(1) \\ &= (n-1)+\frac{\sqrt{4n^2-8n+5}}{n-1} \\ &= \frac{(n-1)^2+\sqrt{4n^2-8n+5}}{n-1}. \end{aligned}$$

□

Theorem 2.3. *Laplacian minimum covering Randić energy of star graph is*

$$LR_C(K_{1,n-1}) = \begin{cases} \sqrt{5} & \text{if } n=2 \\ \frac{(n-2)^2}{n}+\sqrt{n^2-6n+13} & \text{if } n>2. \end{cases}$$

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V=\{v_0,v_1,v_2,\dots,v_{n-1}\}$. Then its minimum covering set is $C=\{v_0\}$.

Case (i): If $n=2$ then characteristic equation is $\rho^2-\rho-1=0$.

Laplacian minimum covering Randić

$$\operatorname{spec} LR_C(K_{1,n-1})=\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}$$

Number of vertices $=2$, number of edges $=1$.

Therefore, average degree $=\frac{2(1)}{2}=1$.

Laplacian minimum covering Randić energy,

$$LRE_C(K_{1,n-1})=\left|\frac{1+\sqrt{5}}{2}-1\right|(1)+\left|\frac{1-\sqrt{5}}{2}-1\right|(1)=\frac{\sqrt{5}-1}{2}+\frac{\sqrt{5}+1}{2}=\sqrt{5}.$$

Case (ii): If $n>2$

$$R_C(K_{1,n-1})=\begin{pmatrix}1 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \dots & \frac{1}{\sqrt{n-1}}\\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \dots & 0\\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \dots & 0\\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\sqrt{n-1}} & 0 & 0 & 0 & \dots & 0\end{pmatrix}_{n\times n},$$

$$D(K_{1,n-1})=\begin{pmatrix}n-1 & 0 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1\end{pmatrix}_{n\times n},$$

$$LR_C(K_{1,n-1})=D(K_{1,n-1})-R_C(K_{1,n-1})=\begin{pmatrix}n-2 & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \frac{-1}{\sqrt{n-1}} & \dots & \frac{-1}{\sqrt{n-1}}\\ \frac{-1}{\sqrt{n-1}} & 1 & 0 & 0 & \dots & 0\\ \frac{-1}{\sqrt{n-1}} & 0 & 1 & 0 & \dots & 0\\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{-1}{\sqrt{n-1}} & 0 & 0 & 0 & \dots & 1\end{pmatrix}_{n\times n}.$$

Characteristic equation is $(-1)^n[\rho-1]^{n-2}[\rho^2-(n-1)\rho n-3]=0$.

Laplacian minimum covering Randić

$$\operatorname{spec} LR_C(K_{1,n-1})=\begin{pmatrix}1 & \frac{(n-1)+\sqrt{n^2-6n+13}}{2} & \frac{(n-1)-\sqrt{n^2-6n+13}}{2}\\ n-2 & 1 & 1\end{pmatrix}.$$

Laplacian minimum covering Randić energy,

$$\begin{aligned}
LRE_C(K_{1,n-1}) &= \left|1-\frac{2(n-1)}{n}\right|(n-2)+\left|\frac{(n-1)+\sqrt{n^2-6n+13}}{2}-\frac{2(n-1)}{n}\right|(1)\\
&\quad+\left|\frac{(n-1)-\sqrt{n^2-6n+13}}{2}-\frac{2(n-1)}{n}\right|(1)\\
&=\left|\frac{-n+2}{n}\right|(n-2)+\left|\frac{(n^2-5n+4)+n\sqrt{n^2-6n+13}}{2n}\right|(1)\\
&\quad+\left|\frac{(n^2-5n+4)-n\sqrt{n^2-6n+13}}{2n}\right|(1)\\
&=\frac{(n-2)^2}{n}+\sqrt{n^2-6n+13}.
\end{aligned}$$

□

Definition 2.2. Crown graph S_n^0 for an integer $n\ge 2$ is the graph with vertex set $\{u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n\}$ and edge set $\{u_iv_j : 1\le i,j\le n, i\ne j\}$.

Theorem 2.4. *For $n\ge 2$, Laplacian minimum covering Randić energy of the crown graph S_n^0 is equal to $\sqrt{5}+\sqrt{n^2-2n+5}$*

Proof. For the crown graph S_n^0 with vertex set $V=\{u_1,u_2,\dots,u_n,v_1,v_2,\dots,v_n\}$, minimum covering set of crown graph S_n^0 is $C=\{u_1,u_2,\dots,u_n\}$. Then

$$R_C(S_n^0)=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n\\ \hline
u_1 & 1 & 0 & 0 & \dots & 0 & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1}\\
u_2 & 0 & 1 & 0 & \dots & 0 & \frac{1}{n-1} & 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1}\\
u_3 & 0 & 0 & 1 & \dots & 0 & \frac{1}{n-1} & \frac{1}{n-1} & 0 & \dots & \frac{1}{n-1}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
u_n & 0 & 0 & 0 & \dots & 1 & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & 0\\ \hline
v_1 & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \dots & \frac{1}{n-1} & 0 & 0 & 0 & \dots & 0\\
v_2 & \frac{1}{n-1} & 0 & \frac{1}{n-1} & \dots & \frac{1}{n-1} & 0 & 0 & 0 & \dots & 0\\
v_3 & \frac{1}{n-1} & \frac{1}{n-1} & 0 & \dots & \frac{1}{n-1} & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
v_n & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \dots & 0 & 0 & 0 & 0 & \dots & 0
\end{array}\right),$$

$$D(S_n^0)=\left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n\\ \hline
u_1 & n-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
u_2 & 0 & n-1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
u_3 & 0 & 0 & n-1 & \dots & 0 & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
u_n & 0 & 0 & 0 & \dots & n-1 & 0 & 0 & 0 & \dots & 0\\ \hline
v_1 & 0 & 0 & 0 & \dots & 0 & n-1 & 0 & 0 & \dots & 0\\
v_2 & 0 & 0 & 0 & \dots & 0 & 0 & n-1 & 0 & \dots & 0\\
v_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & n-1 & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
v_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & n-1
\end{array}\right),$$

$LR_C(S_n^0) = D(S_n^0) - R_C(S_n^0)$

$$= \left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \\
\hline
u_1 & n-2 & 0 & 0 & \dots & 0 & 0 & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} \\
u_2 & 0 & n-2 & 0 & \dots & 0 & \frac{-1}{n-1} & 0 & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} \\
u_3 & 0 & 0 & n-2 & \dots & 0 & \frac{-1}{n-1} & \frac{-1}{n-1} & 0 & \dots & \frac{-1}{n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & 0 & 0 & 0 & \dots & n-2 & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & 0 \\
\hline
v_1 & 0 & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & n-1 & 0 & 0 & \dots & 0 \\
v_2 & \frac{-1}{n-1} & 0 & \frac{-1}{n-1} & \dots & \frac{-1}{n-1} & 0 & n-1 & 0 & \dots & 0 \\
v_3 & \frac{-1}{n-1} & \frac{-1}{n-1} & 0 & \dots & \frac{-1}{n-1} & 0 & 0 & n-1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & \frac{-1}{n-1} & \frac{-1}{n-1} & \frac{-1}{n-1} & \dots & 0 & 0 & 0 & 0 & \dots & n-1
\end{array}\right).$$

Characteristic polynomial is

$$\frac{[\rho^2-(2n-3)\rho+(n^2-3n+1)][(n-1)^2\rho^2-(2n-3)(n-1)^2\rho+(n^4-5n^3+9n^2-7n+1)]^{n-1}}{(n-1)^{2n-2}}.$$

Characteristic equation is

$$\frac{[\rho^2-(2n-3)\rho+(n^2-3n+1)][(n-1)^2\rho^2-(2n-3)(n-1)^2\rho+(n^4-5n^3+9n^2-7n+1)]^{n-1}}{(n-1)^{2n-2}}=0.$$

Laplacian minimum covering Randić

$$\operatorname{spec} S_n^0 = \begin{pmatrix} \frac{(2n-3)+\sqrt{5}}{2} & \frac{(2n-3)-\sqrt{5}}{2} & \frac{(2n^2-5n+3)+\sqrt{n^2-2n+5}}{2(n-1)} & \frac{(2n^2-5n+3)-\sqrt{n^2-2n+5}}{2(n-1)} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Number of vertices $=2n$, number of edges $=n(n-1)$.

Therefore, average degree $=\frac{2n(n-1)}{2n}=n-1$.

Laplacian Minimum covering Randić energy,

$$\begin{aligned}
LRE_C(S_n^0) &= \left|\frac{(2n-3)+\sqrt{5}}{2}-(n-1)\right|(1)+\left|\frac{(2n-3)-\sqrt{5}}{2}-(n-1)\right|(1) \\
&\quad+\left|\frac{(2n^2-5n+3)+\sqrt{n^2-2n+5}}{2(n-1)}-(n-1)\right|(n-1) \\
&\quad+\left|\frac{(2n^2-5n+3)-\sqrt{n^2-2n+5}}{2(n-1)}-(n-1)\right|(n-1) \\
&=\left|\frac{\sqrt{5}-1}{2}\right|(1)+\left|\frac{\sqrt{5}+1}{2}\right|(1)+\left|\frac{(-n+1)+\sqrt{n^2-2n+5}}{2(n-1)}\right|(n-1) \\
&\quad+\left|\frac{(-n+1)-\sqrt{n^2-2n+5}}{2(n-1)}\right|(n-1) \\
&=\sqrt{5}+\sqrt{n^2-2n+5}.
\end{aligned}$$

□

Theorem 2.5. *The minimum covering Randić energy, $R_C(G)$ of the complete bipartite graph $RE_C(K_{m,n})$ is equal to*

$$\left|\frac{n^2-mn-m-n}{m+n}\right|(m-1)+\left|\frac{m(m-n)}{m+n}\right|(n-1)+\sqrt{n^2-(2m+2)n+m^2+2m+5}.$$

Proof. For the complete bipartite graph $K_{m,n}$ $(m \le n)$ with vertex set $V = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$, minimum covering set is $C = \{u_1, u_2, \dots, u_m\}$. Then

$$R_C(K_{m,n}) = \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \dots & u_m & v_1 & v_2 & v_3 & \dots & v_n \\ \hline u_1 & 1 & 0 & 0 & \dots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} \\ u_2 & 0 & 1 & 0 & \dots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} \\ u_3 & 0 & 0 & 1 & \dots & 0 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & 0 & 0 & 0 & \dots & 1 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} \\ \hline v_1 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \dots & 0 \\ v_2 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \dots & 0 \\ v_3 & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \frac{1}{\sqrt{mn}} & \dots & \frac{1}{\sqrt{mn}} & 0 & 0 & 0 & \dots & 0 \end{array}\right),$$

$$D(K_{m,n}) = \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \dots & u_m & v_1 & v_2 & v_3 & \dots & v_n \\ \hline u_1 & n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ u_2 & 0 & n & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ u_3 & 0 & 0 & n & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & 0 & 0 & 0 & \dots & n & 0 & 0 & 0 & \dots & 0 \\ \hline v_1 & 0 & 0 & 0 & \dots & 0 & m & 0 & 0 & \dots & 0 \\ v_2 & 0 & 0 & 0 & \dots & 0 & 0 & m & 0 & \dots & 0 \\ v_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & m \end{array}\right),$$

$$LR_C(K_{m,n}) = D(K_{m,n}) - R_C(K_{m,n})$$

$$= \left(\begin{array}{c|ccccc|ccccc} & u_1 & u_2 & u_3 & \dots & u_m & v_1 & v_2 & v_3 & \dots & v_n \\ \hline u_1 & n-1 & 0 & 0 & \dots & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} \\ u_2 & 0 & n-1 & 0 & \dots & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} \\ u_3 & 0 & 0 & n-1 & \dots & 0 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & 0 & 0 & 0 & \dots & n-1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} \\ \hline v_1 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & m & 0 & 0 & \dots & 0 \\ v_2 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & 0 & m & 0 & \dots & 0 \\ v_3 & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & 0 & 0 & m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \frac{-1}{\sqrt{mn}} & \dots & \frac{-1}{\sqrt{mn}} & 0 & 0 & 0 & \dots & m \end{array}\right)$$

Characteristic equation is

$$(-1)^{m+n}[\rho-(n-1]^{m-1}[\rho-m]^{n-1}[\rho^2-(n+m-1)\rho+(mn-(m+1))]=0.$$

Laplacian minimum covering Randić

$$\text{spec+}(K_{m,n}) = \begin{pmatrix} n-1 & m & \frac{(m+n-1)+\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2} & \frac{(m+n-1)-\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2} \\ m-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Number of vertices $= m+n$, number of edges $= mn$.

Therefore, average degree $= \frac{2mn}{m+n}$.

Laplacian minimum covering Randić energy,

$$\begin{aligned} LRE_C(K_{m,n}) &= \left|(n-1)-\frac{2mn}{m+n}\right|(m-1)+\left|m-\frac{2mn}{m+n}\right|(n-1) \\ &+\left|\frac{(m+n-1)+\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2}-\frac{2mn}{m+n}\right|(1) \\ &+\left|\frac{(m+n-1)-\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2}-\frac{2mn}{m+n}\right|(1) \\ &=\left|\frac{n^2-m-n-mn}{m+n}\right|(m-1)+\left|\frac{m(m-n)}{m+n}\right|(n-1) \\ &+\left|\frac{(m-n)^2-(m+n)+(m+n)\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2(m+n)}\right|(1) \\ &+\left|\frac{(m-n)^2-(m+n)-(m+n)\sqrt{n^2-(2m+2)n+m^2+2m+5}}{2(m+n)}\right|(1) \\ &=\left|\frac{n^2-mn-m-n}{m+n}\right|(m-1)+\left|\frac{m(m-n)}{m+n}\right|(n-1) \\ &+\sqrt{n^2-(2m+2)n+m^2+2m+5}. \end{aligned}$$

□

2.2 Properties of Laplacian Minimum Covering Randić Eigenvalues

Theorem 2.6. *Let G be a graph with vertex set $V=\{v_1,v_2,\dots,v_n\}$, edge set E and $C=\{u_1,u_2,\dots,u_k\}$ be a minimum covering set. If $\rho_1,\rho_2,\dots,\rho_n$ are the eigenvalues of Laplacian minimum covering Randić matrix $LR_C(G)$ then*

(i) $\sum_{i=1}^{n}\rho_i = 2|E|-|C|$

(ii) $\sum_{i=1}^{n}\rho_i^2 = 2M$ *where* $M=\frac{1}{2}\sum_{i=1}^{n}(d_i-c_i)^2+\sum_{i<j}\frac{1}{d_id_j}$ *and* $c_i=\begin{cases}1 & if\ v_i\in C\\ 0 & if\ v_i\notin C.\end{cases}$

Proof. (i) We know that the sum of the eigenvalues of $LR_C(G)$ is the trace of $LR_C(G)$.

Therefore, $\sum_{i=1}^{n}\rho_i=\sum_{i=1}^{n}r_{ii}=\sum_{i=1}^{n}d_i-|C|=2|E|-|C|$.

(ii) Similarly the sum of squares of the eigenvalues of $LR_C(G)$ is trace of $[LR_C(G)]^2$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n}\rho_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}r_{ij}r_{ji} \\ &= \sum_{i=1}^{n}(r_{ii})^2+\sum_{i\neq j}r_{ij}r_{ji}\end{aligned}$$

$$= \sum_{i=1}^{n}(r_{ii})^2 + 2\sum_{i<j}(r_{ij})^2$$

$$= \sum_{i=1}^{n}(r_{ii})^2 + 2\sum_{i<j}\left(\frac{-1}{\sqrt{d_i d_j}}\right)^2$$

$$= \sum_{i=1}^{n}(d_i - c_i)^2 + 2\sum_{i<j}\frac{1}{d_i d_j}, \quad \text{where } c_i = \begin{cases} 1 & \text{if } v_i \in C \\ 0 & \text{if } v_i \notin C \end{cases}$$

$$= 2M, \quad \text{where } M = \frac{1}{2}\sum_{i=1}^{n}(d_i - c_i)^2 + \sum_{i<j}\frac{1}{d_i d_j}. \qquad \square$$

The question of when does the graph energy becomes a rational number was answered by Bapat and Pati in their article [4]. Similar result for Laplacian minimum covering Randić energy is obtained in the following theorem.

Theorem 2.7. *Let G be a graph with a minimum covering set C. If the sum of the absolute values of Laplacian minimum covering Randić eigenvalues is a rational number, then* $\sum_{i=1}^{n} |\rho_i| \equiv |C| (\text{mod}\, 2)$.

Proof. Let $\rho_1, \rho_2, \dots, \rho_n$ be Laplacian minimum covering Randić eigenvalues of a graph G, of which $\rho_1, \rho_2, \dots, \rho_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^{n} |\rho_i| = (\rho_1 + \rho_2 + \ldots + \rho_r) - (\rho_{r+1} + \ldots + \rho_n) = 2(\rho_1 + \rho_2 + \ldots + \rho_r) - (\rho_1 + \rho_2 + \ldots + \rho_n)$$

$$= 2(\rho_1 + \rho_2 + \ldots + \rho_r) - \sum_{i=1}^{n} \rho_i = 2(\rho_1 + \rho_2 + \ldots + \rho_r) - (2|E| - |C|)$$

$$= 2(\rho_1 + \rho_2 + \ldots + \rho_r - |E|) + |C|$$

Therefore,

$$\sum_{i=1}^{n} |\rho_i| \equiv |C| (\text{mod}\, 2). \tag{2.1}$$

$\square$

Theorem 2.8. *Let G be a graph with n vertices, m edges and C is a minimum covering set of G. If the sum of the absolute Laplacian minimum covering Randić eigenvalues is a rational number, then*

$$LRE_C(G) \in (|C| + 2t - 2m, |C| + 2t + 2m),$$

where t is an integer such that

$$\sum_{i=1}^{n} |\rho_i| \equiv |C| (\text{mod}\, 2).$$

Proof. We know that

$$\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right| \le \sum_{i=1}^{n} |\rho_i| + 2m$$

i.e.,

$$LRE_C(G) \le \sum_{i=1}^{n} |\rho_i| + 2m = |C| + 2t + 2m \qquad \text{(from (2.1))}$$

Also,

$$LRE_C(G) \ge \sum_{i=1}^{n} |\rho_i| - 2m = |C| + 2t - 2m \qquad \text{(from (2.1))}$$

i.e.,

$$LRE_C(G) \in (|C| + 2t - 2m, |C| + 2t + 2m).$$

The above result is similar to Parity Theorem 3.7 of [1]. □

2.3 Bounds for Laplacian Minimum Covering Randić Energy

MClelland's [23] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for $LRE_C(G)$ are given in the following theorem.

Theorem 2.9 (Upper bound)**.** *Let G be a graph with n vertices, m edges and C is a minimum covering set of a graph G. Then* $LRE_C(G) \le \sqrt{2Mn} + 2m$.

Proof. Cauchy-Schwarz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Put $a_i = 1$, $b_i = |\rho_i|$ then

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\rho_i|^2\right)$$

i.e.,

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \le n2M.$$

Therefore,

$$\sum_{i=1}^{n} |\rho_i| \le \sqrt{2Mn}.$$

By Triangle inequality,

$$\left|\rho_i - \frac{2m}{n}\right| \le |\rho_i| + \left|\frac{2m}{n}\right|, \quad \forall\ i = 1, 2 \dots, n$$

i.e.,

$$\left|\rho_i - \frac{2m}{n}\right| \le |\rho_i| + \frac{2m}{n}, \quad \forall\ i.$$

Thus

$$\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right| \le \sum_{i=1}^{n} |\rho_i| + \sum_{i=1}^{n} \frac{2m}{n} \le \sqrt{2Mn} + 2m.$$

Therefore, $LRE_C(G) \le \sqrt{2Mn} + 2m$. □

Theorem 2.10 (Upper bound)**.** *Let G be a graph with n vertices, m edges and C is a minimum covering set of G. Then $LRE_C(G) \leq \sqrt{2Mn + 4m(|C| - m)}$.*

Proof. Using Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Put $a_i = 1$, $b_i = \left|\rho_i - \frac{2m}{n}\right|$, then

$$\left(\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right|^2\right)$$

i.e.,

$$[LRE_C(G)]^2 \leq n\left[\sum_{i=1}^{n} \rho_i^2 + \sum_{i=1}^{n} \frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n} \rho_i\right] = n\left[2M + \frac{4m^2}{n^2} \cdot n - \frac{4m}{n}(2m - |C|)\right]$$

$$= n\left[2M + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m|C|}{n}\right] = 2Mn + 4m(|C| - m).$$

Therefore, $LRE_C(G) \leq \sqrt{2Mn + 4m(|C| - m)}$. □

Theorem 2.11 (Lower bound)**.** *Let G be a graph with n vertices and m edges and C is a minimum covering set of G. If $D = |\det LR_C(G)|$, then*

$$LRE_C(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}} - 2m}.$$

Proof. Consider

$$\left[\sum_{i=1}^{n} |\rho_i|\right]^2 = \left(\sum_{i=1}^{n} |\rho_i|\right) \cdot \left(\sum_{j=1}^{n} |\rho_j|\right) = \sum_{i=1}^{n} |\rho_i|^2 + \sum_{i \neq j} |\rho_i||\rho_j|.$$

Therefore,

$$\sum_{i \neq j} |\rho_i|\,|\rho_j| = \left(\sum_{i=1}^{n} |\rho_i|\right)^2 - \sum_{i=1}^{n} |\rho_i|^2. \tag{2.2}$$

Applying inequality between the arithmetic and geometric means for $n(n-1)$ terms, we have

$$\frac{\sum_{i \neq j} |\rho_i||\rho_j|}{n(n-1)} \geq \left[\prod_{i \neq j} |\rho_i||\rho_j|\right]^{\frac{1}{n(n-1)}}$$

i.e.,

$$\sum_{i \neq j} |\rho_i||\rho_j| \geq n(n-1)\left[\prod_{i \neq j} |\rho_i|\,|\rho_j|\right]^{\frac{1}{n(n-1)}}.$$

Using (2.2), we get

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 - \sum_{i=1}^{n} |\rho_i|^2 \geq n(n-1)\left[\prod_{i=1}^{n} |\rho_i|^{2(n-1)}\right]^{\frac{1}{n(n-1)}},$$

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 - 2M \geq n(n-1)\left[\prod_{i=1}^{n} |\rho_i|\right]^{\frac{2}{n}},$$

$$\left(\sum_{i=1}^{n}|\rho_i|\right)^2 \geq 2M + n(n-1)\left[\prod_{i=1}^{n}\rho_i\right]^{\frac{2}{n}}.$$

Therefore,

$$\sum_{i=1}^{n}|\rho_i| \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}}.$$

We know that

$$|\rho_i| - \left|\frac{2m}{n}\right| \leq \left|\rho_i - \frac{2m}{n}\right|, \quad \forall\ i \tag{2.3}$$

i.e.,

$$|\rho_i| - \frac{2m}{n} \leq \left|\rho_i - \frac{2m}{n}\right|, \quad \forall\ i,$$

$$\sum_{i=1}^{n}|\rho_i| - \sum_{i=1}^{n}\frac{2m}{n} \leq \sum_{i=1}^{n}\left|\rho_i - \frac{2m}{n}\right|$$

i.e.,

$$\sum_{i=1}^{n}|\rho_i| - 2m \leq LRE_C(G)$$

i.e.,

$$LRE_C(G) \geq \sum_{i=1}^{n}|\rho_i| - 2m \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m \qquad \text{(from (2.3))}$$

Therefore,

$$LRE_C(G) \geq \sqrt{2M + n(n-1)D^{\frac{2}{n}}} - 2m.$$

□

Theorem 2.12. *If $\rho_1(G)$ is the largest minimum covering Randić eigen value of $LR_C(G)$, then $\rho_1(G) \geq \frac{2m-|C|-R(G)}{n}$.*

Proof. For any nonzero vector X, we have by [3],

$$\rho_1(A) = \max_{X\neq 0}\left\{\frac{X'AX}{X'X}\right\}.$$

Therefore,

$$\rho_1(G) \geq \frac{J'AJ}{J'J} = \frac{2m - |C| - 2\sum i<j\frac{1}{\sqrt{d_id_j}}}{n} = \frac{2m - |C| - R(G)}{n},$$

where J is a unit column matrix. □

Just like Koolen and Moulton's [26] upper bound for energy of a graph, an upper bound for $LRE_C(G)$ is given in the following theorem.

Theorem 2.13. *If G is a graph with n vertices and m edges and $\left(|C| + 2\sum_{i<j}\frac{1}{d_id_j}\right) \geq n$ then*

$$LRE_C(G) \leq \frac{2M}{n} + \sqrt{(n-1)\left[2M - \left(\frac{|C| + 2\sum_{i<j}\frac{1}{d_id_j}}{n}\right)^2\right]}.$$

Proof. Cauchy-Schwartz inequality is

$$\left[\sum_{i=2}^{n} a_i b_i\right]^2 \le \left(\sum_{i=2}^{n} a_i^2\right)\left(\sum_{i=2}^{n} b_i^2\right).$$

Put $a_i = 1$, $b_i = |\rho_i|$, then

$$\left(\sum_{i=2}^{n} |\rho_i|\right)^2 = \sum_{i=2}^{n} 1 \sum_{i=2}^{n} |\rho_i|^2$$

$$\Rightarrow \quad [LRE_C(G) - \rho_1]^2 \le (n-1)(2M - \rho_1^2)$$

$$\Rightarrow \quad LRE_C(G) \le \rho_1 + \sqrt{(n-1)(2M - \rho_1^2)}$$

Let $f(x) = x + \sqrt{(n-1)(2M - x^2)}$.

For decreasing function

$$f'(x) \le 0 \quad \Rightarrow \quad 1 - \frac{x(n-1)}{\sqrt{(n-1)\left(|C| + 2\sum_{i<j} \frac{1}{d_i d_j} - x^2\right)}} \le 0$$

$$\Rightarrow \quad x \ge \sqrt{\frac{2M}{n}}$$

Since $(2M) \ge n$, we have $\sqrt{\frac{2M}{n}} \le \frac{2M}{n} \le \rho_1$.

Therefore

$$f(\rho_1) \le f\left(\frac{2M}{n}\right)$$

i.e.,

$$LRE_C(G) \le f(\rho_1) \le f\left(\frac{2M}{n}\right)$$

i.e.,

$$LRE_C(G) \le f\left(\frac{2M}{n}\right)$$

i.e.,

$$LRE_C(G) \le \frac{2M}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2M}{n}\right)^2\right]}. \qquad \square$$

Milovanović [24] bounds for Laplacian minimum covering Randić energy of a graph are given in the following theorem.

Theorem 2.14. *Let G be a graph with n vertices and m edges. Let $|\rho_1| \ge |\rho_2| \ge \ldots \ge |\rho_n|$ be a non-increasing order of eigenvalues of $LR_C(G)$. If C is minimum covering set then*

$$LRE_C(G) \ge \sqrt{2nM - \alpha(n)(|\rho_1| - |\rho_n|)^2 - 2m},$$

where $\alpha(n)= n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$, $[x]$ *denotes greatest integer part of real number and*

$$M = \frac{1}{2}\sum_{i=1}^{n}(d_i - c_i)^2 + \sum_{i<j} \frac{1}{d_i d_j},$$

where $c_i = \begin{cases} 1 & if \ v_i \in C \\ 0 & if \ v_i \notin C. \end{cases}$

Proof. Let $a, a_1, a_2, \dots, a_n, A$ and $b, b_1, b_2, \dots, b_n, B$ be real numbers such that $a \le a_i \le A$ and $b \le b_i \le B \ \forall \ i = 1, 2, \dots, n$ then the following inequality is valid.

$$\left| n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le \alpha(n)(A-a)(B-b).$$

If $a_i = |\rho_i|$, $b_i = |\rho_i|$, $a = b = |\rho_n|$ and $A = b = |\rho_1|$

$$\left| n \sum_{i=1}^{n} |\rho_i|^2 - \left(\sum_{i=1}^{n} |\rho_i| \right)^2 \right| \le \alpha(n)(|\rho_1| - |\rho_n|)^2.$$

But

$$\left(\sum_{i=1}^{n} |\rho_i| \right)^2 \le 2nM$$

$$\Rightarrow \quad n2M - \left(\sum_{i=1}^{n} |\rho_i| \right)^2 \le \alpha(n)(|\rho_1| - |\rho_n|)^2$$

$$\left(\sum_{i=1}^{n} |\rho_i| \right) \ge \sqrt{2Mn - \alpha(n)(|\rho_1| - |\rho_n|)^2}$$

Since

$$LRE_C(G) = \sum_{i=1}^{n} \left| \rho_i - \frac{2m}{n} \right| \ge \sum_{i=1}^{n} \left(|\rho_i| - \left| \frac{2m}{n} \right| \right).$$

Hence

$$LRE_C(G) \ge \sqrt{2nM - \alpha(n)(|\rho_1| - |\rho_n|)^2} - 2m. \qquad \square$$

Theorem 2.15. *Let G be a graph with n vertices and m edges. Let $|\rho_1| \ge |\rho_2| \ge \dots \ge |\rho_n| > 0$ be a non-increasing order of eigenvalues of $LR_C(G)$ and C is minimum covering set then*

$$LRE_C(G) \ge \frac{2M + n|\rho_1||\rho_n|}{(|\rho_1| + |\rho_n|)} - 2m$$

where $M = \frac{1}{2} \sum_{i=1}^{n} (d_i - c_i)^2 + \sum_{i<j} \frac{1}{d_i d_j}$. *Here* $c_i = \begin{cases} 1 & if \ v_i \in C \\ 0 & if \ v_i \notin C. \end{cases}$

Proof. Let $a_i \ne 0$, b_i, r and R be real numbers satisfying $ra_i \le b_i \le Ra_i$, then we have the following inequality

$$\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i \le (r+R) \sum_{i=1}^{n} a_i b_i.$$

Put $b_i = |\rho_i|$, $a_i = 1$, $r = |\rho_n|$ and $R = |\rho_1|$

$$\sum_{i=1}^{n} |\rho_i|^2 + |\rho_1| \, |\rho_n| \sum_{i=1}^{n} 1 \le (|\rho_1| + |\rho_n|) \sum_{i=1}^{n} |\rho_i|$$

i.e.,

$$2M + |\rho_1| \, |\rho_n| n \le (|\rho_1| + |\rho_n|) \sum_{i=1}^{n} |\rho_i|$$

$$\Rightarrow \quad \sum_{i=1}^{n}|\rho_i| \geq \frac{2M+n|\rho_1||\rho_n|}{(|\rho_1|+|\rho_n|)}$$

We know that

$$LRE_C(G)=\sum_{i=1}^{n}\left|\rho_i-\frac{2m}{n}\right|.$$

Therefore

$$LRE_C(G) \geq \sum_{i=1}^{n}|\rho_i| - \left|\frac{2m}{n}\right|$$

$$\Rightarrow \quad LRE_C(G) \geq \frac{2M+n|\rho_1||\rho_n|}{(|\rho_1|+|\rho_n|)} - 2m. \qquad \square$$

3. Conclusions

It was proved in this article that the minimum covering Randić energy of a graph G depends on the covering set that we take for consideration. Upper and lower bounds for minimum covering Randić energy are established. A generalized expression for minimum covering Randić energies for star graph, complete graph, thorn graph of complete graph, crown graph, complete bipartite graph, cocktail party graph and friendship graphs are also computed.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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