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Research Article

The New Approach for Some Coupled Fixed Point Results

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Abstract. The aim of this paper is to stand for new approach of the results of Jain et al. [14] and Berinde [4]. The proofs of all our results are without mixed monotone property. An example are also given to confirm the effectiveness of the presented work.

Keywords. Coupled coincidence point; Generalized compatibility; Mixed monotone property; Ordered set

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1. Introduction and Some Important Definitions

In the sequel, let K be a non-empty set. Throughout this paper, we use indifferently the notation K^2 to denote the product space $K \times K$. $\preccurlyeq$ will denote a partial order on K, and η will be a metric on K.

The existence of fixed points of nonlinear contraction mappings in metric spaces endowed with a partial ordering has been considered recently by Ran and Reurings [17] to obtain a solution of a matrix equation in 2004. Fixed point theorems in partially ordered metric spaces have been studied by some authors since 2004 (see [1], [3], [8], [10], [15], [16], [18]). Nietto and Lopez [15] extended the results in [17] by removing the continuity condition of the mapping. They applied their results to get a solution of a boundary value problem. The efficiency of these kind of extensions of fixed point theorems in such kind of problems, as it is well known, is due to the fact that most real valued function spaces are partially ordered metric spaces.

The work of coupled fixed points in partially ordered metric spaces was initiated by Guo and Lakshmikantham [9]. Subsequently, Bhaskar and Lakshmikantham [5] introduced the concept of the mixed monotone property as follows.

Definition 1 ([5]). Let $(K,\preccurlyeq)$ be a partially ordered set and $F:K^2\to K$ be a mapping. Then a map F is said to have the mixed monotone property if $F(a,b)$ is monotone nondecreasing in a and is monotone non-increasing in b; that is, for any a, $b\in K$, a_1, $a_2\in K$, $a_1\preccurlyeq a_2$ implies $F(a_1,b)\preccurlyeq F(a_2,b)$ and b_1, $b_2\in K$, $b_1\preccurlyeq b_2$ implies $F(a,b_1)\succcurlyeq F(a,b_2)$.

Definition 2 ([5]). An element $(a,b)\in K^2$ is said to be a coupled fixed point of the mapping $F:K^2\to K$ if $F(a,b)=a$ and $F(b,a)=b$.

Lakshmikantham and Ćirić [12] introduced concepts of a mixed g-monotone mapping and a coupled coincidence point.

Definition 3 ([12]). Let $(K,\preccurlyeq)$ be a partially ordered set and $F:K^2\to K$ and $g:K\to K$. We say F has the mixed g-monotone property if for any a, $b\in K$, a_1, $a_2\in K$, $ga_1\preccurlyeq ga_2$ implies $F(a_1,b)\preccurlyeq F(a_2,b)$ and b_1, $b_2\in K$, $gb_1\preccurlyeq gb_2$ implies $F(a,b_1)\succcurlyeq F(a,b_2)$.

Definition 4 ([12]). An element $(a,b)\in K^2$ is said to be a coupled coincidence point of a mapping $F:K^2\to K$ and $g:K\to K$ if $F(a,b)=ga$ and $F(b,a)=gb$.

Definition 5 ([12]). Let K be a nonempty set and $F:K^2\to K$ and $g:K\to K$. We say F and g are commutative if $gF(a,b)=F(ga,gb)$ for all a, $b\in K$.

Definition 6 ([7]). Let (K,η) be a metric space, $F:K^2\to K$ a mapping and g a self mapping on K. A hybrid pair F, g is compatible if $\eta(g(F(a_n,b_n)),F(ga_n,gb_n))\to 0$ as $n\to\infty$ and $\eta(g(F(b_n,a_n)),F(gb_n,ga_n))\to 0$ as $n\to\infty$ whenever (a_n) and (b_n) are sequences in K such that $\lim\limits_{n\to\infty}F(a_n,b_n)=\lim\limits_{n\to\infty}ga_n=a$, $\lim\limits_{n\to\infty}F(b_n,a_n)=\lim\limits_{n\to\infty}gb_n=b$ with a, $b\in K$.

Definition 7 ([4]). Let Φ denote the class of functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfying

(φi) φ is continuous and (strictly) increasing;

(φii) $\varphi(t)<t$ for all $t>0$;

(φiii) $\varphi(t+s)\le\varphi(t)+\varphi(s)$ for all t, $s\in[0,\infty)$.

Note that by (φi) and (φii), we have $\varphi(t)=0$ if and only if $t=0$.

Definition 8 ([13]). Let Ψ denote the class of functions $\psi:[0,\infty)\to[0,\infty)$ which satisfying

(ψi) $\lim\limits_{t\to r}\psi(t)>0$ for all $r>0$ and $\lim\limits_{t\to 0_+}\psi(t)=0$.

Theorem 1 ([14]). Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Let $F: K^2 \to K$ and $g: K \to K$ be two maps with F having the mixed g-monotone property on K such that there exists two elements a_0, $b_0 \in K$ with $ga_0 \preccurlyeq F(a_0,b_0)$ and $gb_0 \succcurlyeq F(b_0,a_0)$. Suppose there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right) \leq \varphi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right)-\psi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right) \tag{1.1}$$

for all a, b, c, $d \in K$ with $ga \succcurlyeq gc$ and $gb \preccurlyeq gd$.

Suppose $F(K^2) \subseteq g(K)$, g is continuous and the pair (F,g) is compatible.

Also suppose either

(a) F is continuous or;

(b) K has the following properties:

(b1) if a non-decreasing sequence $\{a_n\} \to a$, then $ga_n \preccurlyeq ga$ for all n,

(b2) if a non-increasing sequence $\{b_n\} \to b$, then $gb \preccurlyeq b_n$ for all n.

Then there exists a, $b \in K$ such that $ga = F(a,b)$ and $gb = F(b,a)$, that is, F and g have a coupled coincidence point in K.

The concept of G-increasing and $\{F,G\}$ generalized compatiblity has been established by Hussain et al. [11].

Definition 9 ([11]). Suppose that F, $G: K^2 \to K$ are two mappings. F is said to be G-increasing with respect to $\preccurlyeq$ if for all a, b, c, $d \in K$, with $G(a,b) \preccurlyeq G(c,d)$ we have $F(a,b) \preccurlyeq F(c,d)$.

Definition 10 ([11]). An element $(a,b) \in K^2$ is said to be a coupled coincidence point of a mappings F, $G: K^2 \to K$ if $F(a,b) = G(a,b)$ and $F(b,a) = G(b,a)$.

Definition 11 ([11]). Let F, $G: K^2 \to K$. We say that pair $\{F,G\}$ is generalized compatible if

$$\begin{cases} \lim_{n\to\infty} \eta(F(G(a_n,b_n),G(b_n,a_n)),G(F(a_n,b_n),F(b_n,a_n))) = 0 \\ \lim_{n\to\infty} \eta(F(G(b_n,a_n),G(a_n,b_n)),G(F(b_n,a_n),F(a_n,b_n))) = 0, \end{cases}$$

whenever (a_n) and (b_n) are sequences in K such that

$$\begin{cases} \lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} G(a_n,b_n) = x_1, \\ \lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} G(b_n,a_n) = x_2. \end{cases}$$

Definition 12 ([11]). Let F, $G: K^2 \to K$ be two maps. We say that the pair $\{F,G\}$ is commuting if $F(G(a,b),G(b,a)) = G(F(a,b),F(b,a))$ for all a, $b \in K$.

Remark 1 ([11]). A commuting pair is a generalized compatible but not conversely in general.

The aim of this paper is to stand for new approach of the results of Jain et al. [14] and Berinde [4]. We utilize generalized compatibility of a pair $\{F,G\}$, of mapping F, $G: K^2 \to K$ to obtain coupled coincidence point results for such a pair of mappings involving (φ,ψ) contractive condition without mixed G-monotone property of F. Thus, the derived coupled fixed point results do not have the mixed monotone property of F. An example are also given to confirm the effectiveness of the presented work.

2. Main Results

Theorem 2. *Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Assume that F, $G: K^2 \to K$ are two generalized compatible mappings such that F is G-increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exists two elements a_0, $b_0 \in K$ with*

$$G(a_0,b_0) \preccurlyeq F(a_0,b_0) \quad \text{and} \quad G(b_0,a_0) \succcurlyeq F(b_0,a_0).$$

Suppose there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\begin{aligned}\varphi&\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right)\\ &\le \varphi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(b,a),G(d,c))}{2}\right)\\ &\quad -\psi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(b,a),G(d,c))}{2}\right)\end{aligned} \tag{2.1}$$

for all a, b, c, $d \in K$ with $G(a,b) \preccurlyeq G(c,d)$ and $G(b,a) \succcurlyeq G(d,c)$. Suppose that for any a, $b \in K$, there exists c, $d \in K$ such that

$$\begin{cases} F(a,b) = G(c,d), \\ F(b,a) = G(d,c). \end{cases} \tag{2.2}$$

Also suppose that either

(a) *F is continuous or;*

(b) *K has the following properties:*

  (b1) *if a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preccurlyeq a$ for all n,*

  (b2) *if a non-increasing sequence $\{b_n\} \to b$, then $b \preccurlyeq b_n$ for all n.*

Then F and G have a coupled coincidence point in K.

Proof. Let a_0, $b_0 \in K$ be such that $G(a_0,b_0) \preccurlyeq F(a_0,b_0)$ and $G(b_0,a_0) \succcurlyeq F(b_0,a_0)$. By (2.2), there exists $(a_1,b_1) \in K^2$ such that $F(a_0,b_0) = G(a_1,b_1)$ and $F(b_0,a_0) = G(b_1,a_1)$. Continuing in the same way, we can easly construct sequences $\{a_n\}$ and $\{b_n\}$ in K such that

$$F(a_n,b_n) = G(a_{n+1},b_{n+1}) \text{ and } F(b_n,a_n) = G(b_{n+1},a_{n+1}) \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

First, we show that for all $n \in \mathbb{N}$, we have

$$G(a_n,b_n) \preccurlyeq G(a_{n+1},b_{n+1}) \text{ and } G(b_{n+1},a_{n+1}) \preccurlyeq G(b_n,a_n). \tag{2.4}$$

Since $G(a_0,b_0) \preccurlyeq F(a_0,b_0)$ and $F(b_0,a_0) \preccurlyeq G(b_0,a_0)$ and since $F(a_0,b_0) = G(a_1,b_1)$ and $F(b_0,a_0) = G(b_1,a_1)$, we get $G(a_0,b_0) \preccurlyeq G(a_1,b_1)$ and $G(b_1,a_1) \preccurlyeq G(b_0,a_0)$. Hence (2.4) holds for $n=0$. Assume that (2.4) holds for some fixed $n \in \mathbb{N}$. As F is G-increasing with respect to $\preccurlyeq$, we get

$$G(a_{n+1},b_{n+1}) = F(a_n,b_n) \preccurlyeq F(a_{n+1},b_{n+1}) = G(a_{n+2},b_{n+2})$$

and

$$G(b_{n+2},a_{n+2}) = F(b_{n+1},a_{n+1}) \preccurlyeq F(b_n,a_n) = G(b_{n+1},a_{n+1}).$$

Thus (2.4) holds for all $n \in \mathbb{N}$. Denote

$$t_n = \frac{\eta(G(a_n,b_n),G(a_{n+1},b_{n+1})) + \eta(G(b_n,a_n),G(b_{n+1},a_{n+1}))}{2}$$

for all $n \in \mathbb{N}$. Since $G(a_n,b_n) \preccurlyeq G(a_{n+1},b_{n+1})$ and $G(b_n,a_n) \succcurlyeq G(b_{n+1},a_{n+1})$, from (2.1) and (2.3), we have

$$\begin{aligned}
&\varphi\left(\frac{\eta(G(a_{n+1},b_{n+1}),G(a_{n+2},b_{n+2})) + \eta(G(b_{n+1},a_{n+1}),G(b_{n+2},a_{n+2}))}{2}\right)\\
&\quad = \varphi\left(\frac{\eta(F(a_n,b_n),F(a_{n+1},b_{n+1})) + \eta(F(b_n,a_n),F(b_{n+1},a_{n+1}))}{2}\right)\\
&\quad \le \varphi\left(\frac{\eta(G(a_n,b_n),G(a_{n+1},b_{n+1})) + \eta(G(b_n,a_n),G(b_{n+1},a_{n+1}))}{2}\right)\\
&\qquad - \psi\left(\frac{\eta(G(a_n,b_n),G(a_{n+1},b_{n+1})) + \eta(G(b_n,a_n),G(b_{n+1},a_{n+1}))}{2}\right).
\end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned}
&\varphi\left(\frac{\eta(G(b_{n+2},a_{n+2}),G(b_{n+1},a_{n+1})) + \eta(G(a_{n+2},b_{n+2}),G(a_{n+1},b_{n+1}))}{2}\right)\\
&\quad = \varphi\left(\frac{\eta(F(b_{n+1},a_{n+1}),F(b_n,a_n)) + \eta(F(a_{n+1},b_{n+1}),F(a_n,b_n))}{2}\right)\\
&\quad \le \varphi\left(\frac{\eta(G(b_{n+1},a_{n+1}),G(b_n,a_n)) + \eta(G(a_{n+1},b_{n+1}),G(a_n,b_n))}{2}\right)\\
&\qquad - \psi\left(\frac{\eta(G(b_{n+1},a_{n+1}),G(b_n,a_n)) + \eta(G(a_{n+1},b_{n+1}),G(a_n,b_n))}{2}\right).
\end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6), we have

$$2\varphi(t_{n+1}) \le 2\varphi(t_n) - 2\psi(t_n)$$

or equivalently,

$$\varphi(t_{n+1}) \le \varphi(t_n) - \psi(t_n). \tag{2.7}$$

Since ψ is non-negative and the monotonicity of φ, it follows that the sequence $\{t_n\}$ is monotone decreasing. Hence, there is some $t \ge 0$ such that $\lim_{n\to\infty} t_n = t$. We claim that $t = 0$. Suppose, to the contrary, that $t > 0$. Taking limit as $n \to \infty$ on both sides of (2.7) and using the property of φ

and ψ, we have

$$\begin{aligned}\varphi(t)&=\lim_{n\to+\infty}\varphi(t_n)\le\lim_{n\to+\infty}\left[\varphi(t_{n-1})-\psi(t_{n-1})\right]\\&=\varphi(t)-\lim_{t_{n-1}\to t}\psi(t_{n-1})<\varphi(t)\end{aligned}$$

a contradiction. Hence $t=0$, that is

$$\lim_{n\to+\infty}\left[\frac{\eta(G(a_n,b_n),G(a_{n+1},b_{n+1}))+\eta(G(b_n,a_n),G(b_{n+1},a_{n+1}))}{2}\right]=\lim_{n\to+\infty}\varphi(t_n)=0. \tag{2.8}$$

Now, we show that $(G(a_n,b_n),G(b_n,a_n))$ is Cauchy sequence in K^2 endowed with the metric Υ defined by

$$\Upsilon((a,b),(c,d))=\eta(a,c)+\eta(b,d) \tag{2.9}$$

for all (a,b), $(c,d)\in K^2$. If $(G(a_n,b_n),G(b_n,a_n))$ is not a Cauchy sequence in $\left(K^2,\Upsilon\right)$. Then there exists $\varepsilon>0$ for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integer k with $n(k)>m(k)>k$, we have

$$\begin{cases}\dfrac{\Upsilon\left(\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(b_{m(k)},a_{m(k)}\right)\right),\left(G\left(a_{n(k)},b_{n(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)\right)}{2}>\varepsilon,\\[2ex]\dfrac{\Upsilon\left(\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(b_{m(k)},a_{m(k)}\right)\right),\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)\right)}{2}\le\varepsilon.\end{cases} \tag{2.10}$$

By (2.9), we get

$$\eta_k=\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}>\varepsilon \tag{2.11}$$

and

$$\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)}{2}\le\varepsilon. \tag{2.12}$$

By (2.11), (2.12) and using triangle inequality, for $k\ge 0$, we have

$$\begin{aligned}\varepsilon&<\eta_k\\&\le\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)-1},b_{n(k)-1}\right)\right)+\eta\left(G\left(a_{n(k)-1},b_{n(k)-1}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)}{2}\\&\quad+\frac{\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)-1},a_{n(k)-1}\right)\right)+\eta\left(G\left(b_{n(k)-1},a_{n(k)-1}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}\\&\le\varepsilon+t_{n(k)-1}.\end{aligned}$$

Letting $k\to\infty$ and using (2.8) in the last inequality, we obtain

$$\lim_{n\to\infty}\eta_k=\varepsilon. \tag{2.13}$$

Again, from the triangle inequality,

$$\begin{aligned}\eta_k&=\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}\\&\le\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)+1},b_{m(k)+1}\right)\right)+\eta\left(G\left(a_{m(k)+1},b_{m(k)+1}\right),G\left(a_{n(k)+1},b_{n(k)+1}\right)\right)}{2}\end{aligned}$$

$$+\frac{\eta\left(G\left(a_{n(k)+1},b_{n(k)+1}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)+1},a_{m(k)+1}\right)\right)}{2}$$

$$+\frac{\eta\left(G\left(b_{m(k)+1},a_{m(k)+1}\right),G\left(b_{n(k)+1},a_{n(k)+1}\right)\right)+\eta\left(G\left(b_{n(k)+1},a_{n(k)+1}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}$$

$$=\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{m(k)+1},b_{m(k)+1}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{m(k)+1},a_{m(k)+1}\right)\right)}{2}$$

$$+\frac{\eta\left(G\left(a_{m(k)+1},b_{m(k)+1}\right),G\left(a_{n(k)+1},b_{n(k)+1}\right)\right)+\eta\left(G\left(b_{m(k)+1},a_{m(k)+1}\right),G\left(b_{n(k)+1},a_{n(k)+1}\right)\right)}{2}$$

$$+\frac{\eta\left(G\left(a_{n(k)+1},b_{n(k)+1}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{n(k)+1},a_{n(k)+1}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}$$

$$=t_{m(k)}+t_{n(k)}$$

$$+\frac{\eta\left(G\left(a_{m(k)+1},b_{m(k)+1}\right),G\left(a_{n(k)+1},b_{n(k)+1}\right)\right)+\eta\left(G\left(b_{m(k)+1},a_{m(k)+1}\right),G\left(b_{n(k)+1},a_{n(k)+1}\right)\right)}{2}.$$

From the monotonicity of φ and (φiii), we get

$$\varphi\left(\eta_k\right)\le\varphi\left(t_{m(k)}\right)+\varphi\left(t_{n(k)}\right)+\varphi\left(\begin{array}{c}\dfrac{\eta\left(G\left(a_{m(k)+1},b_{m(k)+1}\right),G\left(a_{n(k)+1},b_{n(k)+1}\right)\right)}{2}\\ +\dfrac{\eta\left(G\left(b_{m(k)+1},a_{m(k)+1}\right),G\left(b_{n(k)+1},a_{n(k)+1}\right)\right)}{2}\end{array}\right). \tag{2.14}$$

Then by (2.1) and (2.4), we have

$$\begin{aligned}
&\varphi\left(\frac{\eta\left(G\left(a_{m(k)+1},b_{m(k)+1}\right),G\left(a_{n(k)+1},b_{n(k)+1}\right)\right)+\eta\left(G\left(b_{m(k)+1},a_{m(k)+1}\right),G\left(b_{n(k)+1},a_{n(k)+1}\right)\right)}{2}\right)\\
&=\varphi\left(\frac{\eta\left(F\left(a_{m(k)},b_{m(k)}\right),F\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(F\left(b_{m(k)},a_{m(k)}\right),F\left(b_{n(k)},a_{n(k)}\right)\right)}{2}\right)\\
&\le\varphi\left(\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}\right)\\
&\quad-\psi\left(\frac{\eta\left(G\left(a_{m(k)},b_{m(k)}\right),G\left(a_{n(k)},b_{n(k)}\right)\right)+\eta\left(G\left(b_{m(k)},a_{m(k)}\right),G\left(b_{n(k)},a_{n(k)}\right)\right)}{2}\right)\\
&=\varphi\left(\eta_k\right)-\psi\left(\eta_k\right).
\end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we have

$$\varphi\left(\eta_k\right)\le\varphi\left(t_{m(k)}\right)+\varphi\left(t_{n(k)}\right)+\varphi\left(\eta_k\right)-\psi\left(\eta_k\right). \tag{2.16}$$

Letting $k\to\infty$ and by (2.8), (2.13) and the properties of φ and ψ in (2.16), we get

$$\begin{aligned}
\varphi(\varepsilon)&\le\varphi(0)+\varphi(0)+\varphi(\varepsilon)-\lim_{k\to\infty}\psi\left(\eta_k\right)\\
&=\varphi(\varepsilon)-\lim_{\eta_k\to\varepsilon}\psi\left(\eta_k\right)<\varphi(\varepsilon)
\end{aligned}$$

which is a contradiction. Therefore, $(G(a_n,b_n),G(b_n,a_n))$ is Cauchy sequence in $\left(K^2,\Upsilon\right)$ which implies that $(G(a_n,b_n))$ and $(G(b_n,a_n))$ are Cauchy sequence in (K,η). Since K is a complete metric space, there exists a, $b \in K$ such that

$$\lim_{n\to\infty} G(a_n,b_n) = \lim_{n\to\infty} F(a_n,b_n) = a, \quad \lim_{n\to\infty} G(b_n,a_n) = \lim_{n\to\infty} F(b_n,a_n) = b. \tag{2.17}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility, by (2.17), we have

$$\eta(F(G(a_n,b_n),G(b_n,a_n)),G(F(a_n,b_n),F(b_n,a_n))) \to 0 \text{ as } n\to\infty \tag{2.18}$$

$$\eta(F(G(b_n,a_n),G(a_n,b_n)),G(F(b_n,a_n),F(a_n,b_n))) \to 0 \text{ as } n\to\infty. \tag{2.19}$$

Suppose the assumption (a) holds. For all $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\eta(G(a,b),F(G(a_n,b_n),G(b_n,a_n))) \\
&\quad \le \eta(G(a,b),G(F(a_n,b_n),F(b_n,a_n))) \\
&\qquad + \eta(G(F(a_n,b_n),F(b_n,a_n)),F(G(a_n,b_n),G(b_n,a_n))).
\end{aligned}$$

Letting $n\to\infty$, by (2.17) and (2.19), and since F and G are continuous, we have

$$G(a,b) = F(a,b). \tag{2.20}$$

Similarly, we show that

$$G(b,a) = F(b,a). \tag{2.21}$$

Hence (a,b) is a coupled coincidence point of F and G.

Next, suppose the assumption (b) holds. From (2.4) and (2.17), we obtain $(G(a_n,b_n))$ is non-decreasing sequence, $G(a_n,b_n) \to a$ as $n\to\infty$ and $(G(b_n,a_n))$ is non-increasing sequence, $G(b_n,a_n) \to b$ as $n\to\infty$. Hence, we obtain

$$G(a_n,b_n) \preccurlyeq a \text{ and } G(b_n,a_n) \succcurlyeq b. \tag{2.22}$$

Since the pair $\{F,G\}$ satisfies the generalized compatibility and G is continuous, from (2.19), we get

$$\begin{aligned}
\lim_{n\to\infty} G(G(a_n,b_n),G(b_n,a_n)) &= G(a,b) \\
&= \lim_{n\to\infty} G(F(a_n,b_n),F(b_n,a_n)) \\
&= \lim_{n\to\infty} F(G(a_n,b_n),G(b_n,a_n))
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
\lim_{n\to\infty} G(G(b_n,a_n),G(a_n,b_n)) &= G(b,a) \\
&= \lim_{n\to\infty} G(F(b_n,a_n),F(a_n,b_n)) \\
&= \lim_{n\to\infty} F(G(b_n,a_n),G(a_n,b_n)).
\end{aligned} \tag{2.24}$$

Next, we have

$$\begin{aligned}
\eta(G(a,b),F(a,b)) &\le \lim_{n\to\infty} \eta(G(F(a_n,b_n),F(b_n,a_n)),F(a,b)) \\
&= \lim_{n\to\infty} \eta(F(G(a_n,b_n),G(b_n,a_n)),F(a,b)).
\end{aligned}$$

Since G has the mixed monotone property, it follows from (2.22) that $G(G(a_n,b_n),G(b_n,a_n)) \preccurlyeq G(a,b)$ and $G(G(b_n,a_n),G(a_n,b_n)) \succcurlyeq G(b,a)$. By (2.1), (2.23) and (2.24), we have

$$\begin{aligned}
&\varphi\left(\frac{\eta\big(F(a,b),G(a,b)+\eta(F(b,a),G(b,a))\big)}{2}\right)\\
&\le \lim_{n\to\infty}\varphi\left(\frac{\eta\big(F(a,b),F(G(a_n,b_n),G(b_n,a_n))+\eta(F(b,a),F(G(b_n,a_n),G(a_n,b_n)))\big)}{2}\right)\\
&\le \lim_{n\to\infty}\varphi\left(\frac{\eta\big(G(a,b),G(G(a_n,b_n),G(b_n,a_n))+\eta(G(b,a),G(G(b_n,a_n),G(a_n,b_n)))\big)}{2}\right)\\
&\quad -\lim_{n\to\infty}\psi\left(\frac{\eta\big(G(a,b),G(G(a_n,b_n),G(b_n,a_n))+\eta(G(b,a),G(G(b_n,a_n),G(a_n,b_n)))\big)}{2}\right).
\end{aligned}$$

By $\lim\limits_{t\to 0+}\psi(t)=0$, we have

$$\begin{aligned}
&\varphi\left(\frac{\eta\big(F(a,b),G(a,b)+\eta(F(b,a),G(b,a))\big)}{2}\right)\\
&\le \lim_{n\to\infty}\varphi\left(\frac{\eta\big(G(a,b),G(G(a_n,b_n),G(b_n,a_n))+\eta(G(b,a),G(G(b_n,a_n),G(a_n,b_n)))\big)}{2}\right)\\
&=\varphi(0)=0.
\end{aligned}$$

As φ is nonnegative and $\varphi(0)=0$, we get

$$\eta(F(a,b),G(a,b))=0 \quad \text{and} \quad \eta(F(b,a),G(b,a))=0;$$

that is;

$$F(a,b)=G(a,b) \quad \text{and} \quad F(b,a)=G(b,a).$$

This completes the proof. □

The commuting maps $\{F,G\}$ are generalized compatible. Therefore, we have the following Corollary.

Corollary 1. *Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Assume that F, $G:K^2\to K$ are two commuting mappings such that F is $G-$increasing with respect to $\preccurlyeq$, G is continuous and has the mixed monotone property, and there exists two elements a_0, $b_0\in K$ with*

$$G(a_0,b_0)\preccurlyeq F(a_0,b_0) \quad \textit{and} \quad G(b_0,a_0)\succcurlyeq F(b_0,a_0).$$

Suppose that the inequalities (2.1) *and* (2.2) *hold and either*

(a) *F is continuous or;*

(b) *K has the following properties:*

  (b1) *if a non-decreasing sequence $\{a_n\}\to a$, then $a_n\preccurlyeq a$ for all n,*

  (b2) *if a non-increasing sequence $\{b_n\}\to b$, then $b\preccurlyeq b_n$ for all n.*

Then F and G have a coupled coincidence point in K.

Definition 13 ([11]). Let $(K,\preccurlyeq)$ be a partially ordered set and $F:K^2\to K$ and $g:K\to K$. We say that F is g-increasing with respect to $\preccurlyeq$ if for any a, $b\in K$,

$$ga_1\preccurlyeq ga_2 \quad \text{implies} \quad F(a_1,b)\preccurlyeq F(a_2,b)$$

and

$$gb_1\preccurlyeq gb_2 \quad \text{implies} \quad F(a,b_1)\preccurlyeq F(a,b_2).$$

Next, we infer an analogous result to Theorem 2.1 of Jain et al. [14, Theorem 1] without $g-$mixed monotone property of F as follows.

Corollary 2. *Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Let $F:K^2\to K$ and $g:K\to K$ be two maps with F is $g-$increasing with respect to $\preccurlyeq$, and there exists $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\begin{aligned}\varphi&\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right)\\ &\le\varphi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right)-\psi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right)\end{aligned}$$

for all a, b, c, $d\in K$ with $ga\preccurlyeq gc$ and $gb\succcurlyeq gd$. Suppose $F\left(K^2\right)\subseteq g(K)$, g is continuous and monotone increasing with respect to $\preccurlyeq$, and the pair $\{F,g\}$ is compatible. Also suppose that either

(a) *F is continuous or;*

(b) *K has the following properties:*

  (b1) *if a non-decreasing sequence $\{a_n\}\to a$, then $ga_n\preccurlyeq ga$ for all n,*

  (b2) *if a non-increasing sequence $\{b_n\}\to b$, then $gb\preccurlyeq gb_n$ for all n.*

If there exists two elements a_0, $b_0\in K$ with $ga_0\preccurlyeq F(a_0,b_0)$ and $gb_0\succcurlyeq F(b_0,a_0)$. F and g have a coupled coincidence point in K.

Corollary 3. *Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Let $F:K^2\to K$ and $g:K\to K$ be two maps with F is $g-$increasing with respect to $\preccurlyeq$, and there exists $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\begin{aligned}\varphi&\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right)\\ &\le\varphi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right)-\psi\left(\frac{\eta(ga,gc)+\eta(gb,gd)}{2}\right)\end{aligned}$$

for all a, b, c, $d\in K$ with $ga\preccurlyeq gc$ and $gb\succcurlyeq gd$. Suppose $F\left(K^2\right)\subseteq g(K)$, g is continuous and monotone increasing with respect to $\preccurlyeq$, and the pair $\{F,g\}$ is commuting. Also suppose that either

(a) *F is continuous or;*

(b) *K has the following properties:*

  (b1) *if a non-decreasing sequence $\{a_n\}\to a$, then $ga_n\preccurlyeq ga$ for all n,*

  (b2) *if a non-increasing sequence $\{b_n\}\to b$, then $gb\preccurlyeq gb_n$ for all n.*

If there exists two elements a_0, $b_0 \in K$ with $ga_0 \preccurlyeq F(a_0,b_0)$ and $gb_0 \succcurlyeq F(b_0,a_0)$. F and g have a coupled coincidence point.

Definition 14 ([11])**.** Let $(K,\preccurlyeq)$ be a partially ordered set and $F: K^2 \to K$. We say that F is increasing with respect to $\preccurlyeq$ if for any a, $b \in K$,

$$a_1 \preccurlyeq a_2 \quad \text{implies} \quad F(a_1,b) \preccurlyeq F(a_2,b)$$

and

$$b_1 \preccurlyeq b_2 \quad \text{implies} \quad F(a,b_1) \preccurlyeq F(a,b_2).$$

Corollary 4 ([4])**.** *Let $(K,\preccurlyeq)$ be a partially ordered set, and suppose there is a metric η on K such that (K,η) is a complete metric space. Let $F: K^2 \to K$ is an increasing map with respect to $\preccurlyeq$ and there exists two elements a_0, $b_0 \in K$ with $a_0 \preccurlyeq F(a_0,b_0)$ and $b_0 \succcurlyeq F(b_0,a_0)$. Suppose there exists $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi\left(\frac{\eta(F(a,b),F(c,d)) + \eta(F(b,a),F(d,c))}{2}\right) \le \varphi\left(\frac{\eta(a,c)+\eta(b,d)}{2}\right) - \psi\left(\frac{\eta(a,c)+\eta(b,d)}{2}\right)$$

for all a, b, c, $d \in K$ with $a \preccurlyeq c$ and $b \succcurlyeq d$. Also suppose that either

(a) *F is continuous or;*

(b) *K has the following properties:*

   (b1) *if a non-decreasing sequence $\{a_n\} \to a$, then $a_n \preccurlyeq a$ for all n,*

   (b2) *if a non-increasing sequence $\{b_n\} \to b$, then $b \preccurlyeq b_n$ for all n.*

Then F has a coupled fixed point.

Remark 2. (i) Theorem 2 generalized and improved Theorem 15 in [11].

(ii) Theorem 2 generalized and improved Theorem 3.1 of Alotaibi and Alsulami [2] without g-mixed monotone property of F.

(iii) Theorem 2 generalized and improved Luong and Thuan [13] and Bhaskar and Lakshmikantham [5] without the mixed monotone property of the concerned mapping F.

Now, we shall prove the uniqueness of coupled fixed point. Note that if $(K,\preccurlyeq)$ is a partially ordered set, then we endow the product K^2 with the following partial order relation:

$$(a,b) \preccurlyeq (c,d) \Leftrightarrow G(a,b) \preccurlyeq G(c,d) \text{ and } G(b,a) \succcurlyeq G(d,c),$$

where $G: K^2 \to K^2$ is one-one.

Theorem 3. *In addition to the hypotheses of Theorem 2, suppose that for every (a,b), $(m,n) \in K^2$, there exists another $(c,d) \in K^2$ which is comparable to (a,b) and (m,n). Then F and G have a unique coupled coincidence point.*

Proof. By virtue of Theorem 2, the set of coupled coincidence points of F and G is non-empty. Suppose (a,b) and (m,n) are coupled coincidence points of F and G, that is,

$$\begin{cases} F(a,b)=G(a,b), \\ F(b,a)=G(b,a) \end{cases} \quad \text{and} \quad \begin{cases} F(m,n)=G(m,n), \\ F(n,m)=G(n,m). \end{cases}$$

By assumption, there exists $(c,d)\in K^2$ such that (c,d) is comparable to (a,b) and (m,n). We define sequences $\{c_n\}$, $\{d_n\}$ as follows

$$c_0=c,\, d_0=d,\, F(c_n,d_n)=G(c_{n+1},d_{n+1}) \text{ and } F(d_n,c_n)=G(d_{n+1},c_{n+1}) \quad \text{for all } n\in\mathbb{N}.$$

Since (c,d) is comparable with (a,b), we assume that $(a,b)\preccurlyeq(c,d)=(c_0,d_0)$ which implies $G(a,b)\preccurlyeq G(c_0,d_0)$ and $G(b,a)\succcurlyeq G(d_0,c_0)$. We assume that $(a,b)\preccurlyeq(c_n,d_n)$ for some $n\in\mathbb{N}$. We prove that

$$(a,b)\preccurlyeq(c_{n+1},d_{n+1}) \text{ for all } n\in\mathbb{N}.$$

Since F is G increasing, we have $G(a,b)\preccurlyeq G(c_n,d_n)$ implies $F(a,b)\preccurlyeq F(c_n,d_n)$ and $G(b,a)\succcurlyeq G(d_n,c_n)$ implies $F(b,a)\preccurlyeq F(d_n,c_n)$. Then, we have

$$G(a,b)=F(a,b)\preccurlyeq F(c_n,d_n)=G(c_{n+1},d_{n+1})$$

and

$$G(b,a)=F(b,a)\succcurlyeq F(d_n,c_n)=G(d_{n+1},c_{n+1}).$$

Hence we obtain

$$(a,b)\preccurlyeq(c_n,d_n) \text{ for all } n\in\mathbb{N}. \tag{2.25}$$

Denote

$$\lambda_n=\frac{\eta(G(a,b),G(c_n,d_n))+\eta(G(b,a),G(d_n,c_n))}{2}$$

for all $n\in\mathbb{N}$.

Using (2.1) and (2.25), we have

$$\begin{aligned}
&\varphi\left(\frac{\eta(G(a,b),G(c_{n+1},d_{n+1}))+\eta(G(b,a),G(d_{n+1},c_{n+1}))}{2}\right)\\
&\quad=\varphi\left(\frac{\eta(F(a,b),F(c_n,d_n))+\eta(F(b,a),F(d_n,c_n))}{2}\right)\\
&\quad\le\varphi\left(\frac{\eta(G(a,b),G(c_n,d_n))+\eta(G(b,a),G(d_n,c_n))}{2}\right)\\
&\qquad-\psi\left(\frac{\eta(G(a,b),G(c_n,d_n))+\eta(G(b,a),G(d_n,c_n))}{2}\right).
\end{aligned} \tag{2.26}$$

Similarly, we obtain

$$\begin{aligned}
&\varphi\left(\frac{\eta(G(d_{n+1},c_{n+1}),G(b,a))+\eta(G(c_{n+1},d_{n+1}),G(a,b))}{2}\right)\\
&\quad=\varphi\left(\frac{\eta(F(d_n,c_n),F(b,a))+\eta(F(c_n,d_n),F(a,b))}{2}\right)
\end{aligned}$$

$$\leq \varphi\left(\frac{\eta(G(d_n,c_n),G(b,a))+\eta(G(c_n,d_n),G(a,b))}{2}\right)$$
$$-\psi\left(\frac{\eta(G(d_n,c_n),G(b,a))+\eta(G(c_n,d_n),G(a,b))}{2}\right). \tag{2.27}$$

Adding (2.26) and (2.27), we have

$$2\varphi(\lambda_{n+1}) \leq 2\varphi(\lambda_n) - 2\psi(\lambda_n)$$

or equivalently,

$$\varphi(\lambda_{n+1}) \leq \varphi(\lambda_n) - \psi(\lambda_n). \tag{2.28}$$

Since ψ is non-negative and the monotonicity of φ, it follows that the sequence $\{\lambda_n\}$ is monotone decreasing.Thus, there is some $\lambda_n \geq 0$ such that $\lim\limits_{n\to\infty} \lambda_n = \lambda$. We claim that $\lambda = 0$. Suppose, to the contrary, that $\lambda > 0$. Taking limit as $n \to \infty$ on both sides of (2.28) and using the property of φ and ψ, we have

$$\varphi(\lambda) \leq \varphi(\lambda) - \lim_{n\to\infty} \psi\left(\frac{\eta(G(a,b),G(c_n,d_n))+\eta(G(b,a),G(d_n,c_n))}{2}\right) < \varphi(\lambda)$$

which is a contradiction. Therefore $\lambda = 0$, i.e.,

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{\eta(G(a,b),G(c_n,d_n))+\eta(G(b,a),G(d_n,c_n))}{2} = 0. \tag{2.29}$$

Hence, it follows that

$$\lim_{n\to\infty} \eta(G(a,b),G(c_n,d_n)) = \lim_{n\to\infty} \eta(G(b,a),G(d_n,c_n)) = 0. \tag{2.30}$$

Similarly

$$\lim_{n\to\infty} \eta(G(m,n),G(c_n,d_n)) = \lim_{n\to\infty} \eta(G(n,m),G(d_n,c_n)) = 0. \tag{2.31}$$

From (2.29) and (2.31), we have $G(a,b) = G(m,n)$ and $G(b,a) = G(n,m)$. This completes the proof. □

Example 1. Let $K = [0,1]$ endowed with the natural ordering of real numbers. Let $\eta(a,b) = |a-b|$, for all a, $b \in K$. Then (K,η) is a complete metric space. Let F, $G : K^2 \to K$ be defined by

$$F(a,b) = \begin{cases} \dfrac{a^2-b^2}{16} & \text{if } a \geq b, \\ 0 & \text{if } a < b \end{cases}$$

and

$$G(a,b) = \begin{cases} a^2-b^2 & \text{if } a \geq b, \\ 0 & \text{if } a < b. \end{cases}$$

Clearly, G is continuous. Also, F is $G-$increasing.

Now, we prove that for any a, $b \in K$, there exists c, $d \in K$ such that $F(a,b) = G(c,d)$ and $F(b,a) = G(d,c)$. Let $(a,b) \in K^2$ be fixed. It is easy to see the following cases.

Case 1: If $a = b$, then we have $F(a,b) = 0 = G(a,b)$ and $F(b,a) = 0 = G(b,a)$.

Case 2: If $a > b$, then we have $F(a,b) = \dfrac{a^2-b^2}{16} = G\left(\dfrac{a}{4},\dfrac{b}{4}\right)$ and $F(b,a) = 0 = G\left(\dfrac{b}{4},\dfrac{a}{4}\right)$.

Case 3: If $a < b$, then we have $F(a,b) = 0 = G\left(\frac{a}{4},\frac{b}{4}\right)$ and $F(b,a) = \frac{b^2-a^2}{16} = G\left(\frac{b}{4},\frac{a}{4}\right)$.

Now, we show that the pair $\{F,G\}$ satisfies the generalized compatibility hypothesis. Let (a_n) and (b_n) be two sequences in K such that

$$\begin{cases} \lim_{n\to\infty} F(a_n,b_n) = \lim_{n\to\infty} G(a_n,b_n) = x_1, \\ \lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} G(b_n,a_n) = x_2. \end{cases}$$

Then we must have $x_1 = 0 = x_2$ and one can easily prove that

$$\begin{cases} \eta(F(G(a_n,b_n),G(b_n,a_n)),G(F(a_n,b_n),F(b_n,a_n))) \to 0 \ \text{ as } n \to +\infty, \\ \eta(F(G(b_n,a_n),G(a_n,b_n)),G(F(b_n,a_n),F(a_n,b_n))) \to 0 \ \text{ as } n \to +\infty. \end{cases}$$

Let $\varphi,\psi : [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \frac{3t}{4}$, $\psi(t) = \frac{3t}{8}$ for all $t \geq 0$. Now, we verify the contraction (2.1) for all a, b, c, $d \in K$, with $G(a,b) \preccurlyeq G(c,d)$ and $G(d,c) \preccurlyeq G(b,a)$. We have the following cases.

Case 1: $a \geq b$, $c \geq d$. Then

$$\begin{aligned}
&\varphi\left(\frac{\eta(F(a,b),F(c,d)) + \eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{4}\left(\frac{\eta(F(a,b),F(c,d)) + \eta(0,0)}{2}\right) \\
&= \frac{3}{8}\eta\left(\frac{a^2-b^2}{16},\frac{c^2-d^2}{16}\right) = \frac{3}{8}\left|\frac{a^2-b^2}{16} - \frac{c^2-d^2}{16}\right| \\
&\leq \frac{3}{8}\frac{\left|(a^2-b^2)-(c^2-d^2)\right| + \left|(a^2-b^2)-(c^2-d^2)\right|}{16} \\
&\leq \frac{3}{8}\frac{\left|(a^2-b^2)-(c^2-d^2)\right| + \left|(a^2-b^2)-(c^2-d^2)\right|}{2} \\
&= \frac{3}{4}\frac{\left|(a^2-b^2)-(c^2-d^2)\right| + |-1|\left|(b^2-a^2)-(d^2-c^2)\right|}{2} \\
&\quad - \frac{3}{8}\frac{\left|(a^2-b^2)-(c^2-d^2)\right| + |-1|\left|(b^2-a^2)-(d^2-c^2)\right|}{2} \\
&= \varphi\left(\frac{\eta(G(a,b),G(c,d)) + \eta(G(b,a),G(d,c))}{2}\right) \\
&\quad - \psi\left(\frac{\eta(G(a,b),G(c,d)) + \eta(G(b,a),G(d,c))}{2}\right).
\end{aligned}$$

Case 2: $a \geq b$, $c < d$. Then

$$\begin{aligned}
&\varphi\left(\frac{\eta(F(a,b),F(c,d)) + \eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{4}\left(\frac{\eta(F(a,b),F(c,d)) + \eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{8}\left\{\eta\left(\frac{a^2-b^2}{16},0\right) + \eta\left(0,\frac{d^2-c^2}{16}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{8}\left\{\frac{a^2-b^2}{16}+\frac{d^2-c^2}{16}\right\}=\frac{3}{8}\left\{\frac{(a^2-b^2)-(c^2-d^2)}{16}\right\} \\
&\le \frac{3}{8}\frac{\{(a^2-b^2)-(c^2-d^2)\}+\{(a^2-b^2)-(c^2-d^2)\}}{16} \\
&\le \frac{3}{8}\frac{\{(a^2-b^2)-(c^2-d^2)\}+\{(a^2-b^2)-(c^2-d^2)\}}{2} \\
&= \frac{3}{8}\frac{\{(a^2-b^2)-(c^2-d^2)\}+\{(d^2-c^2)-(-a^2+b^2)\}}{2} \\
&= \frac{3}{4}\frac{\{(a^2-b^2)-(c^2-d^2)\}+\{(d^2-c^2)-(-a^2+b^2)\}}{2} \\
&\quad -\frac{3}{8}\frac{\{(a^2-b^2)-(c^2-d^2)\}+\{(d^2-c^2)-(-a^2+b^2)\}}{2} \\
&= \varphi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(d,c),G(b,a))}{2}\right) \\
&\quad -\psi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(d,c),G(b,a))}{2}\right).
\end{aligned}
$$

Case 3: $a<b$, $c\ge d$. Then

$$
\begin{aligned}
&\varphi\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{4}\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{8}\left\{\eta\left(0,\frac{c^2-d^2}{16}\right)+\eta\left(\frac{b^2-a^2}{16},0\right)\right\}=\frac{3}{8}\left\{\frac{c^2-d^2}{16}+\frac{b^2-a^2}{16}\right\} \\
&\le \frac{3}{8}\frac{\{(c^2-d^2)+(b^2-a^2)\}+\{(c^2-d^2)+(b^2-a^2)\}}{16} \\
&\le \frac{3}{8}\frac{\{(c^2-d^2)+(b^2-a^2)\}+\{(c^2-d^2)+(b^2-a^2)\}}{2} \\
&= \frac{3}{4}\frac{\{(c^2-d^2)-(a^2-b^2)\}+\{(b^2-a^2)-(d^2-c^2)\}}{2} \\
&\quad -\frac{3}{8}\frac{\{(c^2-d^2)-(a^2-b^2)\}+\{(b^2-a^2)-(d^2-c^2)\}}{2} \\
&= \varphi\left(\frac{\eta(G(c,d),G(a,b))+\eta(G(b,a),G(d,c))}{2}\right) \\
&\quad -\psi\left(\frac{\eta(G(c,d),G(a,b))+\eta(G(b,a),G(d,c))}{2}\right).
\end{aligned}
$$

Case 4: $a<b$, $c<d$. Then

$$
\varphi\left(\frac{\eta(F(a,b),F(c,d))+\eta(F(b,a),F(d,c))}{2}\right)
$$

$$
\begin{aligned}
&= \frac{3}{4}\left(\frac{\eta(0,0)+\eta(F(b,a),F(d,c))}{2}\right) \\
&= \frac{3}{8}\eta\left(\frac{b^2-a^2}{16},\frac{d^2-c^2}{16}\right) = \frac{3}{8}\left|\frac{b^2-a^2}{16}-\frac{d^2-c^2}{16}\right| \\
&\le \frac{3}{8}\frac{\left|(b^2-a^2)-(d^2-c^2)\right|+\left|(b^2-a^2)-(d^2-c^2)\right|}{16} \\
&\le \frac{3}{8}\frac{\left|(b^2-a^2)-(d^2-c^2)\right|+\left|(b^2-a^2)-(d^2-c^2)\right|}{2} \\
&= \frac{3}{4}\frac{|-1|\left|(a^2-b^2)-(c^2-d^2)\right|+\left|(b^2-a^2)-(d^2-c^2)\right|}{2} \\
&\quad - \frac{3}{8}\frac{|-1|\left|(a^2-b^2)-(c^2-d^2)\right|+\left|(b^2-a^2)-(d^2-c^2)\right|}{2} \\
&= \varphi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(b,a),G(d,c))}{2}\right) \\
&\quad - \psi\left(\frac{\eta(G(a,b),G(c,d))+\eta(G(b,a),G(d,c))}{2}\right).
\end{aligned}
$$

Thus, the inequality (2.1) of Theorem 2 holds.

Therefore, all the conditions of Theorem 2 are satisfied and $(0,0)$ is a coupled coincidence point of F and G.

3. Conclusion

Our theorems and corollaries improve the coupled common fixed point theorems for a generalized compatible pair of mappings in Jain et. (2012), Hussain et al. (2014), Alotaibi and Alsulami (2011), Luong and Thuan (2011) and Bhaskar and Lakshmikantham (2006). Farther, we deduce coupled fixed point results without mixed monotone property of F. Within the future scope of the idea, reader may indicate

(1) the existence of a coupled coincidence point theorem for a α -(φ,ψ)-contractive mapping in partially metric spaces via the notion of M-invariant set (for the definition of this notion, see [6]),

(2) some existence and uniqueness results for coupled coincidence point and common fixed point of (φ,ψ)-contractive mappings in complete metric spaces involving a graph.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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