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Research Article

Generalized Derivations on Prime Rings with Involution

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Abstract. Let R be an associative ring. An additive mapping $F : R \to R$ is called a generalized derivation with an associated derivation δ of R if it satisfies $F(st) = F(s)t + s\delta(t)$ for all $s,t \in R$. In the present paper, we obtain description of the structure of R and information about the generalized derivation F which satisfies certain $*$-differential identities on prime rings with involution.

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1. Introduction

In all that follows, unless specially stated, R always denotes an associative ring with centre $Z(R)$. As usual the symbols $s \circ t$ and $[s,t]$ will denote the anti-commutator $st+ts$ and commutator $st-ts$, respectively. A ring equipped with involution $*$ is called ring with involution or $*$-ring (cf. [18]). Also $H(R)$ and $S(R)$ represent the sets of all hermitian and skew-hermitian elements of R.

An additive mapping $\delta : R \to R$ is said to be a derivation of R if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s,t \in R$. A derivation δ is said to be inner if there exists $a \in R$ such that $\delta(s) = as - sa$ for all $s \in R$. Following Brešar [14], an additive mapping $F : R \to R$ is said to be a generalized derivation of R

with an associated derivation δ if $F(st) = F(s)t + s\delta(t)$ for all $s, t \in R$. It is clear that the concept of generalized derivation covers both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $T : R \to R$ satisfies $T(st) = T(s)t$ for all $s, t \in R$). Over the last 30 years, several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps on R. This line of study was originated by Posner [20], who proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Several other papers studying commutativity of prime and semiprime rings admitting derivations or generalized derivations satisfying certain identities can be found in [5], [6], [7], [8], [9],[10], [12], [14], [19], [21], where further references can be found. In [17], Herstein proved that a prime ring R of characteristic not two with a nonzero derivation δ satisfying $\delta(s)\delta(t) = \delta(t)\delta(s)$ for all $s, t \in R$, must be commutative. Further, Daif [15] studied this result in the setting of semiprime rings. In [10], Bell and Daif showed that if R is a prime ring admitting a nonzero derivation δ such that $\delta(st) = \delta(ts)$ for all $s, t \in R$, then R is commutative. This result was extended for semiprime rings by Daif [15]. Bell and Kappe [11] proved that if a derivation δ of a prime ring R can act as homomorphism or anti-homomorphism on a nonzero right ideal of R, then $\delta = 0$ on R. These results were studied in the setting of generalized derivations by many authors (viz.; [1], [13], [16], [22]). Very recently, Ali et al. [2, 3], studied these results in the setting of rings with involution involving derivations.

In this paper, our intent is to investigate certain identities involving generalized derivations in prime rings with involution. Finally, an example is given to demonstrate that the restrictions imposed on the hypothesis of our result are not superfluous.

Throughout the paper, we denote by I_{id} the identity map of a ring R (i.e., the map $I_{id} : R \to R$ defined by $I_{id}(s) = s$ for all $s \in R$). At the same time, the map $-I_{id} : R \to R$ defined by $(-I_{id})(s) = -s$ for all $s \in R$. We will use the following facts in the proofs below.

Fact 1 ([2, Lemma 2.1])**.** Let R be a prime ring with involution such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.

Fact 2. The center of a prime ring is free from zero divisors.

Fact 3. Let R be a 2-torsion free ring with involution. Then every $s \in R$ can be uniquely represented as $2s = h + k$, where $h \in H(R)$ and $k \in S(R)$.

Fact 4. Let n be any integer. If $F : R \to R$ is a generalized derivation with an associated derivation δ, then $F \pm nI_{id}$ is also a generalized derivation on R.

2. The Results

In 1995, Bell and Daif [10] showed that if R is a prime ring admitting a nonzero derivation δ such that $\delta([s, t]) = 0$ for all $s, t \in R$, then R is commutative. This result was extended for semiprime rings in [15] by Daif. Further, for semiprime rings, Andima and Pajoohesh [4] showed that an inner derivation satisfying the above mentioned condition on a nonzero ideal of R must be zero on that ideal. Moreover, for semiprime rings with identity, they generalized this result to inner derivations of powers of s and t. Recently, Ali et al. [3], studied the above mentioned result in the setting of prime rings with involution by replacing t by s^*. Precisely, they proved the following theorem:

Theorem 2.1. *Let R be a prime ring with involution such that char$(R) \neq 2$. If δ is a nonzero derivation of R such that $\delta([s,s^*]) = 0$ for all $s \in R$ and $S(R) \cap Z(R) \neq (0)$, then R is commutative.*

We extend the above theorem for generalized derivation in rings with involution as follows:

Theorem 2.2. *Let R be a prime ring with involution of the second kind such that char$(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ such that $F([s,s^*]) = 0$ for all $s \in R$, then one of the following holds:*

(i) *$F(s) = 0$ for all $s \in R$;*

(ii) *R is commutative.*

Proof. In view of our hypothesis, we have

$$F([s,s^*]) = 0 \quad \text{for all } s \in R. \tag{2.1}$$

Replacing s by $a+b$ in (2.1), where $a \in H(R)$ and $b \in S(R)$, we get

$$F([a,b]) = 0 \quad \text{for all } a \in H(R) \text{ and } b \in S(R). \tag{2.2}$$

Taking $a = b_0 b_1$ in above expression, where $b_0 \in S(R)$ and $b_1 \in S(R) \cap Z(R)$, we get

$$F([b_0,b])b_1 + [b_0,b]\delta(b_1) = 0 \tag{2.3}$$

for all $b_0, b \in S(R)$ and $b_1 \in S(R) \cap Z(R)$. Substituting $a_0 b_1$ for b in (2.3), we obtain

$$F([b_0,a_0])b_1^2 + 2[b_0,a_0]\delta(b_1)b_1 = 0 \tag{2.4}$$

for all $b_0 \in S(R)$, $a_0 \in H(R)$ and $b_1 \in S(R) \cap Z(R)$. In view of (2.2), we have

$$2[b_0,a_0]\delta(b_1)b_1 = 0$$

for all $b_0 \in S(R)$, $a_0 \in H(R)$ and $b_1 \in S(R) \cap Z(R)$. Since *char*$(R) \neq 2$, so the last expression yields that

$$[b_0,a_0]\delta(b_1)b_1 = 0 \tag{2.5}$$

for all $b_0 \in S(R)$, $a_0 \in H(R)$ and $b_1 \in S(R) \cap Z(R)$. In view of Fact 2, we conclude that either $[b_0,a_0] = 0$ or $\delta(b_1)b_1 = 0$. First assume that $[b_0,a_0] = 0$ for all $b_0 \in S(R)$, $a_0 \in H(R)$ and hence R is commutative by Fact 1. On the other hand, we assume that $\delta(b_1)b_1 = 0$ for all $b_1 \in S(R) \cap Z(R)$. This further implies that $\delta(b_1) = 0$ or $b_1 = 0$. Since $b_1 = 0$ also implies that $\delta(b_1) = 0$ for some $b_1 \in S(R) \cap Z(R)$. Thus the relation (2.3) reduces to

$$F([b_0,b])b_1 = 0 \quad \text{for all } b_0, b \in S(R) \text{ and } b_1 \in S(R) \cap Z(R). \tag{2.6}$$

Application of Fact 2 yields that

$$F([b_0, b]) = 0 \quad \text{for all } b_0, b \in S(R). \tag{2.7}$$

In view of Fact 3, (2.2) and (2.7), we are force to conclude that

$$F([s, b]) = 0 \quad \text{for all } b \in S(R) \text{ and } s \in R.$$

Substituting hb_1 for b in the above expression, where $a \in H(R)$ and $b_1 \in S(R) \cap Z(R)$, and proceeding as above we arrive at $F([s, t]) = 0$ for all $s, t \in R$. This gives $0 = F([s, ts]) = [s, t]\delta(s)$ for all $s, t \in R$. This implies that $[s, t]R\delta(s) = (0)$ for all $s, t \in R$. Hence, by the primeness of R, we have either R is commutative or $\delta(s) = 0$ for all $s \in R$. Thus we assume that $\delta(s) = 0$ for all $s \in R$. This implies that $F(st) = F(s)t$ for all $s, t \in R$. Hence $0 = F([s, tu]) = F([s, t]u + t[s, u]) = F(t)[s, u]$ for all $s, y, u \in R$ and hence $F(t)R[s, u] = (0)$ for all $s, t, u \in R$. Primeness of R forces that either $F(t) = 0$ for all $t \in R$ or R is commutative. This proves the theorem completely. □

The following are immediate consequences of Theorem 2.2.

Corollary 2.3. *Let R be a prime ring with involution of the second kind such that* char$(R) \neq 2$. *If R admits a nonzero derivation $d : R \to R$ such that $\delta([s, s^*]) = 0$ for all $s \in R$, then R is commutative.*

Corollary 2.4. *Let R be a prime ring with involution of the second kind such that* char$(R) \neq 2$. *If R admits a nonzero left multiplier $T : R \to R$ such that $T([s, s^*]) = 0$ for all $s \in R$, then R is commutative.*

If we replace commutator by anti-commutator in Theorem 2.2, we get the following result:

Theorem 2.5. *Let R be a prime ring with involution of the second kind such that* char$(R) \neq 2$. *If R admits a generalized derivation $F : R \to R$ such that $F(s \circ s^*) = 0$ for all $s \in R$, then $F = 0$.*

Proof. By the given hypothesis, we have

$$F(s \circ s^*) = 0 \quad \text{for all } s \in R. \tag{2.8}$$

This can be further written as

$$F(s)s^* + s\delta(s^*) + F(s^*)s + s^*\delta(s) = 0 \quad \text{for all } s \in R. \tag{2.9}$$

Replacing s by a_1 in (2.9), where $a_1 \in H(R) \cap Z(R)$ and using the fact that *char*$(R) \neq 2$, we get

$$(F(a_1) + \delta(a_1))a_1 = 0 \quad \text{for all } a_1 \in H(R) \cap Z(R). \tag{2.10}$$

In view of Fact 2, we conclude that $F(a_1) + \delta(a_1) = 0$ for all $a_1 \in H(R) \cap Z(R)$. Substituting b_1^2 for a_1 in the last expression, where $b_1 \in S(R) \cap Z(R)$, we obtain

$$F(b_1)b_1 + 3\delta(b_1)b_1 = 0 \quad \text{for all } b_1 \in S(R) \cap Z(R). \tag{2.11}$$

Now taking $s = b_1$ in (2.9), where $b_1 \in S(R) \cap Z(R)$ and using the fact that $char(R) \neq 2$, we get

$$F(b_1)b_1 + \delta(b_1)b_1 = 0 \quad \text{for all } b_1 \in S(R) \cap Z(R). \tag{2.12}$$

Combining (2.11) and (2.12), we obtain

$$2\delta(b_1)b_1 = 0 \quad \text{for all } b_1 \in S(R) \cap Z(R). \tag{2.13}$$

Since $char(R) \neq 2$, the last expression yields that

$$\delta(b_1)b_1 = 0 \quad \text{for all } b_1 \in S(R) \cap Z(R). \tag{2.14}$$

This further implies that $\delta(b_1) = 0$ for all $b_1 \in S(R) \cap Z(R)$. Thus (2.12) reduces to $F(b_1)b_1 = 0$ and hence $F(b_1) = 0$ for all $b_1 \in S(R) \cap Z(R)$. Next, linearize (2.9), we find that

$$0 = F(s)t^* + F(t)s^* + s\delta(t^*) + t\delta(s^*) + F(s^*)t + F(t^*)s + s^*\delta(t) + t^*\delta(s) \quad \text{for all } s,t \in R. \tag{2.15}$$

Substituting b_1 for s in (2.15), where $b_1 \in S(R) \cap Z(R)$, we get

$$(F(t^* - t) + \delta(t^* - t))b_1 = 0 \tag{2.16}$$

for all $t \in R$ and $b_1 \in S(R) \cap Z(R)$. Application of Fact 2 yields that

$$F(t^* - t) + \delta(t^* - t) = 0 \quad \text{for all } t \in R. \tag{2.17}$$

Substituting $a - b$ for t in (2.17), where $a \in H(R)$ and $b \in S(R)$ and using the fact that $char(R) \neq 2$, we get $F(b) + \delta(b) = 0$ for all $b \in S(R)$. Now replacing b by b_1a in the last expression, where $a \in H(R)$ and $b_1 \in S(R) \cap Z(R)$ and making use of $F(b_1) = \delta(b_1) = 0$, we obtain $\delta(a)b_1 = 0$ for all $a \in H(R)$ and $b_1 \in S(R) \cap Z(R)$ and hence $\delta(a) = 0$ for all $a \in H(R)$. Since $bb_1 \in H(R)$, where $b \in S(R)$ and $b_1 \in S(R) \cap Z(R)$, so we obtain $\delta(bb_1) = 0$. Hence, $\delta(b) = 0$ for all $b \in S(R)$. Application of Fact 3 yields that $\delta(s) = 0$ for all $s \in R$. Thus in view of (2.15), we have

$$F(s)t^* + F(t)s^* + F(s^*)t + F(t^*)s = 0 \quad \text{for all } s,t \in R. \tag{2.18}$$

Taking $t = b_1$ in (2.18), where $b_1 \in S(R) \cap Z(R)$, we get

$$F(s^* - s)b_1 = 0 \quad \text{for all } s \in R \text{ and } b_1 \in S(R) \cap Z(R). \tag{2.19}$$

Again application of Fact 2 forces that $F(s^* - s) = 0$ for all $s \in R$. Substituting $a - b$ for s in the last expression, we arrive at $F(b) = 0$ for all $b \in S(R)$. This further yields that $F(a) = 0$ for all $a \in H(R)$. By Fact 3, we conclude that $F(s) = 0$ for all $s \in R$. This proves the theorem completely. □

Application of Theorem 2.5 and Fact 4 leads the following:

Corollary 2.6. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ such that $F(ss^*) + ss^* = 0$ for all $s \in R$ or $F(ss^*) - ss^* = 0$ for all $s \in R$, then either $F = -I_{id}$ or $F = I_{id}$.*

Proof. By the assumption we have $F(ss^*) + ss^* = 0$ for all $s \in R$ or $F(ss^*) - ss^* = 0$ for all $s \in R$.

In view of Fact 4, we easily see that the map $G(s)=F(s)+s$ (respectively, $G(s)=F(s)-s$) for all $s\in R$ is a generalized derivation and satisfying $G(ss^*)=0$ for all $s\in R$. This also implies that $G(s\circ s^*)=0$ for all $s\in R$. By Theorem 2.5, we conclude that $G(s)=0$ for all $s\in R$, i.e., $F(s)=\mp s$ for all $s\in R$. Hence, we obtain either $F=-I_{id}$ or $F=I_{id}$. This completes the proof. □

Corollary 2.7. *Let R be a prime ring with involution of the second kind such that char$(R)\neq 2$. If $\delta: R\to R$ is a derivation on R such that $\delta(ss^*)+ss^*=0$ for all $s\in R$ or $\delta(ss^*)-ss^*=0$ for all $s\in R$, then either R is commutative or $\delta=0$.*

3. Generalized Derivation Acting as Homomorphism or Anti-Homomorphism in Rings with Involution

An additive mapping $f: R\to R$ is called a homomorphism (resp. anti-homomorphism) of R if $f(st)=f(s)f(t)$ (resp. $f(st)=f(t)f(s)$) for all $s,t\in R$. In [11], Bell and Kappe initiated the study of mapping which acts as a homomorphism or as an anti-homomorphism on a prime ring and proved that if R is a semiprime ring and δ a derivation on R, which is either an endomorphism or an anti-endomorphism, then $\delta=0$. This result was generalized by Rehman [22] by replacing derivation δ with a generalized derivation F. Recently, Gusic [16] proved the result in more complete form by replacing the generalized derivation F by multiplicative(generalized)-derivation of R. Here, we study a similar problem in more general setting in case of rings with involution.

Theorem 3.1. *Let R be a prime ring with involution of the second kind such that char$(R)\neq 2$. If R admits a generalized derivation $F: R\to R$ such that $F(ss^*)=F(s)F(s^*)$ for all $s\in R$, then one of the following hold:*

(i) $F=I_{id}$;

(ii) *R is commutative.*

Proof. By the assumption, we have

$$F(ss^*)-F(s)F(s^*)=0 \quad \text{for all } s\in R. \tag{3.1}$$

Taking $s=a+b$ in (3.1), where $a\in H(R)$ and $b\in S(R)$, we get

$$F([b,a])-[F(b),F(a)]=0 \quad \text{for all } a\in H(R) \text{ and } b\in S(R). \tag{3.2}$$

Let $b_1\in S(R)\cap Z(R)$ and substituting $b+b_1$ for b in (3.2), we have

$$[F(b_1),F(a)]=0 \quad \text{for all } a\in H(R) \text{ and } b_1\in S(R)\cap Z(R). \tag{3.3}$$

Again replacing a by $a+a_1$ in (3.2), where $a_1\in H(R)\cap Z(R)$, we arrive at

$$[F(b),F(a_1)]=0 \quad \text{for all } b\in S(R) \text{ and } a_1\in H(R)\cap Z(R). \tag{3.4}$$

Since $b_1\in S(R)\cap Z(R)$, so $b_1^2\in H(R)\cap Z(R)$. Thus, the last expression yields that $0=[F(b),F(b_1^2)]=[F(b),F(b_1)b_1+b_1\delta(b_1)]=[F(b),F(b_1)]b_1$. That is, $[F(b),F(b_1)]b_1=0$ for all

$b \in S(R)$ and $b_1 \in S(R) \cap Z(R)$. In view of Fact 2, we obtain

$$[F(b), F(b_1)] = 0 \quad \text{for all } b \in S(R) \text{ and } b_1 \in S(R) \cap Z(R). \tag{3.5}$$

In view of Fact 3, (3.3) and (3.5), we are force to conclude that

$$[F(s), F(b_1)] = 0 \quad \text{for all } s \in R \text{ and } b_1 \in S(R) \cap Z(R). \tag{3.6}$$

Replacing s by sb_1 in (3.6), where $b_1 \in S(R) \cap Z(R)$, we get $\delta(b_1)[s, F(b_1)] = 0$ for all $s \in R$ and $b_1 \in S(R) \cap Z(R)$. Using the primeness of R, we have either $F(b_1) \in Z(R)$ or $\delta(b_1) = 0$. Suppose $F(b_1) \in Z(R)$ and replacing $b_1 b$ for a, in (3.2), where $b \in S(R)$ and $b_1 \in S(R) \cap Z(R)$, we get

$$[F(b), b]\delta(b_1) = 0 \quad \text{for all } b \in S(R) \text{ and } b_1 \in S(R) \cap Z(R). \tag{3.7}$$

This further implies that either $[F(b), b] = 0$ or $\delta(b_1) = 0$. Suppose that $[F(b), b] = 0$ for all $b \in S(R)$. Substituting $b_1 a$ for b, where $a \in H(R)$ and $b_1 \in S(R) \cap Z(R)$, we get

$$[F(b_1)a + b_1\delta(a), b_1 a] = 0 \quad \text{for all } a \in H(R) \text{ and } b_1 \in S(R) \cap Z(R). \tag{3.8}$$

Since $F(b_1) \in Z(R)$ and $b_1 \in S(R) \cap Z(R)$, so we have

$$[a, \delta(a)] = 0 \quad \text{for all } a \in H(R). \tag{3.9}$$

On linearizing (3.9), we obtain

$$[\delta(a), a_0] + [\delta(a_0), a] = 0 \quad \text{for all } a, a_0 \in H(R). \tag{3.10}$$

Which can be further written as

$$[\delta(a_0), a] = [a_0, \delta(a)] \quad \text{for all } a, a_0 \in H(R). \tag{3.11}$$

Substituting a^2 for a in the above expression, we obtain

$$[\delta(a_0), a^2] = [a_0, \delta(a)]a + a[a_0, \delta(a)] + \delta(a)[a_0, a] + [a_0, a]\delta(a) \tag{3.12}$$

for all $a, a_0 \in H(R)$. Also, we have

$$[\delta(a_0), a^2] = [\delta(a_0), a]a + a[\delta(a_0), a] = [a_0, \delta(a)]a + a[a_0, \delta(a)] \tag{3.13}$$

for all $a, a_0 \in H(R)$. Combining (3.12) and (3.13), we obtain

$$\delta(a)[a_0, a] + [a_0, a]\delta(a) = 0 \quad \text{for all } a, a_0 \in H(R). \tag{3.14}$$

Now, taking $a_0 = bb_1$ in (3.14), where $b \in S(R)$ and $b_1 \in S(R) \cap Z(R)$, and using the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$\delta(a)[b, a] + [b, a]\delta(a) = 0 \quad \text{for all } a \in H(R) \text{ and } b \in S(R). \tag{3.15}$$

Replacing a by $a + a_1$ in (3.15), where $a_1 \in H(R) \cap Z(R)$, we get

$$\delta(a)[b, a] + \delta(a_1)[b, a] + [b, a]\delta(a) + [b, a]\delta(a_1) = 0 \tag{3.16}$$

for all $a \in H(R)$ and $b \in S(R)$. In view of (3.15), the last relation reduces to

$$2[b, a]\delta(a_1) = 0$$

for all $a \in H(R)$, $b \in S(R)$ and $a_1 \in H(R) \cap Z(R)$. Since $char(R) \neq 2$, the above expression gives us

$$[b,a]\delta(a_1) = 0 \tag{3.17}$$

for all $a \in H(R)$, $b \in S(R)$ and $a_1 \in H(R) \cap Z(R)$. Since R is prime, this yields that either $[b,a] = 0$ or $\delta(a_1) = 0$. If $[b,a] = 0$ for all $a \in H(R)$ and $b \in S(R)$, then in view of Fact 1, R must be commutative. Now assume that $\delta(a_1) = 0$ for all $a_1 \in H(R) \cap Z(R)$. This further implies that $\delta(b_1^2) = 0$ and hence $\delta(b_1) = 0$ for all $b_1 \in S(R) \cap Z(R)$. Finally assume that $\delta(b_1) = 0$ for all $b_1 \in S(R) \cap Z(R)$. Replacing a by $b_0 b_1$ in (3.2) and using $\delta(b_1) = 0$, we get

$$F([b,b_0]) - [F(b),F(b_0)] = 0 \tag{3.18}$$

for all $b, b_0 \in S(R)$. Since $char(R) \neq 2$, every $t \in R$ can be written as $2t = a + b_0$, where $a \in H(R)$ and $b_0 \in S(R)$, so in view of (3.2) and (3.18), we obtain

$$F([b,t]) - [F(b),F(t)] = 0 \quad \text{for all } t \in R \text{ and } b \in S(R). \tag{3.19}$$

Next, replacing b by $a_0 b_1$ in (3.19), where $a_0 \in H(R)$ and $b_1 \in S(R) \cap Z(R)$ and proceeding as above, we arrive at

$$F([s,t]) - [F(s),F(t)] = 0 \quad \text{for all } s,t \in R. \tag{3.20}$$

Hence, in view of [1, Theorem 1] either R is commutative or $F(s) = s$ for all $s \in R$ i.e., $F = I_{id}$. This proves the theorem. □

Theorem 3.2. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If F is a nonzero generalized derivation of R such that $F(ss^*) = F(s^*)F(s)$ for all $s \in R$, then $F = -I_{id}$.*

Proof. By the assumption, we have

$$F(ss^*) - F(s^*)F(s) = 0 \quad \text{for all } s \in R. \tag{3.21}$$

Replacing s by $a + b$, where $a \in H(R)$ and $b \in S(R)$, we get

$$F([b,a]) + [F(b),F(a)] = 0 \quad \text{for all } a \in H(R) \text{ and } b \in S(R). \tag{3.22}$$

for all $a \in H(R)$ and $b \in S(R)$. Proceeding on similar lines as in the proof of Theorem 3.1, we arrive at

$$F([s,t]) + [F(s),F(t)] = 0 \quad \text{for all } s,t \in R. \tag{3.23}$$

Hence, in view of [1, Theorem 1] we get the required result. This completes the proof of the theorem. □

Corollary 3.3. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $\delta : R \to R$ such that $\delta(ss^*) = \delta(s)\delta(s^*)$ for all $s \in R$ (δ acts as a homomorphism), then either R is commutative or $\delta = 0$.*

Corollary 3.4. *Let R be a prime ring with involution of the second kind such that* char$(R) \neq 2$. *If R admits a derivation $\delta : R \to R$ such that $\delta(ss^*) = \delta(s^*)\delta(s)$ for all $s \in R$ (δ acts as an anti-homomorphism), then either R is commutative or $\delta = 0$.*

The following example shows that the restriction of the second kind involution in the hypotheses of Theorem 2.2 cannot be relaxed.

Example 3.1. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in Z \right\}.$$

Of course, R is a prime ring with matrix addition and matrix multiplication. Define mappings $\delta : R \to R$, $F : R \to R$ and $* : R \to R$ such that

$$\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$$

and

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R.$$

Obviously, $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in Z \right\}$. Then $s^* = s$ for all $s \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution is of the first kind. This implies that $S(R) \cap Z(R) = (0)$. It is straightforward to check that the mappings δ and F are nonzero derivation and generalized derivation on R. Further, the condition $F([s,s^*]) = 0$ for all $s \in R$, is satisfied. However, neither $F = 0$ nor R is commutative. Hence, the condition of the second kind involution in Theorem 2.2 is essential.

4. Conclusion

The paper deals with the study of some commutativity criteria for rings with involution. The main objective of the this paper is to solve some $*$-differential identities involving generalized derivations in prime rings. In particular, we describe the structure of prime rings and the form of generalized derivations satisfying certain $*$-differential identities. Further, we provide an example to show that the restriction imposed on the hypothesis of our main theorem is not superfluous.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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