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Research Article

# On Generalization of Fejér Type Inequalities

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**Abstract.** Some generalizations of Fejér type inequalities related to $\eta$-convex functions are investigated. Also applications for trapezoid and mid-point type inequalities are given.

**Keywords.** $\eta$-convex function; Hermite-Hadamard-Fejér inequality; Trapezoid and mid-point type inequality

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## 1. Introduction and Preliminaries

This paper generalizes some well-known results for Hermite-Hadamard-Fejér (simply Fejér) integral inequality by generalizing the convex function factor of the integrand to be an $\eta$-convex function. The obtained results have as particular cases those previously obtained for convex functions in the integrand.

The Hermite-Hadamard-Fejér integral inequality has been proved in [5] as the following.

**Theorem 1.1.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function. Then*

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.1}$$

*where $g:[a,b]\to\mathbb{R}^+=[0,+\infty)$ is integrable and symmetric about $x=\frac{a+b}{2}$ ($g(x)=g(a+b-x)$, $\forall\ x\in[a,b]$).*

For more results about (1.1), see [1, 6, 8, 9, 11, 12] and references therein.

Let $I$ be an interval in real line $\mathbb{R}$. Consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

**Definition 1.2** ([3, 4])**.** A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta$ (briefly $\eta$-convex), if

$$f\big(tx + (1-t)y\big) \le f(y) + t\eta\big(f(x), f(y)\big), \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$.

In fact the above definition geometrically says that if a function is $\eta$-convex on $I$, then it's graph between any $x, y \in I$ is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. Note that by, taking $x = y$ in (1.2), we get $t\eta(f(x), f(x)) \ge 0$ for any $x \in I$ and $t \in [0,1]$ which implies that

$$\eta(f(x), f(x)) \ge 0$$

for any $x \in I$. Also if we take $t = 1$ in (1.2) we get

$$f(x) - f(y) \le \eta(f(x), f(y))$$

for any $x, y \in I$.

There are simple examples about $\eta$-convexity of a function.

**Example 1.3** ([3, 4])**.** (1) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x, & x \ge 0; \\ x, & x < 0, \end{cases}$$

and define a bifunction $\eta$ as $\eta(x, y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that $f$ is an $\eta$-convex function but not a convex one.

(2) Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & x > 1, \end{cases}$$

and a bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\eta(x, y) = \begin{cases} x + y, & x \le y; \\ 2(x + y), & x > y. \end{cases}$$

Then $f$ is $\eta$-convex but is not convex.

The following theorem is an important result:

**Theorem 1.4** ([4, 10])**.** *Suppose that $f : I \to \mathbb{R}$ is a $\eta$-convex function and $\eta$ is bounded from above on $f(I) \times f(I)$. Then $f$ satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I^\circ$ of $I$. Hence, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^\circ$.*

**Note.** As a consequence of Theorem 1.4, an $\eta$-convex function $f:[a,b]\to\mathbb{R}$ where $\eta$ is bounded from above on $f([a,b])\times f([a,b])$ is integrable. For more results about $eta$-convex functions, see [3, 4, 10].

The following simple lemma is required.

**Lemma 1.5.** *Suppose that $a,b,c\in\mathbb{R}$. Then*

(i) $\min\{a,b\}\le\frac{a+b}{2}$.

(ii) *if* $c\ge 0$, $c\cdot\min\{a,b\}=\min\{ca,cb\}$.

*Proof.* Assertion (i) and (ii) are consequences of this fact:

$$\min\{a,b\}=\frac{a+b-|a-b|}{2}.$$

□

## 2. Hermite-Hadamard-Fejér Type Inequalities

In this section we obtain some Hermite-Hadamard-Fejér type integral inequalities which improve right and left side of (1.1) respectively.

**Theorem 2.1.** *Let $f:[a,b]\to\mathbb{R}$ be an $\eta$-convex function with $\eta$ bounded from above on $f([a,b])\times f([a,b])$. If $g:[a,b]\to\mathbb{R}^+$ is integrable on $[a,b]$, then we have inequalities*

$$\begin{aligned}
&\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]g(x)dx\\
&\quad\le\min\left\{f(b)+\frac{1}{2}\eta\big(f(a),f(b)\big),f(a)+\frac{1}{2}\eta\big(f(b),f(a)\big)\right\}\int_a^b g(x)dx\\
&\quad\le\frac{1}{2}[f(a)+f(b)]\int_a^b g(x)dx+\frac{1}{4}\big[\eta\big(f(a),f(b)\big)+\eta\big(f(b),f(a)\big)\big]\int_a^b g(x)dx,
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
&\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]g(x)dx\\
&\quad\le\frac{1}{2}[f(a)+f(b)]\int_a^b g(x)dx+\frac{1}{2}\left[\frac{\eta\big(f(a),f(b)\big)+\eta\big(f(b),f(a)\big)}{b-a}\right]\int_a^b(x-a)g(x)dx,
\end{aligned}\tag{2.2}$$

*and*

$$\begin{aligned}
\int_a^b f(x)g(x)dx&\le\frac{1}{2}[f(a)+f(b)]\int_a^b g(x)dx+\frac{\eta\big(f(a),f(b)\big)}{2(b-a)}\int_a^b(x-a)g(x)dx\\
&\quad+\frac{\eta\big(f(b),f(a)\big)}{2(b-a)}\int_a^b(b-x)g(x)dx.
\end{aligned}\tag{2.3}$$

*If $g$ is symmetric on $[a,b]$, then from inequalities* (2.1)-(2.3) we get

$$\begin{aligned}
\int_a^b f(x)g(x)dx&\le\min\left\{f(b)+\frac{1}{2}\eta\big(f(a),f(b)\big),f(a)+\frac{1}{2}\eta\big(f(b),f(a)\big)\right\}\int_a^b g(x)dx\\
&\le\frac{1}{2}[f(a)+f(b)]\int_a^b g(x)dx+\frac{1}{4}\big[\eta\big(f(a),f(b)\big)+\eta\big(f(b),f(a)\big)\big]\int_a^b g(x)dx,
\end{aligned}\tag{2.4}$$

and

$$\int_a^b f(x)g(x)dx \le \frac{1}{2}[f(a)+f(b)]\int_a^b g(x)dx + \frac{1}{2}\left[\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{b-a}\right]\int_a^b (x-a)g(x)dx. \tag{2.5}$$

*Proof.* If in (1.2) we put $t$ instead of $1-t$ and then add that inequality with (1.2) we have:

$$\frac{1}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] \le f(b)+\frac{1}{2}\eta\left(f(a),f(b)\right), \tag{2.6}$$

for all $t\in[0,1]$.

If in (2.6) we replace $a$ with $b$, add the result with (2.6) and then using assertion (i) of Lemma 1.5

$$\begin{aligned}\frac{1}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] &\le \min\left\{f(b)+\frac{1}{2}\eta\left(f(a),f(b)\right),f(a)+\frac{1}{2}\eta\left(f(b),f(a)\right)\right\}\\ &\le \frac{1}{2}[f(a)+f(b)]+\frac{1}{4}\left[\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right].\end{aligned} \tag{2.7}$$

Now, if in (1.2) we put $a$ instead of $b$ and then add that inequality with (1.2) we get:

$$f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right) \le f(b)+f(a)+t\left[\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right]$$

for all $t\in[0,1]$, which is equivalent to

$$\frac{1}{2}\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right] \le \frac{1}{2}[f(a)+f(b)]+t\frac{1}{2}\left[\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right]. \tag{2.8}$$

If we change $a$ with $b$, and $t$ with $1-t$ in (1.2) and then add that inequality with (1.2) we obtain

$$2f\left(ta+(1-t)b\right) \le f(b)+f(a)+t\eta\left(f(a),f(b)\right)+(1-t)\eta\left(f(b),f(a)\right),$$

for all $t\in[0,1]$ and the following inequality is proved.

$$f\left(ta+(1-t)b\right) \le \frac{1}{2}[f(a)+f(b)]+\frac{1}{2}[t\eta(f(a),f(b))+(1-t)\eta(f(b),f(a))]. \tag{2.9}$$

Since $\eta$ is bounded from above on $f([a,b])\times f([a,b])$, the note after Theorem 1.4, guarantees that the function $f$ is integrable. By multiplying the inequalities (2.7)-(2.9) with $g\left((1-t)a+tb\right)\ge 0$ and then integrating over $t\in[0,1]$, we have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]g\left((1-t)a+tb\right)dt\\ &\le \min\left\{f(b)+\frac{1}{2}\eta\left(f(a),f(b)\right),f(a)+\frac{1}{2}\eta\left(f(b),f(a)\right)\right\}\int_0^1 g\left((1-t)a+tb\right)dt\\ &\le \frac{1}{2}[f(a)+f(b)]\int_0^1 g\left((1-t)a+tb\right)dt\\ &\quad+\frac{1}{4}\left[\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right]\int_0^1 g\left((1-t)a+tb\right)dt,\end{aligned} \tag{2.10}$$

$$\frac{1}{2}\int_0^1 [f(ta+(1-t)b)+f((1-t)a+tb)]g((1-t)a+tb)dt$$
$$\le \frac{1}{2}[f(a)+f(b)]\int_0^1 g((1-t)a+tb)dt$$
$$+\frac{1}{2}[\eta(f(a),f(b))+\eta(f(b),f(a))]\int_0^1 tg((1-t)a+tb)dt, \tag{2.11}$$

and

$$\int_0^1 f(ta+(1-t)b)g((1-t)a+tb)dt \le \frac{1}{2}[f(a)+f(b)]\int_0^1 g((1-t)a+tb)dt$$
$$+\frac{1}{2}\eta(f(a),f(b))\int_0^1 tg((1-t)a+tb)dt$$
$$+\frac{1}{2}\eta(f(b),f(a))\int_0^1 (1-t)g((1-t)a+tb)dt. \tag{2.12}$$

If the function $g$ is symmetric on $[a,b]$, then

$$\frac{1}{2}\int_0^1 [f(ta+(1-t)b)+f((1-t)a+tb)]g((1-t)a+tb)dt$$
$$=\frac{1}{2}\int_0^1 f(ta+(1-t)b)g((1-t)a+tb)dt+\frac{1}{2}\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt$$
$$=\frac{1}{2}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt+\frac{1}{2}\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt$$
$$=\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt.$$

In the last equality we used changing the variable $u=1-t$ for $t\in[0,1]$.

Then, by relations (2.10) and (2.11) (or (2.12)), we get

$$\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt$$
$$\le \min\left\{f(b)+\frac{1}{2}\eta(f(a),f(b)),f(a)+\frac{1}{2}\eta(f(b),f(a))\right\}\int_0^1 g((1-t)a+tb)dt$$
$$\le \frac{1}{2}[f(a)+f(b)]\int_0^1 g((1-t)a+tb)dt$$
$$+\frac{1}{4}[\eta(f(a),f(b))+\eta(f(b),f(a))]\int_0^1 g((1-t)a+tb)dt, \tag{2.13}$$

and

$$\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt$$
$$\le \frac{1}{2}[f(a)+f(b)]\int_0^1 g((1-t)a+tb)dt+\frac{1}{2}[\eta(f(a),f(b))+\eta(f(b),f(a))]\int_0^1 tg((1-t)a+tb)dt. \tag{2.14}$$

Now it is enough to apply the change of variable $x = ta + (1-t)b$ in (2.10)-(2.14). Note that if $g$ is symmetric we have

$$\int_a^b (x-a)g(x)dx = \int_a^b (b-x)g(x)dx. \qquad \square$$

**Remark 2.2.** Inequalities (2.4) and (2.5) give a refinement for the right side of (1.1). If in Theorem 2.1, we consider

(1) $g(x) \equiv 1$, then we obtain a refinement for the right side of Hermite-Hadamard type inequality related to $\eta$-convex functions (see [4, Theorem 3.6]).

(2) If we suppose that $\eta(x,y) = x - y$, then we recapture right side of (1.1).

In the following theorem we improve the left side of (1.1).

**Theorem 2.3.** *Let $f : [a,b] \to \mathbb{R}$ be an $\eta$-convex function with $\eta$ bounded from above on $f([a,b]) \times f([a,b])$. If $g : [a,b] \to \mathbb{R}^+$ is integrable on $[a,b]$, then*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&\le \int_a^b g(x)\min\left\{f(a+b-x) + \frac{1}{2}\eta\big(f(x), f(a+b-x)\big), f(x) + \frac{1}{2}\eta\big(f(a+b-x), f(x)\big)\right\}dx \\
&\le \min\left\{\int_a^b g(x)f(a+b-x)dx + \frac{1}{2}\int_a^b g(x)\eta\big(f(x), f(a+b-x)\big)dx,\right. \\
&\left.\int_a^b g(x)f(x)dx + \frac{1}{2}\int_a^b g(x)\eta\big(f(a+b-x), f(x)\big)dx\right\} \\
&\le \int_a^b g(x)\frac{f(a+b-x)+f(x)}{2}dx + \frac{1}{4}\int_a^b g(x)\big[\eta\big(f(x), f(a+b-x)\big) + \eta\big(f(a+b-x), f(x)\big)\big]dx.
\end{aligned} \tag{2.15}$$

*Moreover, if $g$ is symmetric on $[a,b]$, then*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \\
&\le \int_a^b g(x)\min\left\{f(a+b-x) + \frac{1}{2}\eta\big(f(x), f(a+b-x)\big), f(x) + \frac{1}{2}\eta\big(f(a+b-x), f(x)\big)\right\}dx \\
&\le \int_a^b g(x)f(x)dx + \frac{1}{2}\int_a^b g(x)\eta\big(f(x), f(a+b-x)\big)dx.
\end{aligned} \tag{2.16}$$

*Proof.* Since we have

$$f\Big(\frac{a+b}{2}\Big) = f\Big(\frac{ta+(1-t)b+tb+(1-t)a}{2}\Big), \tag{2.17}$$

then by using the concept of $\eta$-convexity

$$f\left(\frac{a+b}{2}\right) \le \min\left\{f\big(ta+(1-t)b\big) + \frac{1}{2}\eta\big(f\big((1-t)a+tb\big), f(ta+(1-t)b)\big),\right.$$

$$f\big((1-t)a+tb\big)+\frac{1}{2}\eta\big(f(ta+(1-t)b),f\big((1-t)a+tb\big)\big)\Big\}, \tag{2.18}$$

for any $t\in[0,1]$. Multiplying inequality (2.18) with $g\big((1-t)a+tb\big)\geq 0$ and integrating over $t$ we have

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)\int_0^1 g\big((1-t)a+tb\big)\\
&\leq \int_0^1 g\big((1-t)a+tb\big)\times\min\Big\{f\big(ta+(1-t)b\big)+\frac{1}{2}\eta\big(f\big((1-t)a+tb\big),f(ta+(1-t)b)\big),\\
&\qquad f\big((1-t)a+tb\big)+\frac{1}{2}\eta\big(f(ta+(1-t)b),f\big((1-t)a+tb\big)\big)\Big\}\,dt.
\end{aligned}$$

Using assertion (ii) of lemma 1.5 implies that

$$\begin{aligned}
&\int_0^1 g\big((1-t)a+tb\big)\times\min\Big\{f\big(ta+(1-t)b\big)+\frac{1}{2}\eta\big(f\big((1-t)a+tb\big),f(ta+(1-t)b)\big),\\
&\qquad f\big((1-t)a+tb\big)+\frac{1}{2}\eta\big(f(ta+(1-t)b),f\big((1-t)a+tb\big)\big)\Big\}\,dt\\
&\leq \min\Big\{\int_0^1 g\big((1-t)a+tb\big)f\big(ta+(1-t)b\big)dt\\
&\qquad +\frac{1}{2}\int_0^1 g\big((1-t)a+tb\big)\eta\big(f\big((1-t)a+tb\big),f\big(ta+(1-t)b\big)\big)dt,\\
&\qquad \int_0^1 g\big((1-t)a+tb\big)f\big((1-t)a+tb\big)dt\\
&\qquad +\frac{1}{2}\int_0^1 g\big((1-t)a+tb\big)\eta\big(f\big(ta+(1-t)b\big),f\big((1-t)a+tb\big)\big)dt\Big\}.
\end{aligned}$$

Also from assertion (i) of Lemma 1.5

$$\begin{aligned}
&\min\Big\{\int_0^1 g\big((1-t)a+tb\big)f\big(ta+(1-t)b\big)dt\\
&\qquad +\frac{1}{2}\int_0^1 g\big((1-t)a+tb\big)\eta\big(f\big((1-t)a+tb\big),f\big(ta+(1-t)b\big)\big)dt,\\
&\qquad \int_0^1 g\big((1-t)a+tb\big)f\big((1-t)a+tb\big)dt\\
&\qquad +\frac{1}{2}\int_0^1 g\big((1-t)a+tb\big)\eta\big(f\big(ta+(1-t)b\big),f\big((1-t)a+tb\big)\big)dt\Big\}\\
&\leq \int_0^1 g\big((1-t)a+tb\big)\frac{f\big(ta+(1-t)b\big)+f\big((1-t)a+tb\big)}{2}dt\\
&\qquad +\frac{1}{4}\int_0^1 g\big((1-t)a+tb\big)\big[\eta\big(f\big((1-t)a+tb\big),f\big(ta+(1-t)b\big)\big)\\
&\qquad +\eta\big(f\big(ta+(1-t)b\big),f\big((1-t)a+tb\big)\big)\big]dt.
\end{aligned}$$

Now it is enough to apply the change of variable $x=(1-t)a+tb$ to obtain (2.15).

In the case that $g$ is symmetric on $[a,b]$, it is clear that

$$\int_a^b g(x)\eta\big(f(x),f(a+b-x)\big)dx=\int_a^b g(x)\eta\big(f(a+b-x),f(x)\big)dx,$$

$$\int_a^b g(x)\frac{f(a+b-x)+f(x)}{2}dx=\int_a^b g(x)f(x)dx.$$

So we have (2.16). □

**Remark 2.4.** Inequality (2.16) gives a refinement for the left side of (1.1). Also if in Theorem 2.3 we consider

(2) $g(x)\equiv 1$, then we obtain a refinement for the left side of Hermite-Hadamard type inequality related to $\eta$-convex functions (see [4, Theorem 3.6]).

(3) If we suppose that $\eta(x,y)=x-y$, then we recapture right side of (1.1).

## 3. Trapezoid and mid-point type inequalities

An interesting question in (1.1), is estimating the difference between left and middle terms and between right and middle terms. Respectively we investigate about first and then second part of this question, when the absolute value of the derivative of a function is $\eta$-convex.

The following lemma is generalization of Lemma 2.1 in [7].

**Lemma 3.1.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping, $g:[a,b]\to\mathbb{R}^+$ is a continuous mapping and $f'$ is integrable on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx$$
$$=(b-a)\Big[\int_0^{\frac12}M_1(t)f'\big(ta+(1-t)b\big)dt+\int_{\frac12}^1 M_2(t)f'\big(ta+(1-t)b\big)dt\Big],$$

*where*

$$M_1(t)=\int_0^t g\big(ua+(1-u)b\big)du \quad \textit{and} \quad M_2(t)=-\int_t^1 g\big(ua+(1-u)b\big)du.$$

*Proof.* Using the change of variable $x=ta+(1-t)b$, we have

$$I_1=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx$$
$$=\int_0^1 f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)dt-f\Big(\frac{a+b}{2}\Big)\int_0^1 g\big(ta+(1-t)b\big)dt=I_2.$$

It follows that

$$I_2=\int_0^{\frac12} f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)dt-f\Big(\frac{a+b}{2}\Big)\int_0^{\frac12} g\big(ta+(1-t)b\big)dt$$
$$+\int_{\frac12}^1 f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)dt-f\Big(\frac{a+b}{2}\Big)\int_{\frac12}^1 g\big(ta+(1-t)b\big)dt.$$

On the other hand Leibniz integral rule gives

$$g\big(ta+(1-t)b\big)=\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)'=\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)'.$$

So

$$\begin{aligned}I_2&=\int_0^{\frac{1}{2}} f\big(ta+(1-t)b\big)\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)'dt-f\Big(\frac{a+b}{2}\Big)\int_0^{\frac{1}{2}} g\big(ta+(1-t)b\big)dt\\&\quad+\int_{\frac{1}{2}}^1 f\big(ta+(1-t)b\big)\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)'dt-f\Big(\frac{a+b}{2}\Big)\int_{\frac{1}{2}}^1 g\big(ta+(1-t)b\big)dt.\end{aligned}\tag{3.1}$$

Using integration by parts in (3.1) implies that

$$\begin{aligned}I_2&=\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)\cdot f\big(ta+(1-t)b\big)\Big|_0^{\frac{1}{2}}\\&\quad-\int_0^{\frac{1}{2}}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)(a-b)dt\\&\quad-f\Big(\frac{a+b}{2}\Big)\int_0^{\frac{1}{2}} g\big(ta+(1-t)b\big)dt+\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)f\big(ta+(1-t)b\big)\Big|_{\frac{1}{2}}^1\\&\quad+\int_{\frac{1}{2}}^1\Big(\int_t^1 g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)(a-b)dt-f\Big(\frac{a+b}{2}\Big)\int_{\frac{1}{2}}^1 g\big(ta+(1-t)b\big)dt.\end{aligned}$$

If we apply the limits in above we have

$$\begin{aligned}I_2&=(b-a)\Big[\int_0^{\frac{1}{2}}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)dt\\&\quad+\int_{\frac{1}{2}}^1\Big(-\int_t^1 g\big(ua+(1-u)b\big)du\Big)f'\big(ta+(1-t)b\big)dt\Big].\end{aligned}$$

Since $I_1=I_2$, the result is obtained. □

**Remark 3.2.** *In Lemma* 3.1*, if we use the change of variable* $x=tb+(1-t)a$*, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\\&\quad=(b-a)\Big[\int_0^{\frac{1}{2}} M_1(t)f'\big(tb+(1-t)a\big)dt+\int_{\frac{1}{2}}^1 M_2(t)f'\big(tb+(1-t)a\big)dt\Big],\end{aligned}$$

*where*

$$M_1(t)=-\int_0^t g\big(ub+(1-u)a\big)du\quad\text{and}\quad M_2(t)=\int_t^1 g\big(ub+(1-u)a\big)du.$$

Using Lemma 3.1 and Remark 3.2, we can prove the following theorem to estimate the difference between the middle and left terms in (1.1).

**Theorem 3.3.** *Suppose that* $f:[a,b]\to\mathbb{R}$ *is a differentiable mapping,* $g:[a,b]\to\mathbb{R}^+$ *is a continuous mapping symmetric about* $\frac{a+b}{2}$ *and* $|f'|$ *is a* $\eta$*-convex mapping on* $[a,b]$ *with* $\eta$

*bounded from above on* $f([a,b])\times f([a,b])$. *Then*

$$\left|\int_a^b f(x)g(x)dx - f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right| \le \frac{1}{(b-a)}\int_{\frac{a+b}{2}}^b \big[(x-a)^2-(b-x)^2\big]g(x)Kdx,$$

*where*

$$K=\min\Big\{\big|f'(b)\big|+\frac{\eta(|f'(a)|,|f'(b)|)}{2},\big|f'(a)\big|+\frac{\eta(|f'(b)|,|f'(a)|)}{2}\Big\}.$$

*Proof.* From Lemma 3.1 and $\eta$-convexity of $|f'|$ it follows that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|\\ &\le (b-a)\Big\{\int_0^{\frac12}\Big(\int_0^t g\big(ua+(1-u)b\big)du\Big)\Big(|f'(b)|+t\eta\big(|f'(a)|,|f'(b)|\big)\Big)dt\\ &\qquad+\int_{\frac12}^1\Big(\int_t^1 g\big(ua+(1-u)b\big)du\Big)\Big(|f'(b)|+t\eta\big(|f'(a)|,|f'(b)|\big)\Big)dt\Big\}=J_1\end{aligned}$$

Changing the order of integrals and calculation of internal integrals in $J_1$ imply that

$$\begin{aligned}J_1&=(b-a)\Big\{\int_0^{\frac12}\int_u^{\frac12} g\big(ua+(1-u)b\big)\Big(|f'(b)|+t\eta\Big(|f'(a)|,|f'(b)|\Big)\Big)dtdu\\ &\qquad+\int_{\frac12}^1\int_{\frac12}^u g\big(ua+(1-u)b\big)\Big(|f'(b)|+t\eta\Big(|f'(a)|,|f'(b)|\Big)\Big)dtdu\Big\}\\ &=(b-a)\Big\{\int_0^{\frac12}\Big(t|f'(b)|+\frac12 t^2\eta\Big(|f'(a)|,|f'(b)|\Big)\Big|_u^{\frac12}\Big)g\Big(ua+(1-u)b\Big)du\\ &\qquad+\int_{\frac12}^1\Big(t|f'(b)|+\frac12 t^2\eta\Big(|f'(a)|,|f'(b)|\Big)\Big|_{\frac12}^{u}\Big)g\Big(ua+(1-u)b\Big)du\Big\}\\ &=(b-a)\Big\{\int_0^{\frac12}\Big(\frac12-u\Big)|f'(b)|+\Big(\frac18-\frac12u^2\Big)\eta\big(|f'(a)|,|f'(b)|\big)g\big(ua+(1-u)b\big)du\\ &\qquad+\int_{\frac12}^1\Big(u-\frac12\Big)|f'(b)|+\Big(\frac12u^2-\frac18\Big)\eta\big(|f'(a)|,|f'(b)|\big)g\big(ua+(1-u)b\big)du\Big\}=J_2.\end{aligned}$$

Changing the variable by $x=ua+(1-u)b$ in $J_2$ implies that

$$\begin{aligned}J_2&=\int_{\frac{a+b}{2}}^b\Big(\frac12-\frac{x-b}{a-b}\Big)|f'(b)|+\Big(\frac18-\frac12\Big(\frac{x-b}{a-b}\Big)^2\Big)\eta\Big(|f'(a)|,|f'(b)|\Big)g(x)dx\\ &\qquad+\int_a^{\frac{a+b}{2}}\Big(\frac{x-b}{a-b}-\frac12\Big)|f'(b)|+\Big(\frac12\Big(\frac{x-b}{a-b}\Big)^2-\frac18\Big)\eta\Big(|f'(a)|,|f'(b)|\Big)g(x)dx\\ &=\int_{\frac{a+b}{2}}^b\Big(\frac{a+b-2x}{2(a-b)}\Big)|f'(b)|+\Big(\frac{(a-b)^2-4(x-b)^2}{8(a-b)^2}\Big)\eta\Big(|f'(a)|,|f'(b)|\Big)g(x)dx\\ &\qquad+\int_a^{\frac{a+b}{2}}\Big(\frac{2x-(a+b)}{2(a-b)}\Big)|f'(b)|+\Big(\frac{4(x-b)^2-(a-b)^2}{8(a-b)^2}\Big)\eta\Big(|f'(a)|,|f'(b)|\Big)g(x)dx=J_3.\end{aligned}$$

Since for any $x \in [a,b]$ we have $g(x) = g(a+b-x)$, then

$$J_3 = \int_{\frac{a+b}{2}}^{b} \Big(\frac{a+b-2x}{2(a-b)}\Big)|f'(b)| + \Big(\frac{(a-b)^2 - 4(x-b)^2}{8(a-b)^2}\Big)\eta\Big(|f'(a)|, |f'(b)|\Big)g(x)dx$$
$$+ \int_{\frac{a+b}{2}}^{b} \Big(\frac{a+b-2x}{2(a-b)}\Big)|f'(b)| + \Big(\frac{4(x-a)^2 - (a-b)^2}{8(a-b)^2}\Big)\eta\Big(|f'(a)|, |f'(b)|\Big)g(x)dx$$
$$= \int_{\frac{a+b}{2}}^{b} \Big(\frac{a+b-2x}{(a-b)}\Big)|f'(b)| + \Big(\frac{4(x-a)^2 - 4(x-b)^2}{8(a-b)^2}\Big)\eta\Big(|f'(a)|, |f'(b)|\Big)g(x)dx$$
$$= \frac{1}{(a-b)^2}\int_{\frac{a+b}{2}}^{b} \Big((a-b)(a+b-2x)\Big)|f'(b)| + \Big((x-a)^2 - (x-b)^2\Big)\frac{\eta\big(|f'(a)|, |f'(b)|\big)}{2}g(x)dx$$
$$= \frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} \Big((x-a)^2 - (b-x)^2\Big)\Big(|f'(b)| + \frac{\eta\big(|f'(a)|, |f'(b)|\big)}{2}\Big)g(x)dx = J_4.$$

On the other hand, according to Remark 3.2, if we use the change of variable $x = ub + (1-u)a$ in $J_2$ then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right|$$
$$\le \frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^{b} \Big((x-a)^2 - (b-x)^2\Big)\Big(\Big|f'(a)\Big| + \frac{\eta\big(|f'(b)|, |f'(a)|\big)}{2}\Big)g(x)dx = J_5.$$

Since $J_1 = J_2 = J_3 = J_4$, then we can deduce the result from

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\right| \le \min\{J_4, J_5\}. \qquad \square$$

If in Theorem 3.3 consider $\eta(x,y) = x - y$ and $g \equiv 1$, we get:

**Corollary 3.4** ([7])**.** *Consider $I^*$ as the interior of interval $I \subset \mathbb{R}$. Let $f : I^* \to \mathbb{R}$ be a differentiable mapping on $I^*$, $a, b \in I^*$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{b-a}{8}\Big(|f'(a)| + |f'(b)|\Big)$$

In [3] we proved the following results to estimate the difference between the middle and right terms in (1.1). For more details see [3].

**Lemma 3.5.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $f'$ is an integrable function on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx$$
$$= \frac{(b-a)}{4}\Big\{\int_0^1 \Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Big)f'\Big(\frac{1+t}{2}a + \frac{1-t}{2}b\Big)dt$$
$$+ \int_0^1 \Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)du\Big)f'\Big(\frac{1-t}{2}a + \frac{1+t}{2}b\Big)dt\Big\}. \tag{3.2}$$

**Theorem 3.6.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $|f'|$ is a $\eta$-convex function with $\eta$ bounded from above on $f([a,b])\times f([a,b])$. Then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx\right|\\&\quad\le\frac{(b-a)}{4}\left[2|f'(b)|+\eta\big(|f'(a)|,|f'(b)|\big)\right]\int_0^1\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b}g(u)dudt.\end{aligned}$$

If in Theorem 3.6 consider $\eta(x,y)=x-y$ and $g\equiv1$, we have:

**Corollary 3.7** ([2])**.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a,b\in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{b-a}{8}\big(|f'(a)|+|f'(b)|\big).$$

## 4. Conclusions

The convexity of a function is a basis for many inequalities in mathematics. It should be noticed that in new problems related to convexity, generalized notions about convex functions are required to obtain applicable results. One of this generalizations is the notion of $\eta$-convex functions which can generalize many inequalities related to convex functions specially famous Fejér inequality with estimating the difference between left and middle terms and between right and middle terms of this inequality.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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