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Coupled Best Proximity Points under the Proximally Coupled Contraction in a Complete Ordered Metric Space

Research Article

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Abstract. In this paper, we prove the existence and uniqueness of a coupled best proximity point for mappings satisfying the proximally coupled contraction condition in a complete ordered metric space. Further, our result provides an extension of a result due to Luong and Thuan (*Comput. Math. Appl.* **62** (11) (2011), 4238–4248, *Nonlinear Anal.* **74** (2011), 983–992.

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1. Introduction and Mathematical Preliminaries

Let A be a nonempty subset of a metric space (X,d). A mapping $T : A \to X$ has a fixed point in A if the fixed point equation $Tx = x$ has at least one solution. That is, $x \in A$ is a fixed point of T if $d(x,Tx) = 0$. If the fixed point equation $Tx = x$ does not possess a solution, then $d(x,Tx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x,Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Here we state the following well-known best approximation theorem due to Ky Fan [9].

Theorem 1.1 ([9]). *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx,A) := \inf\{\|Tx - a\| : a \in A\}$.*

Such an element $x \in A$ in Theorem 1.1 is called a best approximant of T in A. Note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x\in A} \|x - Tx\|$ has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets A,B of a metric space X and a mapping $T : A \to B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0,Tx_0) = \min\{d(x,Tx) : x \in A\}$. Since $d(x,Tx) \ge d(A,B)$, the optimal solution to the problem of minimizing the real valued function $x \to d(x,Tx)$ over the domain A of the mapping T will be the one for which the value $d(A,B)$ is attained. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0,Tx_0) = d(A,B)$. Note that if $d(A,B) = 0$, then the best proximity point is nothing but a fixed point of T. Also, best proximity point theory in ordered metric spaces was first studied in [1].

The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [3, 5, 8, 13, 15–17, 24–27]. Also one can find the existence of best proximity point in the setting of partially order metric space in [2, 7, 23, 28].

On the other hand, Bhaskar and Lakshmikantham were introduced the concept called mixed monotone mapping and proved coupled fixed point theorems for mappings satisfying the mixed monotone property which is used to investigate a large class of problems and discussed the existence and uniqueness of a solution for a periodic boundary value problem. One can find the existence of coupled fixed points in the setting of partially order metric space in [10–12, 19–21, 29].

Now we recall the definition of coupled fixed point which was introduced by Sintunavarat and Kumam in [28]. Let X be a non-empty set and $F : X \times X \to X$ be a given mapping. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping F if $F(x,y) = x$ and $F(y,x) = y$.

The authors mentioned above also introduced the notion of mixed monotone mapping. If $(X,\le)$ is a partially ordered set, the mapping F is said to have the mixed monotone property if

$$x_1, x_2 \in X, x_1 \le x_2 \Rightarrow F(x_1, y) \le F(x_2, y), \quad \forall\ y \in X$$

and

$$y_1, y_2 \in X, y_1 \le y_2 \Rightarrow F(x, y_1) \ge F(x, y_2), \quad \forall\ x \in X.$$

In [22], Luong and Thuan obtained a more general result. For this, let Φ denote all functions $\phi : [0,\infty) \to [0,\infty)$ which satisfy

(i) ϕ is continuous and non-decreasing;

(ii) $\phi(t) = 0$ if and only if $t = 0$;

(iii) $\phi(t+s) \le \phi(t) + \phi(s)$, $\forall\ t,s \in (0,\infty]$.

Again, let Ψ denote all functions $\psi : (0,\infty] \to (0,\infty]$ which satisfy $\lim\limits_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim\limits_{t\to 0^+} \psi(t) = 0$.

The main theoretical results of Nguyen Van Luong and Nguyen Xuan Thuan, in [22] is the following.

Theorem 1.2 ([22]). *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be mapping having the mixed monotone property on X such that*

$$\phi(d(F(x,y),F(u,v))) \le \frac{1}{2}\phi(d(x,u)+d(y,v)) - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right), \tag{1.1}$$

for all $x,y,u,v \in X$ with $x \ge u$ and $y \le v$, where $\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0, y_0 \in X$ such that $x_0 \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$. Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

 (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all n,*

 (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \ge y_n$ for all n.*

Then there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Motivated by the above theorems, we introduce the concept of proximal mixed monotone property and proximally coupled weak (ψ,ϕ) contraction on A. We also explore the existence and uniqueness of coupled best proximity points in the setting of partially ordered metric spaces. Further, we attempt to give the generalization of Theorem 1.2.

Let X be a non-empty set such that (X,d) is a metric space. Unless otherwise specified, it is assumed throughout this section that A and B are non-empty subsets of the metric space (X,d), the following notions are used subsequently:

$$\begin{aligned} &d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},\\ &A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},\\ &B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}. \end{aligned}$$

In [16], the authors discussed sufficient conditions which guarantee the non-emptiness of A_0 and B_0. Also, in [8], the authors proved that A_0 is contained in the boundary of A. Moreover, the authors proved that A_0 is contained in the boundary of A in the setting of normed linear spaces.

Definition 1.3. Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. A mapping $F : A \times A \to B$ is said to have proximal mixed monotone property if $F(x,y)$ is proximally nondecreasing in x and is proximally non-increasing in y, that is, for all $x, y \in A$.

$$\left.\begin{array}{r} x_1 \leq x_2 \\ d(u_1, F(x_1, y)) = d(A,B) \\ d(u_2, F(x_2, y)) = d(A,B) \end{array}\right\} \implies u_1 \leq u_2$$

and

$$\left.\begin{array}{r} y_1 \leq y_2 \\ d(u_3, F(x, y_1)) = d(A,B) \\ d(u_4, F(x, y_2)) = d(A,B) \end{array}\right\} \implies u_4 \leq u_3$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximal mixed monotone property reduces to that of mixed monotone property.

Lemma 1.4. *Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ whenever x_0, x_1, x_2, y_0, y_1 in A_0 such that*

$$\left.\begin{array}{r} x_0 \leq x_1 \text{ and } y_0 \geq y_1 \\ d(x_1, F(x_0, y_0)) = d(A,B) \\ d(x_2, F(x_1, y_1)) = d(A,B) \end{array}\right\} \implies x_1 \leq x_2. \tag{1.2}$$

Proof. By hypothesis $F(A_0 \times A_0) \subseteq B_0$, therefore $F(x_1, y_0) \in B_0$. Hence there exists $x_1^* \in A$ such that

$$d(x_1^*, F(x_1, y_0)) = d(A,B). \tag{1.3}$$

Using F is proximal mixed monotone (In particular F is proximally non decreasing in x) to (1.2) and (1.3), we get

$$\left.\begin{array}{r} x_0 \leq x_1 \\ d(x_1, F(x_0, y_0)) = d(A,B) \\ d(x_1^*, F(x_1, y_0)) = d(A,B) \end{array}\right\} \implies x_1 \leq x_1^*. \tag{1.4}$$

Analogously, using F is proximal mixed monotone (In particular F is proximally non increasing in y) to (1.2) and (1.3), we get

$$\left.\begin{array}{r} y_1 \leq y_0 \\ d(x_2, F(x_1, y_1)) = d(A,B) \\ d(x_1^*, F(x_1, y_0)) = d(A,B) \end{array}\right\} \implies x_1^* \leq x_2. \tag{1.5}$$

From (1.4) and (1.5), one can conclude the $x_1 \leq x_2$. Hence the proof. □

Lemma 1.5. *Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ whenever x_0, x_1, y_0, y_1, y_2 in A_0 such that*

$$\left.\begin{aligned} x_0 \le x_1 \ and \ y_0 \ge y_1 \\ d(y_1, F(y_0, x_0)) = d(A,B) \\ d(y_2, F(y_1, x_1)) = d(A,B) \end{aligned}\right\} \implies y_1 \ge y_2. \tag{1.6}$$

Proof. The proof is same as the Lemma 1.4. □

In 2014, A.H. Ansari [4] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 1.6 ([4]). A continuous function $f : [0,\infty)^2 \to \mathbb{R}$ is called C-*class* function if for any $s,t \in [0,\infty)$, the following conditions hold:

(1) $f(s,t) \le s$;

(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on f that $f(0,0) = 0$ could be imposed in some cases if required. The letter $\mathscr{C}$ will denote the class of all C- functions.

Example 1.7 ([4]). Following examples show that the class $\mathscr{C}$ is nonempty:

(i) $f(s,t) = s - t$.

(ii) $f(s,t) = ms$.

(iii) $f(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.

(iv) $f(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.

(v) $f(s,t) = \ln(1 + a^s)/2$, for $a > e$. Indeed $f(s,t) = s$ implies that $s = 0$.

(vi) $f(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0,\infty)$.

(vii) $f(s,t) = s\log_{t+a} a$, for $a > 1$.

(viii) $f(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.

(ix) $f(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$ is continuous.

(x) $f(s,t) = s - \frac{t}{k+t}$.

(xi) $f(s,t) = s - \varphi(s)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(xii) $f(s,t) = sh(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$.

(xiii) $f(s,t) = s - (\frac{2+t}{1+t})t$.

(xiv) $f(s,t) = \sqrt[n]{\ln(1+s^n)}$.

(xv) $f(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

(xvi) $f(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$.

Definition 1.8 ([14]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.9. An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) = 0$.

We denote Φ_u set ultra altering distance functions.

Let Φ_s denote the class of the functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) φ lower semicontinuous;

(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) = 0$.

Lemma 1.10 ([6]). *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)},x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1},x_{n(k)}) < \varepsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)+1}) = \varepsilon$;

(ii) $\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \varepsilon$;

(iii) $\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)}) = \varepsilon$.

Remark 1.11. We note that also can see $\lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \varepsilon$ and $\lim_{k\to\infty} d(x_{m(k)},x_{n(k)-1}) = \varepsilon$.

Definition 1.12. Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. A mapping $F : A \times A \to B$ is said to be proximally coupled weak (ψ,ϕ,f) contraction on A, whenever

$$\left.\begin{array}{r} x_1 \leq x_2 \text{ and } y_1 \geq y_2 \\ d(u_1,F(x_1,y_1)) = d(A,B) \\ d(u_2,F(x_2,y_2)) = d(A,B) \end{array}\right\}$$
$$\implies \phi(d(u_1,u_2)) \leq \frac{1}{2} f\left(\phi(d(x_1,x_2)+d(y_1,y_2)), \psi\left(\frac{d(x_1,x_2)+d(y_1,y_2)}{2}\right)\right) \tag{1.7}$$

where $x_1,x_2,y_1,y_2,u_1,u_2 \in A$.

Definition 1.13 ([18]). Let $(X,d,\leq)$ be a partially ordered metric space and A, B are nonempty subsets of X. A mapping $F : A \times A \to B$ is said to be proximally coupled weak (ψ,ϕ) contraction on A, whenever

$$\left.\begin{array}{r} x_1 \leq x_2 \text{ and } y_1 \geq y_2 \\ d(u_1,F(x_1,y_1)) = d(A,B) \\ d(u_2,F(x_2,y_2)) = d(A,B) \end{array}\right\}$$
$$\implies \phi(d(u_1,u_2)) \leq \frac{1}{2}\phi(d(x_1,x_2)+d(y_1,y_2)) - \psi\left(\frac{d(x_1,x_2)+d(y_1,y_2)}{2}\right) \tag{1.8}$$

where $x_1,x_2,y_1,y_2,u_1,u_2 \in A$.

Remark 1.14. With note to $a-2b<a-b$, for $f(s,t)=s-t$, Definition 1.12 stronger than Definition 1.13.

One can see that, if $A=B$ in the above definition, the notion of proximally coupled weak (ψ,ϕ) contraction on A reduces to that of coupled weak (ψ,ϕ) contraction. Let us recall a notion of P-property: The pair (A,B) of nonempty subsets of a metric space (X,d) with $A_0\neq\emptyset$. is said to have the P-property if and only if

$$\left.\begin{aligned} d(x_1,y_1)=d(A,B)\\ d(x_2,y_2)=d(A,B)\end{aligned}\right\} \Longrightarrow (d(x_1,x_2)=d(y_1,y_2)) \tag{1.9}$$

where $x_1,x_2\in A_0$ and $y_1,y_2\in B_0$. It is interesting to note that if the pair (A,B) considered in above definition has the P-property, then the mapping F in Theorem 1.2 satisfies the inequality (1.1).

2. Coupled Best Proximity Point Theorems

Let $(X,d,\leq)$ be a partially ordered complete metric space endowed with the product space $X\times X$ with the following partial order:

$$\text{for } (x,y),(u,v)\in X\times X,\ (u,v)\leq(x,y)\Leftrightarrow x\geq u, y\leq v.$$

Theorem 2.1. *Let $(X,\leq,d)$ be a partially ordered complete metric space. Let A and B be nonempty closed subsets of the metric space (X,d) such that $A_0\neq\emptyset$. Let $F:A\times A\to B$ satisfy the following conditions.*

(i) *F is a continuous proximally coupled weak (ψ,ϕ,f) contraction on A having the proximal mixed monotone property on A such that $F(A_0\times A_0)\subseteq B_0$.*

(ii) *There exist elements (x_0,y_0) and (x_1,y_1) in $A_0\times A_0$ such that*

$$d(x_1,F(x_0,y_0))=d(A,B) \textit{ with } x_0\leq x_1 \textit{ and } d(y_1,F(y_0,x_0))=d(A,B) \textit{ with } y_0\geq y_1.$$

Then there exist $(x,y)\in A\times A$ such that $d(x,F(x,y))=d(A,B)$ and $d(y,F(y,x))=d(A,B)$.

Proof. By hypothesis there exist elements (x_0,y_0) and (x_1,y_1) in $A_0\times A_0$ such that

$$d(x_1,F(x_0,y_0))=d(A,B) \text{ with } x_0\leq x_1 \text{ and } d(y_1,F(y_0,x_0))=d(A,B) \text{ with } y_0\geq y_1.$$

Because of the fact that $F(A_0\times A_0)\subseteq B_0$, there exists an element (x_2,y_2) in $A_0\times A_0$ such that

$$d(x_2,F(x_1,y_1)=d(A,B) \text{ and } d(y_2,F(y_1,x_1)=d(A,B).$$

Hence, by Lemma 1.4 and Lemma 1.5, we obtain $x_1\leq x_2$ and $y_1\geq y_2$.

Continuing this process, we can construct the sequences (x_n) and (y_n) in A_0 such that

$$d(x_{n+1},F(x_n,y_n))=d(A,B),\ \forall\ n\in\mathbb{N} \tag{2.1}$$

$$\text{with } x_0\leq x_1\leq x_2\leq\cdots x_n\leq x_{n+1}\cdots \text{ and}$$

$$d(y_{n+1},F(y_n,x_n))=d(A,B),\ \forall\ n\in\mathbb{N} \tag{2.2}$$

$$\text{with } y_0\geq y_1\geq y_2\geq\cdots y_n\geq y_{n+1}\cdots.$$

Then $d(x_n,F(x_{n-1},y_{n-1}))=d(A,B)$, $d(x_{n+1},F(x_n,y_n))=d(A,B)$ and also we have

$$x_{n-1}\leq x_n, y_{n-1}\geq y_n,\ \forall\ n\in\mathbb{N}.$$

Now using that F is proximally coupled weak (ψ,ϕ) contraction on A we get,

$$\phi(d(x_n,x_{n+1})) \le \frac{1}{2} f\left(\phi(d(x_{n-1},x_n)+d(y_{n-1},y_n)),\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\right), \ \forall\, n\in\mathbb{N}. \tag{2.3}$$

Similarly

$$\phi(d(y_n,y_{n+1})) \le \frac{1}{2} f\left(\phi(d(y_{n-1},y_n)+d(x_{n-1},x_n)),\psi\left(\frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)}{2}\right)\right), \ \forall\, n\in\mathbb{N}. \tag{2.4}$$

Adding (2.3) and (2.4), we get

$$\begin{aligned}&\phi(d(x_n,x_{n+1}))+\phi(d(y_n,y_{n+1}))\\ &\quad\le f\left(\phi(d(x_{n-1},x_n)+d(y_{n-1},y_n)),\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\right).\end{aligned} \tag{2.5}$$

By the property (iii) of ϕ we have,

$$\phi(d(x_n,x_{n+1})+d(y_n,y_{n+1})) \le \phi(d(x_n,x_{n+1}))+\phi(d(y_n,y_{n+1})). \tag{2.6}$$

From (2.5) and (2.6), we get

$$\begin{aligned}&\phi(d(x_n,x_{n+1})+d(y_n,y_{n+1}))\\ &\quad\le f\left(\phi(d(x_{n-1},x_n)+d(y_{n-1},y_n)),\psi\left(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\right)\right).\end{aligned} \tag{2.7}$$

Using the fact that ϕ is non-decreasing, we get

$$d(x_n,x_{n+1})+d(y_n,y_{n+1}) \le d(x_{n-1},x_n)+d(y_{n-1},y_n). \tag{2.8}$$

Set $\delta_n = d(x_n,x_{n+1})+d(y_n,y_{n+1})$ then sequence (δ_n) is decreasing. Therefore, there is some $\delta \ge 0$ such that

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}[d(x_n,x_{n+1})+d(y_n,y_{n+1})] = \delta. \tag{2.9}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n\to\infty$ on both sides of (2.7) and have in mind that we suppose $\lim_{t\to r}\psi(t) > 0$ for all $r > 0$ and ϕ is continuous, we have

$$\phi(\delta) = \lim_{n\to\infty}\phi(\delta_n) \le \lim_{n\to\infty} f\left(\phi(\delta_{n-1}),\psi\left(\frac{\delta_{n-1}}{2}\right)\right) = f\left(\phi(\delta),\lim_{n\to\infty}\psi\left(\frac{\delta_{n-1}}{2}\right)\right) \le \phi(\delta).$$

So $\phi(\delta) = 0$ or $\lim_{n\to\infty}\psi\left(\frac{\delta_{n-1}}{2}\right) = 0$, which is a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}[d(x_{n+1},x_n)+d(y_{n+1},y_n)] = 0. \tag{2.10}$$

Now to prove that (x_n) and (y_n) are Cauchy sequence. Assume that at least one of the sequences (x_n) or (y_n) is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(x_n,x_m) \nrightarrow 0$ or $\lim_{n,m\to\infty} d(y_n,y_m) \nrightarrow 0$, and, consequently,

$$\lim_{n,m\to\infty}[d(x_n,x_m)+d(y_n,y_m)] \nrightarrow 0.$$

Then there exists $\epsilon > 0$ for which we can find subsequences $(x_{n(k)}),(x_{m(k)})$ of (x_n) and $(y_{n(k)})$, $(y_{m(k)})$ of (y_n) such that $n(k)$ is smallest index for which $n(k) > m(k) > k$,

$$[d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})] \ge \epsilon. \tag{2.11}$$

This means that

$$d(x_{n(k)-1},x_{m(k)})+d(y_{n(k)-1},y_{m(k)}) < \epsilon. \tag{2.12}$$

Using (2.11), (2.12) and triangle inequality, we have

$$\begin{aligned}\epsilon \le r_k &:= d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})\\ &\le d(x_{n(k)},x_{n(k)-1})+d(x_{n(k)-1},x_{m(k)})+d(y_{n(k)},y_{n(k)-1})+d(y_{n(k)-1},y_{m(k)})\\ &\le d(x_{n(k)},x_{n(k)-1})+d(y_{n(k)},y_{n(k)-1})+\epsilon.\end{aligned}$$

Letting $k\to\infty$ and using (2.10), we obtain

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}[d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})]=\epsilon. \tag{2.13}$$

By the triangle inequality

$$\begin{aligned}r_k &= d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})\\ &\le d(x_{n(k)},x_{n(k)+1})+d(x_{n(k)+1},x_{m(k)+1})+d(x_{m(k)+1},x_{m(k)})\\ &\quad+d(y_{n(k)},y_{n(k)+1})+d(y_{n(k)+1},y_{m(k)+1})+d(y_{m(k)+1},y_{m(k)})\\ &=\delta_{n(k)}+\delta_{m(k)}+d(x_{n(k)+1},x_{m(k)+1})+d(y_{n(k)+1},y_{m(k)+1}).\end{aligned}$$

Using the property of ϕ, we obtain

$$\begin{aligned}\phi(r_k) &= \phi(\delta_{n(k)}+\delta_{m(k)}+d(x_{n(k)+1},x_{m(k)+1})+d(y_{n(k)+1},y_{m(k)+1}))\\ &\le \phi(\delta_{n(k)})+\phi(\delta_{m(k)})+\phi(d(x_{n(k)+1},x_{m(k)+1}))+\phi(d(y_{n(k)+1},y_{m(k)+1})).\end{aligned} \tag{2.14}$$

Since $x_{n(k)}\ge x_{m(k)}$ and $y_{n(k)}\le y_{m(k)}$, using that F is proximally coupled weak (ψ,ϕ) contraction on A we get,

$$\begin{aligned}&\phi(d(x_{n(k)+1},x_{m(k)+1}))\\ &\quad\le \frac{1}{2}f\left(\phi(d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})),\psi\left(\frac{d(x_{n(k)},x_{m(k)})+d(y_{n(k)},y_{m(k)})}{2}\right)\right)\\ &\quad\le \frac{1}{2}f\left(\phi(r_k),\psi\left(\frac{r_k}{2}\right)\right).\end{aligned} \tag{2.15}$$

Similarly, we also have

$$\begin{aligned}&\phi(d(y_{m(k)+1},y_{n(k)+1}))\\ &\quad\le \frac{1}{2}f\left(\phi(d(y_{m(k)},y_{n(k)})+d(x_{m(k)},x_{n(k)})),\psi\left(\frac{d(y_{m(k)},y_{n(k)})+d(x_{m(k)},x_{n(k)})}{2}\right)\right)\\ &\quad\le \frac{1}{2}f\left(\phi(r_k),\psi\left(\frac{r_k}{2}\right)\right).\end{aligned} \tag{2.16}$$

From (2.14)-(2.16), we have

$$\phi(r_k)\le \phi(\delta_{n(k)}+\delta_{m(k)})+f\left(\phi(r_k),\psi\left(\frac{r_k}{2}\right)\right).$$

Letting $k\to\infty$ and using (2.10) and (2.13), we have

$$\phi(\epsilon)\le\phi(0)+f\left(\phi(\epsilon),\lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right)\right)=f\left(\phi(\epsilon),\lim_{k\to\infty}\psi\left(\frac{r_k}{2}\right)\right)$$

so, $\phi(\epsilon)=0$, $\lim_{k\to\infty}\psi(\frac{r_k}{2})=0$ a contradiction. This shows that (x_n) and (y_n) are Cauchy sequences. Since A is closed subset of a complete metric space X, these sequences have limits. Thus, there exists $x,y\in A$ such that $x_n\to x$ and $y_n\to y$. Therefore $(x_n,y_n)\to(x,y)$ in $A\times A$. Since F is continuous, we have $F(x_n,y_n)\to F(x,y)$ and $F(y_n,x_n)\to F(y,x)$.

Hence the continuity of the metric function d implies that $d(x_{n+1},F(x_n,y_n))\to d(x,F(x,y))$ and $d(y_{n+1},F(y_n,x_n))\to d(y,F(y,x))$. But from equations (2.1) and (2.2) we get, the sequences

$(d(x_{n+1},F(x_n,y_n)))$ and $(d(y_{n+1},F(y_n,x_n)))$ are constant sequences with the value $d(A,B)$. Therefore, $d(x,F(x,y)) = d(A,B)$ and $d(y,F(y,x)) = d(A,B)$. This completes the proof of the theorem. □

Corollary 2.2. *Let $(X,\leq,d)$ be a partially ordered complete metric space. Let A be non-empty closed subsets of the metric space (X,d). Let $F: A\times A \to A$ satisfy the following conditions.*

(i) *F is continuous having the proximal mixed monotone property and proximally coupled weak (ψ,ϕ) contraction on A.*

(ii) *There exist (x_0,y_0) and (x_1,y_1) in $A\times A$ such that $x_1 = F(x_0,y_0)$ with $x_0 \leq x_1$ and $y_1 = F(y_0,x_0)$ with $y_0 \geq y_1$.*

Then there exist $(x,y)\in A\times A$ such that $d(x,F(x,y)) = 0$ and $d(y,F(y,x)) = 0$.

In what follows we prove that Theorem 2.1 is still valid for F not necessarily continuous, assuming the following hypothesis in A. A has the property that

$$(x_n) \text{ is a non-decreasing sequence in } A \text{ such that } x_n \to x, \text{ then } x_n \leq x. \tag{2.17}$$

$$(y_n) \text{ is a non-increasing sequence in } A \text{ such that } y_n \to y, \text{ then } y \leq y_n. \tag{2.18}$$

Theorem 2.3. *Assume the condition (2.17), (2.18) and A_0 is closed in X instead of continuity of F in the Theorem 2.1, then the conclusion of Theorem 2.1 holds.*

Proof. Following the proof of Theorem 2.1, there exist sequences (x_n) and (y_n) in A satisfying the following conditions:

$$d(x_{n+1},F(x_n,y_n)) = d(A,B) \text{ with } x_n \leq x_{n+1}, \quad \forall\, n\in\mathbb{N} \text{ and} \tag{2.19}$$

$$d(y_{n+1},F(y_n,x_n)) = d(A,B) \text{ with } y_n \geq y_{n+1}, \quad \forall\, n\in\mathbb{N}. \tag{2.20}$$

Moreover, x_n converges to x and y_n converges to y in A. From (2.17) and (2.18), we get $x_n \leq x$ and $y_n \geq y$. Note that the sequences (x_n) and (y_n) are in A_0 and A_0 is closed. Therefore, $(x,y)\in A_0\times A_0$. Since $F(A_0\times A_0)\subseteq B_0$, there exists $F(x,y)$ and $F(y,x)$ are in B_0. Therefore, there exists $(x^*,y^*)\in A_0\times A_0$ such that

$$d(x^*,F(x,y)) = d(A,B) \text{ and} \tag{2.21}$$

$$d(y^*,F(y,x)) = d(A,B). \tag{2.22}$$

Since $x_n \leq x$ and $y_n \geq y$. By using F is proximally coupled weak (ψ,ϕ) contraction on A for (2.19) and (2.21) also for (2.22) and (2.20), we get

$$\phi(d(x_{n+1},x^*)) \leq \frac{1}{2} f\left(\phi(d(x_n,x)+d(y_n,y)),\psi\left(\frac{d(x_n,x)+d(y_n,y)}{2}\right)\right), \text{ for all } n \text{ and}$$

$$\phi(d(y^*,y_{n+1})) \leq \frac{1}{2} f\left(\phi(d(y,y_n)+d(x,x_n)),\psi\left(\frac{d(y,y_n)+d(x,x_n)}{2}\right)\right), \text{ for all } n.$$

Since $x_n \to x$ and $y_n \to y$, by taking limit on the above two inequality, we get $x = x^*$ and $y = y^*$. Hence form (2.21) and (2.22), we get $d(x,F(x,y)) = d(A,B)$ and $d(y,F(y,x)) = d(A,B)$. □

Corollary 2.4. *Assume the condition (2.17) and (2.18) instead of continuity of F in the Corollary 2.2, then the conclusion of Corollary 2.2 holds.*

Now, we present an example where it can be appreciated that hypotheses in Theorem 2.1 and Theorem 2.3 do not guarantee uniqueness of the coupled best proximity point.

Example 2.5. Let $X = \{(0,1),(1,0),(-1,0),(0,-1)\} \subset \mathbb{R}^2$ and consider the usual order $(x,y) \preceq (z,t) \Leftrightarrow x \le z$ and $y \le t$.

Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d_2) is a complete metric space considering d_2 the euclidean metric. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,-1),(-1,0)\}$ be a closed subset of X. Then, $d(A,B) = \sqrt{2}, A = A_0$ and $B = B_0$. Let $F : A \times A \to B$ be defined as $F((x_1,x_2),(y_1,y_2)) = (-x_2,-x_1)$. Then, it can be seen that F is continuous such that $F(A_0 \times A_0) \subseteq B_0$. The only comparable pairs of points in A are $x \preceq x$ for $x \in A$, hence proximal mixed monotone property and proximally coupled weak (ψ,ϕ) contraction on A are satisfied trivially.

It can be shown that the other hypotheses of the theorem are also satisfied. However, F has three coupled best proximity points $((0,1),(0,1)), ((0,1),(1,0))$ and $((1,0),(1,0))$.

One can prove that the coupled best proximity point is in fact unique, provided that the product space $A \times A$ endowed with the partial order mentioned earlier has the following property:

$$\text{every pair of elements has either a lower bound or an upper bound.} \tag{2.23}$$

It is known that this condition is equivalent to:
for every pair of (x,y), $(x^*,y^*) \in A \times A$, there exists (z_1,z_2) in $A \times A$,

$$\text{that is comparable to } (x,y) \text{ and } (x^*,y^*). \tag{2.24}$$

Theorem 2.6. *In addition to the hypothesis of Theorem* 2.1 (*resp. Theorem* 2.3)*, suppose that for any two elements* (x,y) *and* (x^*,y^*) *in* $A_0 \times A_0$,

$$\textit{there exists } (z_1,z_2) \in A_0 \times A_0 \textit{ such that } (z_1,z_2) \textit{ is comparable to } (x,y) \textit{ and } (x^*,y^*) \tag{2.25}$$

then F has a unique coupled best proximity point.

Proof. From Theorem 2.1 (resp. Theorem 2.3), the set of coupled best proximity points of F is non-empty. Suppose that there exist (x,y) and (x^*,y^*) in $A \times A$ which are coupled best proximity points. That is,

$$d(x,F(x,y)) = d(A,B), d(y,F(y,x)) = d(A,B) \text{ and}$$
$$d(x^*,F(x^*,y^*)) = d(A,B), d(y^*,F(y^*,x^*)) = d(A,B).$$

We distinguish two cases.

Case 1: Suppose (x,y) is comparable. Let (x,y) is comparable to (x^*,y^*) with respect to the ordering in $A \times A$. Apply F is proximally coupled weak (ψ,ϕ) contraction on A to $d(x,F(x,y)) = d(A,B)$ and $d(x^*,F(x^*,y^*)) = d(A,B)$, we get

$$\phi(d(x,x^*)) \le \frac{1}{2} f\left(\phi(d(x,x^*)+d(y,y^*)), \psi\left(\frac{d(x,x^*)+d(y,y^*)}{2}\right)\right). \tag{2.26}$$

Similarly, one can prove that

$$\phi(d(y,y^*)) \le \frac{1}{2} f\left(\phi(d(y,y^*)+d(x,x^*)), \psi\left(\frac{d(y,y^*)+d(x,x^*)}{2}\right)\right). \tag{2.27}$$

Adding (2.26) and (2.27), we get

$$\phi(d(x,x^*))+\phi(d(y,y^*)) \le f\left(\phi(d(x,x^*)+d(y,y^*)),\psi\left(\frac{d(x,x^*)+d(y,y^*)}{2}\right)\right). \tag{2.28}$$

By the property (iii) of ϕ, we have

$$\phi(d(x,x^*)+d(y,y^*)) \le \phi(d(x,x^*))+\phi(d(y,y^*)). \tag{2.29}$$

From (2.28) and (2.29), we have

$$\phi(d(x,x^*)+d(y,y^*)) \le f\left(\phi(d(x,x^*)+d(y,y^*)),\psi\left(\frac{d(x,x^*)+d(y,y^*)}{2}\right)\right) \tag{2.30}$$

this implies that $\phi(d(x,x^*)+d(y,y^*))=0$, or $\psi\left(\frac{d(x,x^*)+d(y,y^*)}{2}\right)=0$ and using the property of ϕ, ψ, we get $d(x,x^*)+d(y,y^*)=0$, hence $x=x^*$ and $y=y^*$.

Case 2: Suppose (x,y) is not comparable. Let (x,y) is not comparable to (x^*,y^*), then there exists $(u_1,v_1)\in A_0\times A_0$ which is comparable to (x,y) and (x^*,y^*).

Since $F(A_0\times A_0)\subseteq B_0$, there exists $(u_2,v_2)\in A_0\times A_0$ such that $d(u_2,F(u_1,v_1))=d(A,B)$ and $d(v_2,F(v_1,u_1))=d(A,B)$. With out loss of generality assume that $(u_1,v_1)\le(x,y)$(i.e., $x\ge u_1$ and $y\le v_1$.) Note that $(u_1,v_1)\le(x,y)$ implies that $(y,x)\le(v_1,u_1)$. From Lemma 1.4 and Lemma 1.5, we get

$$\left.\begin{aligned} u_1\le x \text{ and } v_1\ge y\\ d(u_2,F(u_1,v_1))=d(A,B)\\ d(x,F(x,y))=d(A,B)\end{aligned}\right\}\implies u_2\le x$$

and

$$\left.\begin{aligned} u_1\le x \text{ and } v_1\ge y\\ d(v_2,F(v_1,u_1))=d(A,B)\\ d(y,F(y,x))=d(A,B)\end{aligned}\right\}\implies v_2\ge y.$$

From the above two inequalities, we obtain $(u_2,v_2)\le(x,y)$. Continuing this process, we get sequences (u_n) and (v_n) such that $d(u_{n+1},F(u_n,v_n))=d(A,B)$ and $d(v_{n+1},F(v_n,u_n))=d(A,B)$ with $(u_n,v_n)\le(x,y)$ $\forall$ $n\in\mathbb{N}$. By using that F is a proximally coupled weak (ψ,ϕ) contraction on A, we get

$$\left.\begin{aligned} u_n\le x \text{ and } v_n\ge y\\ d(u_{n+1},F(u_n,v_n))=d(A,B)\\ d(x,F(x,y))=d(A,B)\end{aligned}\right\}$$

$$\implies \quad \phi(d(u_{n+1},x))\le\frac{1}{2}f\left(\phi(d(u_n,x)+d(v_n,y)),\psi\left(\frac{d(u_n,x)+d(v_n,y)}{2}\right)\right). \tag{2.31}$$

Similarly, we can prove that

$$\left.\begin{aligned} y\le v_n \text{ and } x\ge u_n\\ d(y,F(y,x))=d(A,B)\\ d(v_{n+1},F(v_n,u_n))=d(A,B)\end{aligned}\right\}$$

$$\implies \quad \phi(d(y,v_{n+1}))\le\frac{1}{2}f\left(\phi(d(y,v_n)+d(x,u_n)),\psi\left(\frac{d(y,v_n)+d(x,u_n)}{2}\right)\right). \tag{2.32}$$

Adding (2.31) and (2.32), we obtain

$$\phi(d(u_{n+1},x))+\phi(d(y,v_{n+1}))\le f\left(\phi(d(u_n,x)+d(v_n,y)),\psi\left(\frac{d(u_n,x)+d(v_n,y)}{2}\right)\right). \tag{2.33}$$

But $\phi(d(u_{n+1},x)+d(y,v_{n+1}))\le\phi(d(u_{n+1},x))+\phi(d(y,v_{n+1}))$, hence

$$\begin{aligned}\phi(d(u_{n+1},x)+d(y,v_{n+1}))&\le f\left(\phi(d(u_n,x)+d(v_n,y)),\psi\left(\frac{d(u_n,x)+d(v_n,y)}{2}\right)\right)\\&\le\phi(d(u_n,x)+d(v_n,y)).\end{aligned} \tag{2.34}$$

Using the fact that ϕ is non-decreasing, we get

$$d(u_{n+1},x)+d(y,v_{n+1})\le d(u_n,x)+d(v_n,y). \tag{2.35}$$

That is, the sequence $(d(u_n,x)+d(y,v_n))$ is decreasing. Therefore, there exists $\alpha\ge 0$ such that

$$\lim_{n\to\infty}[d(u_n,x)+d(y,v_n)]=\alpha. \tag{2.36}$$

We shall show that $\alpha=0$. Suppose, to the contrary, that $\alpha>0$. Taking the limit as $n\to\infty$ in (2.34), we have

$$\phi(\alpha)\le f\left(\phi(\alpha),\lim_{n\to\infty}\psi\left(\frac{d(u_n,x)+d(v_n,y)}{2}\right)\right).$$

So, $\phi(\alpha)=0$, $\lim_{n\to\infty}\psi\left(\frac{d(u_n,x)+d(v_n,y)}{2}\right)=0$, a contradiction. Thus, $\alpha=0$, that is,

$$\lim_{n\to\infty}[d(u_n,x)+d(y,v_n)]=0 \tag{2.37}$$

so that $u_n\to x$ and $v_n\to y$. Analogously, one can prove that $u_n\to x^*$ and $v_n\to y^*$.

Therefore, $x=x^*$ and $y=y^*$. Hence the proof. □

The following result, due to Theorem 2.6 in Nguyen Van Luong and Nguyen Xuan Thuan [22] by taking $A=B$.

Corollary 2.7. *In addition to the hypothesis of Corollary* 2.2 (*resp. Corollary* 2.4), *suppose that for any two elements* (x,y) *and* (x^*,y^*) *in* $A\times A$,

$$\text{there exists } (z_1,z_2)\in A\times A \text{ such that } (z_1,z_2) \text{ is comparable to } (x,y) \text{ and } (x^*,y^*) \tag{2.38}$$

then F *has a unique coupled fixed point.*

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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