Communications in Mathematics and Applications
Vol. 7, No. 3, pp. 263–273, 2016
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Special Issue on
Advances in Fixed Point Theory towards Real World Optimization Problems
Proceedings of The 9th Asian Conference on Fixed Point Theory and Optimization
May 18 – 20, 2016, King Mongkut's University of Technology Thonburi, Bangkok, Thailand

On Asymptotically Lacunary Statistical Equivalent Double Sequences of Fuzzy Numbers

Research Article

Chirasak Mongkolkeha[1], Phatiphat Thounthong[2,3] and Poom Kumam[4,5,*]

[1] *Department of Mathematics Statistics and Computer Sciences, Faculty of Liberal Arts and Science, Kasetsart University, Kamphaeng-Saen Campus, Nakhonpathom 73140, Thailand*
[2] *Renewable Energy Research Centre, King Mongkut's University of Technology North Bangkok (KMUTNB), Bangkok, Thailand*
[3] *Department of Teacher Training in Electrical Engineering, Faculty of Technical Education, King Mongkut's University of Technology North Bangkok (KMUTNB), Bangkok, Thailand*
[4] *KMUTTFixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[5] *KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand*
***Corresponding author:** poom.kum@kmutt.ac.th

Abstract. In this paper, we present the asymptotically double lacunary statistical equivalent and strongly asymptotically double lacunary statistical equivalent which is a natural combination of the definition for asymptotically equivalence and lacunary statistical double sequences of fuzzy numbers. In addition, we also give some relations among these new notions.

Keywords. Fuzzy numbers; Asymptotical equivalence; Asymptotical double lacunary statistical equivalence; Double sequences

MSC. 40A05; 40C05

Received: July 1, 2016 **Accepted:** August 4, 2016

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were firstly introduced by Zadeh [24], several authors have discussed various aspects of the theory for applications of fuzzy logic control [21,22] and more applications see [1,2,4,5,13,23]. In 1986, Matloka [7], given some basic theorems for sequences of fuzzy numbers. Afterwards, Nanda [11] studied and discussed the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. In 1993, Marouf [6] presented definitions for asymptotically equivalence sequences and asymptotic regular matrices. Later, Patterson [14] extended by present an asymptotical statistical equivalence analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, Patterson and Savaş [15] extended the definitions of Patterson in [14] to lacunary sequences. In 2007, Savaş [16] presented the new concept which is a natural combination of the notion of asymptotically lacunary statistical equivalence convergence of fuzzy numbers. On the other hand, the notion of double sequences has been defined by Mursaleen and Edely [8,9]. Further of this concept we can see in [10,17–20] and others. The subject of this paper is to presenting the asymptotically double lacunary statistical equivalence, strongly asymptotically double lacunary statistical equivalence of sequences of fuzzy numbers and give some relations among these new notions.

2. Preliminaries

In this section, we give some definitions and basic concept of them for the main results of this paper. A fuzzy number is a function X from $\mathbb{R}^n$ to $[0,1]$ satisfying

(i) X is normal, i.e. there exists an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;

(ii) X is fuzzy convex, i.e. for any $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$, $X(\lambda x + (1-\lambda)y) \ge \min\{X(x), X(y)\}$;

(iii) X is upper semi-continuous;

(iv) The closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by X^o, is compact.

These properties imply that for each $0 < \alpha \le 1$, the α-level set $X^\alpha = \{x \in \mathbb{R}^n : X(x) \ge \alpha\}$ is a nonempty compact convex, subset of $\mathbb{R}^n$ as the support X^o. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $[X+Y]$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of α-level sets by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda [X]^\alpha \quad (X, Y \in L(\mathbb{R}^n),\ \lambda \in \mathbb{R})$$

for each $0 \le \alpha \le 1$. Define for each $0 \le q < \infty$,

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty (X^\alpha, Y^\alpha)^q \right)^{1/q}$$

and $d_\infty = \sup\limits_{0 \le \alpha \le 1} \delta_\infty(X^\alpha, Y^\alpha)$ where d_∞ is Hausdorff metric.

Clearly $d_\infty(X,Y) = \lim\limits_{q \to \infty} d_q(X,Y)$ with $d_q \le d_r$, if $q \le r$. Moreover d_q is a complete, separable and locally compact metric space (for more details see [3]). Throughout the paper, d will denote d_q with $1 \le q \le \infty$.

Definition 2.1 ([16]). A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R}^n)$. The fuzzy number X_n denotes the value of the function at $n \in \mathbb{N}$ and is called the nth term of the sequence.

Definition 2.2 ([16]). A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 written as $\lim_{k\to\infty} X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer N_0 such that $d(X_k, X_0) < \varepsilon$ for all $k > N_0$.

Definition 2.3 ([16]). A sequence $X = (X_k)$ of fuzzy numbers is said to to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers, $c(F)$ the set of all convergent sequences of fuzzy numbers and $l_\infty(F)$ the set of all bounded sequences of fuzzy numbers. It is straightforward to see that $c(F) \subset l_\infty(F) \subset w(F)$. Furthermore, $c(F)$ and $l_\infty(F)$ are complete metric spaces (see also [11]).

3. Definitions and Notations

Definition 3.1. Two sequences $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers are said to be *asymptotically statistical equivalence* if

$$\lim_{k\to\infty} d\left(\frac{X_k}{Y_k}, 1\right) = 0 \qquad (\text{denoted by } X \overset{F}{\sim} Y).$$

Nuray and Savaş [13] defined the concept of statistically convergence of a sequence of fuzzy numbers as follows:

Definition 3.2. A sequence $X = (X_k)$ of fuzzy numbers is said to be *statistical convergent* to a fuzzy number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{\text{the number of} k \le n : d(X_k, L)\} \ge \varepsilon| = 0.$$

By combining the notion of asymptotically equivalence and statistical convergence, we can write the following definition:

Definition 3.3. Two sequences $X = (X_k)$ and $Y = (Y_k)$ of fuzzy numbers are said to be *asymptotically statistical equivalent of multiple* L if provided that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ \text{the number of} k \le n : d\left(\frac{X_k}{Y_k}, L\right) \ge \varepsilon \right\} \right| = 0$$

(denoted by $X \overset{S_L(F)}{\sim} Y$ and simply asymptotically statistical equivalent if $L = 1$.) Note that support Y does not contain 0.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we will mean an increasing sequence of nonneqative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$.

The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and the ratio k_r/k_{r-1} will denoted by q_r.

Nuray [12] introduced the concept of lacunary statistical convergence of fuzzy numbers as follows:

Definition 3.4. A sequence $X = (X_k)$ of fuzzy numbers is said to be *lacunary statistical convergent to a fuzzy number* Lif for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : d(X_k, L) \geq \varepsilon\}| = 0.$$

(denoted by $S_\theta - \lim X = L$ and the set of all lacunary statistical convergent sequences to a fuzzy number L by $S_\theta(X)$).

Savaş and Mursaleen [19] introduced the concept of statistically convergence of a double sequence of fuzzy numbers as follows:

Definition 3.5. A sequence $X = (X_{k,l})$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{(k,l); k \leq m \text{ and } l \leq n : d\left(X_{k,l}, X_0\right) \geq \varepsilon\}\right| = 0.$$

In this case we write $st_2 - \lim X_{k,l} = X_0$ and the set of all statistically convergent double sequences is denoted by st_2.

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called *double lacunary* if there exist two increasing sequences (k_r) and (l_r) of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty$$

and

$$l_0 = 0, \quad \bar{h}_r = l_s - l_{s-1} \to \infty.$$

Notation. $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r$ and $l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_r = \frac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$.

Savaş [17] introduced the concept of lacunary statistically convergence of a double sequence of fuzzy numbers as follows:

Definition 3.6. Let $\theta_{r,s}$ be a double lacunary sequence. A sequence $X = (X_{k,l})$ of fuzzy numbers is said to double lacunary statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\{(k,l) \in I_{r,s} : d\left(X_{k,l}, X_0\right) \geq \varepsilon\}\right| = 0.$$

In this case we write $S_{\theta_{r,s}} - \lim X_{k,l} = X_0$ and the set of all double lacunary statistically convergent double sequences is denoted by $S_{\theta_{r,s}}(F)$.

From these result as above, we introduce the new concept by the following definitions:

Definition 3.7. Let $\theta_{r,s}$ be a double lacunary sequence and $p = (p_{k,l})$ be a bounded double sequence of positive real numbers. Then, the two double sequences of fuzzy numbers $X = (X_{k,l})$ and $Y = (Y_{k,l})$ are said to be *strongly asymptotically double Cesaro statistical equivalent of multiple L*provided that,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} = 0.$$

(denoted by $X \overset{\sigma^{L(p)}}{\sim} Y$ and called strongly asymptotically double Cesaro statistical equivalent, if $L = 1$.)

Definition 3.8. Let $\theta_{r,s}$ be a double lacunary sequence. Then, the two double sequences of fuzzy numbers $X = (X_{k,l})$ and $Y = (Y_{k,l})$ are said to be *asymptotically double lacunary statistical equivalent of multiple L* provided that for every $\varepsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \geq \varepsilon\right\}\right| = 0.$$

(denoted by $X \overset{S^L_{\theta_{r,s}}(F)}{\sim} Y$ and called asymptotically double lacunary statistical equivalent, if $L = 1$).

Definition 3.9. Let $\theta_{r,s}$ be a double lacunary sequence and $p = (p_{k,l})$ be a a bounded double sequence of positive real numbers. Then, the two double sequences of fuzzy numbers $X = (X_{k,l})$ and $Y = (Y_{k,l})$ are said to be *strongly asymptotically double lacunary statistical equivalent of multiple L* provided that,

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} = 0.$$

(denoted by $X \overset{N^{L(p)(F)}_{\theta_{r,s}}}{\sim} Y$ and called strongly asymptotically double lacunary statistical equivalent, if $L = 1$.)

From above definition, if we take $p_{k,l} = p$ for all k,l, we write $X \overset{\sigma^{L_p}(F)}{\sim} Y$ and $X \overset{N^{L_p}_{\theta_{r,s}}(F)}{\sim} Y$ instead $X \overset{\sigma^{L(p)}(F)}{\sim} Y$ and $X \overset{N^{L(p)}_{\theta_{r,s}}(F)}{\sim} Y$ (respectively).

4. Main Result

Theorem 4.1. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X = (X_{k,l})$ and $Y = (Y_{k,l})$ be double sequences of fuzzy numbers. Then*

(i) *If $X \overset{N^{L_p}_{\theta_{r,s}}(F)}{\sim} Y$ then $X \overset{S^{L}_{\theta_{r,s}}(F)}{\sim} Y$.*

(ii) *If $X, Y \in l^2_\infty(F)$ and $X \overset{S^{L}_{\theta_{r,s}}(F)}{\sim} Y$ then $X \overset{N^{L_p}_{\theta_{r,s}}(F)}{\sim} Y$.*

(iii) $S^L_{\theta_{r,s}}(F) \cap l^2_\infty(F) = N^{L_p}_{\theta_{r,s}}(F) \cap l^2_\infty(F)$.

Proof. (i) Let $\varepsilon > 0$ and $X \overset{N^{L_p}_{\theta_{r,s}}}{\sim} Y$. Then we get

$$\begin{aligned}
\sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p &= \sum_{(k,l)\in I_{r,s}, d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)\ge\varepsilon} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p + \sum_{(k,l)\in I_{r,s}, d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)<\varepsilon} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p \\
&\ge \sum_{(k,l)\in I_{r,s}, d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)\ge\varepsilon} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p \\
&\ge \varepsilon^p \left|\left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \ge \varepsilon\right\}\right|
\end{aligned}$$

it follows that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \ge \varepsilon\right\}\right| = 0.$$

Hence $X \overset{S^{L}_{\theta_{r,s}}(F)}{\sim} Y$.

(ii) Let $X, Y \in l^2_\infty(F)$ and $X \overset{S^{L}_{\theta_{r,s}}(F)}{\sim} Y$. Then we can find $M > 0$ such that

$$d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \le M \text{ for all } k \text{ and } l.$$

Let $\varepsilon > 0$ and $N_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{h_{r,s}} \left|\left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)\right\} \ge \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right| \le \frac{\varepsilon}{2M^p},$$

for all $r,s > N_\varepsilon$. Putting $L_{k,l} := \left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \ge \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}$, then for all $r,s > N_\varepsilon$, we have

$$\begin{aligned}
\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p &= \frac{1}{h_{r,s}} \sum_{(k,l)\in L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p + \frac{1}{h_{r,s}} \sum_{(k,l)\notin L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^p \\
&\le \frac{1}{h_{r,s}} \cdot M^p \left|\left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \ge \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right| + \\
&\quad \frac{1}{h_{r,s}} \cdot \frac{\varepsilon}{2} \left|\left\{(k,l)\in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) < \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right| \\
&< \frac{1}{h_{r,s}} M^p \cdot \frac{h_{r,s}\varepsilon}{2M^p} + \frac{1}{h_{r,s}} \cdot h_{r,s} \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $X \overset{N_{\theta_{r,s}}^{L_p}}{\sim} Y$.

(iii) This follows directly from (i) and (ii). Hence the proof is completes. □

Theorem 4.2. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X=(X_{k,l})$ and $Y=(Y_{k,l})$ be double sequences of fuzzy numbers. Suppose that $0<h=\inf p_{k,l}\le p_{k,l}=H<1$. Then*

(i) *If $X \overset{N_{\theta_{r,s}}^{L(p)}(F)}{\sim} Y$, then $X \overset{S_{\theta_{r,s}}^{L}(F)}{\sim} Y$.*

(ii) *If $X \overset{\sigma^{L(p)}(F)}{\sim} Y$, then $X \overset{\sigma^{L_p}(F)}{\sim} Y$.*

Proof. (i) Let $\varepsilon>0$ and $X \overset{N_{\theta_{r,s}}^{L(p)}(F)}{\sim} Y$. Putting $L_{k,l}:=\left\{(k,l)\in I_{r,s}: d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)\ge\varepsilon\right\}$.
Since $0<h=\inf p_{k,l}\le p_{k,l}=H<\infty$, we have

$$\begin{aligned}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} &= \frac{1}{h_{r,s}}\sum_{(k,l)\in L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} + \frac{1}{h_{r,s}}\sum_{(k,l)\notin L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}\\ &\ge \frac{1}{h_{r,s}}\sum_{(k,l)\in L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}\\ &\ge \frac{1}{h_{r,s}}\sum_{(k,l)\in L_{k,l}} \min\{\varepsilon^h,\varepsilon^H\}\\ &\ge \frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}: d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)\ge\varepsilon\right\}\right|\min\{\varepsilon^h,\varepsilon^H\}\end{aligned}$$

which implies that

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}: d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)\right\}\ge\varepsilon\right|=0.$$

Hence $X \overset{S_{\theta_{r,s}}^{L}}{\sim} Y$.

(ii) By the same argument used in proving part (i), we can get this result. □

Theorem 4.3. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X=(X_{k,l})$ and $Y=(Y_{k,l})$ be bounded double sequences of fuzzy numbers. Suppose that $0<h=\inf p_{k,l}\le p_{k,l}=H<1$. Then*

(i) *If $X \overset{S_{\theta_{r,s}}^{L}(F)}{\sim} Y$, then $X \overset{N_{\theta_{r,s}}^{L(p)}(F)}{\sim} Y$.*

(ii) *If $X \overset{\sigma^{L_p}(F)}{\sim} Y$, then $X \overset{\sigma^{L(p)}(F)}{\sim} Y$.*

Proof. (i) Let $\varepsilon>0$. Since $X=(X_{k,l})$ and $Y=(Y_{k,l})$ are bounded, then there exists an integer M such that $d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)\le M$. Putting $L_{k,l}:=\left\{(k,l)\in I_{r,s}: d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)\ge\varepsilon\right\}$, we have

$$\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} = \frac{1}{h_{r,s}}\sum_{(k,l)\in L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} + \frac{1}{h_{r,s}}\sum_{(k,l)\notin L_{k,l}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}$$

$$\leq \frac{1}{h_{r,s}} \sum_{(k,l)\in L_{k,l}} \max\{M^h, M^H\} + \frac{1}{h_{r,s}} \sum_{(k,l)\notin L_{k,l}} (\varepsilon)^{p_{k,l}}$$
$$\leq \max\{M^h, M^H\} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right) \geq \varepsilon \right\} \right|$$
$$+ \max\{\varepsilon^h, \varepsilon^H\}$$

which implies that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} = 0.$$

Therefore $X \overset{N_{\theta_{r,s}}^{L(p)}(F)}{\sim} Y$.

(ii) By applying the argument used in the proof of part (i), we can get this result. □

Theorem 4.4. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X = (X_{k,l})$ and $Y = (Y_{k,l})$ be a double sequences of fuzzy numbers. Suppose that $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$. Then $X \overset{\sigma^{L(p)}(F)}{\sim} Y$ implies $X \overset{N_{\theta_{r,s}}^{L(p)}(F)}{\sim} Y$.*

Proof. Since $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$, then there exist $\delta > 0$ and $\delta_1 > 0$ such that $\delta + 1 < q_r$ and $\delta_1 + 1 < \bar{q}_s$ for all $r, s \geq 1$. This implies that

$$\frac{h_r}{k_r} > \frac{\delta}{1+\delta} \quad \text{and} \quad \frac{\bar{h}_s}{l_s} > \frac{\delta_1}{1+\delta_1}.$$

Then, we can write

$$\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} = \frac{1}{h_{r,s}} \sum_{k\in I_r} \sum_{l\in I_s} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}}$$
$$= \frac{1}{h_{r,s}} \left[\sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} - \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right.$$
$$\left. - \sum_{k=1}^{k_r} \sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} + \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right]$$
$$= \frac{k_r l_s}{h_{r,s}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right)$$
$$- \frac{k_{r-1} l_s}{h_{r,s}} \left(\frac{1}{k_{r-1} l_s} \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right)$$
$$- \frac{k_r l_{s-1}}{h_{r,s}} \left(\frac{1}{k_r l_{s-1}} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right)$$
$$+ \frac{k_{r-1} l_{s-1}}{h_{r,s}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_r-1} \sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} \right).$$

From the fact $X \overset{\sigma^{L(p)}(F)}{\sim} Y$, that the terms

$$\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}, \quad \frac{1}{k_{r-1} l_s}\sum_{k=1}^{k_r-1}\sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}},$$

$$\frac{1}{k_r l_{s-1}}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} \quad \text{and} \quad \frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_r-1}\sum_{l=1}^{l_s-1} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}$$

converge to zero and thus

$$\lim_{r,s\to\infty}\frac{1}{h_{r,s}}\sum_{(k,l)\in I_{r,s}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} = 0.$$

Therefore $X \overset{N^{L(p)}_{\theta_{r,s}}}{\sim} Y$. □

Theorem 4.5. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X=(X_{k,l})$ and $Y=(Y_{k,l})$ be double sequences of fuzzy numbers. Suppose that $\limsup_r q_r<\infty$ and $\limsup_s \bar{q}_s<\infty$. $X \overset{N^{L(p)}_{\theta_{r,s}}(F)}{\sim} Y$ implies $X \overset{\sigma^{L(p)}(F)}{\sim} Y$.*

Proof. Since $\limsup_r q_r<\infty$ and $\limsup_s \bar{q}_s<\infty$, then there exist $B>0$ such that $q_r<B$ and $\bar{q}_s<B$ for all $r,s\ge 1$. Let $\varepsilon>0$ and since $X \overset{N^{L(p)}_{\theta_{r,s}}(F)}{\sim} Y$, also there exist $r_0>0$ and $s_0>0$ such that for every $i\ge r_0$ and $j\ge s_0$

$$\mathscr{A}_{i,j} := \frac{1}{h_{i,j}}\sum_{k\in I_i}\sum_{l\in I_j} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}} < \varepsilon.$$

Let $M=\max\{\mathscr{A}_{r,s}: 1\le r\le r_0 \text{ and } 1\le s\le s_0\}$ and m, n such that $k_{r-1}<m\le k_r$ and $l_{s-1}<n\le l_s$. So, we have

$$\begin{aligned}
&\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}\\
&\le \frac{1}{k_{r-1}l_{s-1}}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}\\
&\le \frac{1}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r,s}\left(\sum_{(k,l)\in I_{p,u}} d\left(\frac{X_{k,l}}{Y_{k,l}},L\right)^{p_{k,l}}\right)\\
&= \frac{1}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r_0,s_0} h_{p,u}\mathscr{A}_{p,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<u\le s)} h_{p,u}\mathscr{A}_{p,u}\\
&\le \frac{M}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r_0,s_0} h_{p,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<u\le s)} h_{p,u}\mathscr{A}_{p,u}\\
&\le \frac{M}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r_0,s_0} h_{p,u} + \left(\sup_{(p\ge r_0)\cup(u\ge s_0)}\mathscr{A}_{p,u}\right)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\le r)\cup(s_0<u\le s)} h_{p,u}\\
&\le \frac{M k_{r_0} l_{s_0} r_0 s_0}{k_{r-1}l_{s-1}} + \frac{\varepsilon}{k_{r-1}l_{s-1}} k_r l_s
\end{aligned}$$

$$\leq \frac{Mk_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \varepsilon B^2.$$

Since k_r and l_s both approach to infinity as both m and n approach to infinity and ε is arbitrary, which implies that

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} d\left(\frac{X_{k,l}}{Y_{k,l}}, L\right)^{p_{k,l}} = 0.$$

Therefore $X \overset{\sigma^{L(p)}(F)}{\sim} Y$. □

Corollary 4.6. *Let $\theta_{r,s}$ be a double lacunary sequence and let $X = (X_{k,l})$ and $Y = (Y_{k,l})$ be double sequences of fuzzy numbers. Suppose that $1 < \liminf_r q_r < \infty$ and $1 < \liminf_s \bar{q}_s < \infty$, then $X \overset{N_{\theta_{r,s}}^{L(p)}}{\sim} Y \Leftrightarrow X \overset{\sigma^{L(p)}(F)}{\sim} Y$.*

Proof. The result follows immediately from Theorem 4.4 and Theorem 4.5. □

Acknowledgement

The first author was supported by Department of Mathematics Statistics and Computer Sciences, Faculty of Liberal Arts and Science, Kasetsart University, also the second author was financially supported by King Mongkuts University of Technology North Bangkok. Contract No. KMUTNB-60-ART-084. and the third author was supported by the Theoretical and Computational Science (TaCS) Center.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] G.A. Anastassiou, Fuzzy approximation by fuzzy convolution type operators, *Comput. Math. Appl.* **48** (2004), 1369–1386.

[2] L.C. Barros, R.C. Bassanezi and P.A. Tonelli, Fuzzy modelling in population dynamics, *Ecol. Model.* **128** (2000), 27–33.

[3] P. Diamond and P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Sets Syst.* **35** (1990), 241–249.

[4] R. Giles, A computer program for fuzzy reasoning, *Fuzzy Sets Syst.* **4** (1980), 22–234.

[5] L. Hong and J.Q. Sun, Bifurcations of fuzzy nonlinear dynamical systems, *Commun. Nonlinear Sci. Numer. Simul.* **1** (2006), 1–12.

[6] M. Marouf, Asymptotic equivalence and summability, *Internat. J. Math. Math. Sci.* **16** (4) (1993), 755–762.

[7] M. Matloka, Sequences of fuzzy numbers, *Busefal* **28** (1986), 28-37.

[8] M. Mursaleen and O.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.* **288** (2003), 223–231.

[9] M. Mursaleen and O.H. Edely, Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.* **293** (2004), 532–540.

[10] M. Mursaleen and S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, *Chaos Solitons Fractals* **41** (2009), 2414–2421.

[11] S. Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989), 123–126.

[12] F. Nuray, Lacunary statisticall convergence of sequence of fuzzy numbers, *Fuzzy Sets Syst.* **99** (1998), 353–356.

[13] F. Nuray and E. Savaş, Statisticall convergence of sequence of fuzzy numbers, *Math. Slovaca* **45** (1995), 269–273.

[14] R.F. Patterson, On asymptotically statistically equivalent sequences, *Demonstratio Math.* **36** (1) (2003), 149–153.

[15] R.F. Patterson and E. Savaş, On asymptotically lacunary statistical equivalent sequences, *Thai J. Math.* **4** (2) (2006), 267–272.

[16] E. Savaş, On extension asymptotically lacunary statistical equivalent sequences of fuzzy numbers, *JCIS 2007 Joint Conference on Information Science*, July 18–24, 2007, USA.

[17] E. Savaş, On lacunary statistically convergent double sequences of fuzzy numbers, *Appl. Math. Lett.* **21** (2008), 134–141.

[18] E. Savaş and S.A. Mohiuddine, λ-statistically convergent double sequences in probabilistic normed spaces, *Math. Slovaca* **62** (1) (2012), 99–108.

[19] E. Savaş and M. Mursaleen, On statistically convergent double sequences of fuzzy numbers, *Inform. Sci.* **162** (2004), 183–192.

[20] E. Savaş and R.F. Patterson, Double sequence spaces characterized by lacunary sequences, *Appl. Math. Lett.* **20** (2007), 964–970.

[21] P. Thounthong, A. Luksanasakul, P. Koseeyaporn and D. Bernard Davat, Intelligent model-based control of a standalone photovoltaic/fuel cell power plant with supercapacitor energy storage, *IEEE Transactions on Sustainable Energy* **4** (1) (January 2013), 240-249.

[22] P. Thounthonga, L. Piegarib, S. Pierfedericic and B. Davatc, Nonlinear intelligent DC grid stabilization for fuel cell vehicle applications with a supercapacitor storage device, *International Journal of Electrical Power & Energy Systems* **64** (January 2015), 723–733.

[23] K. Wu, Convergences of fuzzy sets based on decomposition theory and fuzzy polynomial function, *Fuzzy Sets Syst.* **109** (2000), 173–185.

[24] L.A. Zadeh, Fuzzy sets, *Inform Control* **8** (1965), 338–353.