



Inequalities for Certain Ratios involving the (p, k) -Analogue of the Gamma Function

Research Article

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Abstract. Recently, a (p, k) -analogue of the classical Gamma function was introduced and investigated. In this paper, we establish some inequalities for certain ratios involving the new analogue. The method is based on the monotonicity properties of some functions associated the (p, k) -Gamma function.

Keywords. Gamma function; (p, k) -analogue; Inequality

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1. Introduction

This paper is inspired by the recent paper [13], where the authors introduced and investigated a (p, k) -analogue of the Gamma function. In this paper, our objective is to establish inequalities for certain ratios involving the (p, k) -Gamma function. We begin by recalling the following definitions pertaining to our results.

The classical Euler's Gamma function, $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1)(x+2)\dots(x+n)}$$

satisfying the basic properties:

$$\Gamma(n+1) = n!, \quad n \in \mathbb{Z}^+ \cup \{0\},$$

$$\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R}^+.$$

Closely associated with the Gamma function is the Digamma or Psi function $\psi(x)$, which is defined for $x > 0$ as the logarithmic derivative of the Gamma function. That is,

$$\begin{aligned} \psi(x) &= \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \\ &= -\gamma + (x-1) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+x)} \\ &= -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} \end{aligned} \quad (1.1)$$

where $\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \ln n \right) = 0.577215664\dots$ is the Euler-Mascheroni's constant.

The p -analogue (also known as p -generalization, p -extension or p -deformation) of the Gamma function is defined for $p \in \mathbb{N}$ and $x > 0$ as

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\dots(x+p)}$$

where $\lim_{p \rightarrow \infty} \Gamma_p(x) = \Gamma(x)$ (see [1, p. 270]). It satisfies the identities:

$$\begin{aligned} \Gamma_p(x+1) &= \frac{px}{x+p+1} \Gamma_p(x), \\ \Gamma_p(1) &= \frac{p}{p+1}. \end{aligned}$$

The p -analogue of the Digamma function is defined for $x > 0$ as

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)} = \ln p - \sum_{n=0}^p \frac{1}{n+x}. \quad (1.2)$$

In 2007, Díaz and Pariguan [2] also defined the k -analogue of the Gamma function for $k > 0$ and $x \in \mathbb{C} \setminus k\mathbb{Z}^-$ as

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

where $\lim_{k \rightarrow 1} \Gamma_k(x) = \Gamma(x)$ and $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the Pochhammer k -symbol. It also satisfies the identities:

$$\begin{aligned} \Gamma_k(x+k) &= x\Gamma_k(x), \quad x \in \mathbb{R}^+ \\ \Gamma_k(k) &= 1 \end{aligned}$$

The k -analogue of the Digamma function is also defined for $x > 0$ as

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\Gamma'_k(x)}{\Gamma_k(x)} = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)}. \quad (1.3)$$

Then in the recent paper [13], the authors introduced a (p, k) -analogue of the Gamma function, defined for $p \in \mathbb{N}$, $k > 0$ and $x > 0$ as

$$\begin{aligned} \Gamma_{p,k}(x) &= \int_0^p t^{x-1} \left(1 - \frac{t^k}{pk}\right)^p dt \\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)} \end{aligned}$$

satisfying the identities:

$$\begin{aligned} \Gamma_{p,k}(x+k) &= \frac{pkx}{x+pk+k} \Gamma_{p,k}(x), \\ \Gamma_{p,k}(ak) &= \frac{p+1}{p} k^{a-1} \Gamma_p(a), \quad a \in \mathbb{R}^+, \\ \Gamma_{p,k}(k) &= 1. \end{aligned}$$

Similarly, the (p, k) -analogue of the Digamma function is defined as the logarithmic derivative of the function $\Gamma_{p,k}(x)$. That is

$$\begin{aligned} \psi_{p,k}(x) &= \frac{d}{dx} \ln \Gamma_{p,k}(x) = \frac{\Gamma'_{p,k}(x)}{\Gamma_{p,k}(x)} \\ &= \frac{1}{k} \ln(pk) - \sum_{n=0}^p \frac{1}{(nk+x)} \\ &= \frac{1}{k} \ln(pk) - \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} e^{-xt} dt. \end{aligned} \tag{1.4}$$

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p \rightarrow \infty} & \Gamma_k(x) \\ k \rightarrow 1 \downarrow & & \downarrow k \rightarrow 1 \\ \Gamma_p(x) & \xrightarrow{p \rightarrow \infty} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p \rightarrow \infty} & \psi_k(x) \\ k \rightarrow 1 \downarrow & & \downarrow k \rightarrow 1 \\ \psi_p(x) & \xrightarrow{p \rightarrow \infty} & \psi(x) \end{array}$$

Moreover, $\psi_{p,k}(x)$ is increasing for $x > 0$ since $\psi'_{p,k}(x) = \sum_{n=0}^p \frac{1}{(nk+x)^2} > 0$.

2. Main Results

Let us begin with the following Lemmas, which will be used in the sequel.

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$, $p \in \mathbb{N}$ and $k > 0$. Then,*

$$a\gamma + \frac{b}{k} \ln(pk) + \frac{a}{\alpha + \beta x} + a\psi(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) > 0. \tag{2.1}$$

Proof. Let $x > 0$. Then from (1.1) and (1.4), we obtain

$$a\gamma + \frac{b}{k} \ln(pk) + \frac{a}{x} + a\psi(x) - b\psi_{p,k}(x) = a \sum_{n=1}^{\infty} \frac{x}{n(n+x)} + b \sum_{n=0}^p \frac{1}{nk+x} > 0.$$

Then, replacing x by $\alpha + \beta x$ completes the proof. \square

Lemma 2.2. Let $\alpha > 0$, $\beta > 0$, $a \geq b > 0$, $p \in \mathbb{N}$ and $k \geq 1$. Then,

$$-a \ln p + \frac{b}{k} \ln(pk) + a\psi_p(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) \leq 0. \quad (2.2)$$

Proof. Let $x > 0$. Then from (1.2) and (1.4), we obtain

$$-a \ln p + \frac{b}{k} \ln(pk) + a\psi_p(x) - b\psi_{p,k}(x) = b \sum_{n=0}^p \frac{1}{nk+x} - a \sum_{n=0}^p \frac{1}{n+x} \leq 0.$$

Then, replacing x by $\alpha + \beta x$ completes the proof. \square

Lemma 2.3. Let $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$, $p \in \mathbb{N}$ and $k > 0$. Then,

$$-\frac{a}{k} \ln k + \frac{a\gamma}{k} + \frac{b}{k} \ln(pk) + \frac{a}{\alpha + \beta x} + a\psi_k(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) > 0. \quad (2.3)$$

Proof. Let $x > 0$. Then from (1.3) and (1.4), we obtain

$$-\frac{a}{k} \ln k + \frac{a\gamma}{k} + \frac{b}{k} \ln(pk) + \frac{a}{x} + a\psi_k(x) - b\psi_{p,k}(x) = a \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)} + b \sum_{n=0}^p \frac{1}{nk+x} > 0.$$

Then, replacing x by $\alpha + \beta x$ completes the proof. \square

Lemma 2.4. Let α , β , λ , δ , a and b be positive real numbers such that $b\delta \geq a\beta$ and $\alpha + \beta x \leq \lambda + \delta x$. For $p \in \mathbb{N}$ and $k > 0$, if either:

- (i) $\psi_{p,k}(\alpha + \beta x) > 0$ or
- (ii) $\psi_{p,k}(\lambda + \delta x) > 0$,

then, $a\beta\psi_{p,k}(\alpha + \beta x) - b\delta\psi_{p,k}(\lambda + \delta x) \leq 0$.

Proof. (i) Let $\psi_{p,k}(\alpha + \beta x) > 0$. Then, since $\psi_{p,k}(x)$ is increasing for $x > 0$, we have

$$\psi_{p,k}(\lambda + \delta x) \geq \psi_{p,k}(\alpha + \beta x) > 0.$$

This together with the fact that $b\delta \geq a\beta > 0$ yields

$$b\delta\psi_{p,k}(\lambda + \delta x) \geq a\beta\psi_{p,k}(\lambda + \delta x) \geq a\beta\psi_{p,k}(\alpha + \beta x).$$

Thus $a\beta\psi_{p,k}(\alpha + \beta x) - b\delta\psi_{p,k}(\lambda + \delta x) \leq 0$.

(ii) Let $\psi_{p,k}(\lambda + \delta x) > 0$. Then, there two possible values of $\psi_{p,k}(\alpha + \beta x)$. That is, either $\psi_{p,k}(\alpha + \beta x) \leq 0$ or $\psi_{p,k}(\alpha + \beta x) > 0$. If $\psi_{p,k}(\alpha + \beta x) \leq 0$, then $a\beta\psi_{p,k}(\alpha + \beta x) \leq 0$ and $a\delta\psi_{p,k}(\lambda + \delta x) > 0$ yielding $a\beta\psi_{p,k}(\alpha + \beta x) - b\delta\psi_{p,k}(\lambda + \delta x) \leq 0$ as required. If $\psi_{p,k}(\alpha + \beta x) > 0$, then the procedure coincides with (i) above. \square

Theorem 2.1. Define a function F for $x \in (0, \infty)$, $p \in \mathbb{N}$ and $k > 0$ by

$$F(x) = \frac{e^{a\beta\gamma x}(\alpha + \beta x)^a \Gamma(\alpha + \beta x)^a}{(pk)^{-\frac{b\beta x}{k}} \Gamma_{p,k}(\alpha + \beta x)^b},$$

where a, b, α, β are positive real numbers. Then F is increasing on $x \in (0, \infty)$ and the inequality

$$\begin{aligned} \left(\frac{\alpha}{\alpha + \beta x}\right)^a \frac{e^{-a\beta\gamma x}}{(pk)^{\frac{b\beta x}{k}}} \frac{\Gamma(\alpha)^a}{\Gamma_{p,k}(\alpha)^b} &< \frac{\Gamma(\alpha + \beta x)^a}{\Gamma_{p,k}(\alpha + \beta x)^b} \\ &< \left(\frac{\alpha + \beta}{\alpha + \beta x}\right)^a \frac{e^{a\beta\gamma(1-x)}}{(pk)^{\frac{b\beta}{k}(x-1)}} \frac{\Gamma(\alpha + \beta)^a}{\Gamma_{p,k}(\alpha + \beta)^b} \end{aligned} \tag{2.4}$$

is valid for $x \in (0, 1)$.

Proof. Let $f(x) = \ln F(x)$ for $x \in (0, \infty)$. That is,

$$f(x) = a\beta\gamma x + \frac{b\beta x}{k} \ln(pk) + a \ln(\alpha + \beta x) + a \ln \Gamma(\alpha + \beta x) - b \ln \Gamma_{p,k}(\alpha + \beta x).$$

Then,

$$\begin{aligned} f'(x) &= a\beta\gamma + \frac{b\beta}{k} \ln(pk) + \frac{a\beta}{\alpha + \beta x} + a\beta\psi(\alpha + \beta x) - b\beta\psi_{p,k}(\alpha + \beta x) \\ &= \beta \left[a\gamma + \frac{b}{k} \ln(pk) + \frac{a}{\alpha + \beta x} + a\psi(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) \right] > 0 \end{aligned}$$

as a result of Lemma 2.1. That implies f is increasing on $x \in (0, \infty)$. Thus, F is also increasing and for $x \in (0, 1)$ we have,

$$F(0) < F(x) < F(1)$$

yielding the result (2.4). □

Theorem 2.2. Define a function G for $x \in (0, \infty)$, $p \in \mathbb{N}$ and $k \geq 1$ by

$$G(x) = \frac{(pk)^{\frac{b\beta x}{k}} \Gamma_p(\alpha + \beta x)^a}{p^{a\beta x} \Gamma_{p,k}(\alpha + \beta x)^b},$$

where a, b, α, β are positive real numbers such that $a \geq b$. Then G is decreasing on $x \in (0, \infty)$ and the inequality

$$\begin{aligned} \frac{p^{a\beta(x-1)}}{(pk)^{\frac{b\beta}{k}(x-1)}} \frac{\Gamma(\alpha + \beta)^a}{\Gamma_{p,k}(\alpha + \beta)^b} &\leq \frac{\Gamma_p(\alpha + \beta x)^a}{\Gamma_{p,k}(\alpha + \beta x)^b} \\ &\leq \frac{p^{a\beta x}}{(pk)^{\frac{b\beta x}{k}}} \frac{\Gamma(\alpha)^a}{\Gamma_{p,k}(\alpha)^b} \end{aligned} \tag{2.5}$$

is valid for $x \in (0, 1)$, with equality when $a = b$ and $k = 1$.

Proof. Let $g(x) = \ln G(t)$ for $x \in (0, \infty)$. That is,

$$g(x) = -a\beta x \ln p + \frac{b\beta x}{k} \ln(pk) + a \ln \Gamma_p(\alpha + \beta x) - b \ln \Gamma_{p,k}(\alpha + \beta x).$$

Then,

$$\begin{aligned} g'(x) &= -a\beta \ln p + \frac{b\beta}{k} \ln(pk) + a\beta \psi_p(\alpha + \beta x) - b\beta \psi_{p,k}(\alpha + \beta x) \\ &= \beta \left[-a \ln p + \frac{b}{k} \ln(pk) + a\psi_p(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) \right] \leq 0 \end{aligned}$$

as a result of Lemma 2.2. Thus, G is decreasing and for $x \in (0, 1)$ we have,

$$G(1) \leq G(x) \leq G(0)$$

yielding the result (2.5). □

Theorem 2.3. Define a function H for $x \in (0, \infty)$, $p \in \mathbb{N}$ and $k > 0$ by

$$H(x) = \frac{e^{\frac{a\beta\gamma x}{k}} (\alpha + \beta x)^a \Gamma_k(\alpha + \beta x)^a}{k^{\frac{a\beta x}{k}} (pk)^{-\frac{b\beta x}{k}} \Gamma_{p,k}(\alpha + \beta x)^b},$$

where a, b, α, β are positive real numbers. Then H is increasing on $x \in (0, \infty)$ and the inequality

$$\begin{aligned} \left(\frac{\alpha}{\alpha + \beta x} \right)^a \frac{k^{\frac{a\beta x}{k}}}{e^{\frac{a\beta\gamma x}{k}} (pk)^{\frac{b\beta x}{k}}} \frac{\Gamma_k(\alpha)^a}{\Gamma_{p,k}(\alpha)^b} &< \frac{\Gamma_k(\alpha + \beta x)^a}{\Gamma_{p,k}(\alpha + \beta x)^b} \\ &< \left(\frac{\alpha + \beta}{\alpha + \beta x} \right)^a \frac{k^{\frac{a\beta}{k}(x-1)}}{e^{\frac{a\beta\gamma}{k}(x-1)} (pk)^{\frac{b\beta}{k}(x-1)}} \frac{\Gamma_k(\alpha + \beta)^a}{\Gamma_{p,k}(\alpha + \beta)^b} \end{aligned} \quad (2.6)$$

is valid for $x \in (0, 1)$.

Proof. Let $h(x) = \ln H(t)$ for $x \in (0, \infty)$. That is,

$$h(x) = -\frac{a\beta x}{k} \ln k + \frac{a\beta\gamma x}{k} + \frac{b\beta x}{k} \ln(pk) + a \ln(\alpha + \beta x) + a \ln \Gamma_k(\alpha + \beta x) - b \ln \Gamma_{p,k}(\alpha + \beta x).$$

Then,

$$\begin{aligned} h'(x) &= -\frac{a\beta}{k} \ln k + \frac{a\beta\gamma}{k} + \frac{b\beta}{k} \ln(pk) + \frac{a\beta}{\alpha + \beta x} + a\beta \psi_k(\alpha + \beta x) - b\beta \psi_{p,k}(\alpha + \beta x) \\ &= \beta \left[-\frac{a}{k} \ln k + \frac{a\gamma}{k} + \frac{b}{k} \ln(pk) + \frac{a}{\alpha + \beta x} + a\psi_k(\alpha + \beta x) - b\psi_{p,k}(\alpha + \beta x) \right] \\ &> 0 \end{aligned}$$

as a result of Lemma 2.3. Thus, H is increasing and for $x \in (0, 1)$ we have,

$$H(0) < H(x) < H(1)$$

yielding the result (2.6). □

Remark 2.1. Let $a = b = \beta = 1$ and $k \rightarrow 1$ in either (2.4) or (2.6), then we obtain

$$\left(\frac{\alpha}{\alpha+x}\right) \frac{e^{-\gamma x}}{p^x} \frac{\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+x)}{\Gamma_p(\alpha+x)} < \left(\frac{\alpha+1}{\alpha+x}\right) \frac{e^{\gamma(1-x)}}{p^{(x-1)}} \frac{\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{2.7}$$

which is weaker than the results:

$$\frac{e^{-\gamma x}}{p^x} \frac{\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+x)}{\Gamma_p(\alpha+x)} < \frac{e^{\gamma(1-x)}}{p^{(x-1)}} \frac{\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{2.8}$$

obtained by Krasniqi and Shabani in Theorem 3.3 of [4].

Remark 2.2. Inequalities of type (2.4), (2.5) and (2.6) have been investigated intensively in the papers [5], [6], [8], [9], [10], [11] and [12].

Theorem 2.4. Define a function T for $x \in [0, \infty)$, $p \in \mathbb{N}$ and $k > 0$ by

$$T(x) = \frac{\Gamma_{p,k}(\alpha + \beta x)^a}{\Gamma_{p,k}(\lambda + \delta x)^b}$$

where $\alpha, \beta, \lambda, \delta, a$ and b are positive real numbers such that $\alpha + \beta x \leq \lambda + \delta x$, $a\beta \leq b\delta$ and either $\psi_{p,k}(\alpha + \beta x) > 0$ or $\psi_{p,k}(\lambda + \delta x) > 0$. Then T is decreasing and the inequality

$$\frac{\Gamma_{p,k}(\alpha + \beta)^a}{\Gamma_{p,k}(\lambda + \delta)^b} \leq \frac{\Gamma_{p,k}(\alpha + \beta x)^a}{\Gamma_{p,k}(\lambda + \delta x)^b} \leq \frac{\Gamma_{p,k}(\alpha)^a}{\Gamma_{p,k}(\lambda)^b} \tag{2.9}$$

is valid for $x \in [0, 1]$.

Proof. Let $u(x) = \ln T(x)$ for $x \in [0, \infty)$. That is,

$$u(x) = a \ln \Gamma_{p,k}(\alpha + \beta x) - b \ln \Gamma_{p,k}(\lambda + \delta x)$$

Then,

$$\begin{aligned} u'(x) &= a\beta \frac{\Gamma'_{p,k}(\alpha + \beta x)}{\Gamma_{p,k}(\alpha + \beta x)} - b\delta \frac{\Gamma'_{p,k}(\lambda + \delta x)}{\Gamma_{p,k}(\lambda + \delta x)} \\ &= a\beta\psi_{p,k}(\alpha + \beta x) - b\delta\psi_{p,k}(\lambda + \delta x) \leq 0 \end{aligned}$$

by Lemma 2.4. That implies u is decreasing. Consequently, T is also decreasing and for $x \in [0, 1]$ we have,

$$T(1) \leq T(x) \leq T(0)$$

yielding the result (2.9). □

Remark 2.3. Let $k \rightarrow 1$ in Theorem 2.4, then we obtain the results for the p -analogue as presented in Theorem 3.9 of [4].

Remark 2.4. Let $p \rightarrow \infty$ in Theorem 2.4, then we obtain the results for the k -analogue as presented in Theorem 3.3 of [7].

Remark 2.5. Let $p \rightarrow \infty$ as $k \rightarrow 1$ in Theorem 2.4, then we obtain the results of Theorem 2 of [3].

3. Conclusion

By using some basic analytical techniques, we established some inequalities for certain ratios involving the (p, k) -analogue of the Gamma function, which was recently introduced in [13]. The results provide generalizations of some previous results.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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