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Research Article

Some Remarks on the 2-Distance-Balanced Graphs

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Abstract. The aim of this paper is to investigate the notion of 2-distance-balanced graphs as a generalized form of distance-balanced graphs. Furthermore, we introduce a subclass of such graphs so-called strongly 2-distance-balanced graphs and present some related results based on Cartesian and lexicographic products of two graphs.

Keywords. 2-distance-balanced graph; Strongly 2-distance-balanced graph; Cartesian product; Lexicographic product

MSC. 05C12; 05C40

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1. Introduction

In 1999, Handa [4] considered initially distance-balanced partial cubes. Also, in 2005, distance-balanced graphs in the framework of various kinds of graph products has been studied by Jerebic *et al*. [7]. In 2006, Kutnar *et al*. [9] introduced a subclass of such graphs so-called strongly distance-balanced graphs by the concept of distance partition. Very recently, Kutnar *et al*. [11] also defined another class of such graphs as nicely distance-balanced graphs and classified them for diferent diameters. In this paper, by generalizing the concept of distance-balanced graphs we introduce a new class 2-distance-balanced graphs and study their properties. We also define strongly 2-distance-balanced graphs and discuss about their Cartesian and lexicographic products.

Let G be a finite, undirected and connected graph with diameter d, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u,v \in V(G)$, we put $d(u,v) = d_G(u,v)$ stands for the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of G we denote

$$W_{uv}^G = \{x \in V(G) \mid d(x,u) < d(x,v)\}.$$

Similarly, we can define W_{vu}^G. Also, consider the notion

$${}_uW_v^G = \{x \in V(G) \mid d(x,u) = d(x,v)\}.$$

Note that for any edge $uv \in E(G)$ the sets W_{vu}^G, W_{uv}^G and ${}_uW_v^G$ form a partition of $V(G)$.

Definition 1 ([7])**.** We say that G is distance-balanced (DB) whenever for an arbitrary pair of adjacent vertices u and v of G there exists a positive integer γ_{uv}, such that

$$|W_{uv}^G| = |W_{vu}^G| = \gamma_{uv}.$$

This notion has been studied in several papers (see [1,2], [6–13]).

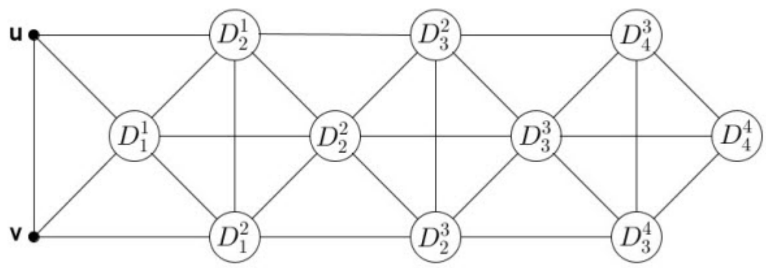

Figure 1. A distance partition of a graph with diameter 4 with respect to edge uv.

For a given graph G, assume that uv is an arbitrary edge of G. For any two integers k, l, we let

$$D_l^k(u,v) = \{x \in V(G) \mid d(u,x) = k \text{ and } d(v,x) = l\}. \tag{1.1}$$

Following Figure 1, the sets $D_l^k(u,v)$ give rise to a "distance partition" of $V(G)$ with respect to the edge uv. The triangle inequality implies that for $k \in \{1,\dots,d\}$ only the sets $D_{k-1}^k(u,v)$, $D_k^k(u,v)$ and $D_k^{k-1}(u,v)$ can be nonempty. Besides, one can easily observe that G is distance-balanced if and only if

$$\sum_{k=1}^{d} |D_{k-1}^k(u,v)| = \sum_{k=1}^{d} |D_k^{k-1}(u,v)| \tag{1.2}$$

holds for every edge $uv \in E(G)$ (see also [9]).

Definition 2 ([9])**.** Graph G is said to be strongly distance-balanced if

$$|D_{k-1}^k(u,v)| = |D_k^{k-1}(u,v)|$$

holds for $1 \le k \le d$ and for every edge $uv \in E(G)$.

2. 2-Distance-Balanced Graphs

In the following we present some results inspired by the results of Frelih *et al*. [3] and present some examples about it.

Definition 3. A connected graph G is called 2-*distanced-balanced* (2-DB for short) if and only if for each $u,v \in V(G)$ with $d(u,v)=2$ we have $|W^G_{u\underline{2}v}| = |W^G_{v\underline{2}u}|$, where

$$W^G_{u\underline{2}v} = \{x \in V(G) | d(x,u) < d(x,v)\}.$$

Similarly, we can define $W^G_{v\underline{2}u}$. Also, consider the notion

$$_u\underset{\underline{2}}{W}{}^G_v = \{x \in V(G) | d(x,u) = d(x,v)\}.$$

We note that the notion $u\underline{2}v$ means a path with length two.

Similar to the distance-balanced property and with respect to vertices u and v at distance 2 we easily see that all the sets $W^G_{u\underline{2}v}$, $W^G_{v\underline{2}u}$ and $_u\underset{\underline{2}}{W}{}^G_v$ form a partition for $V(G)$ (see Figure 2).

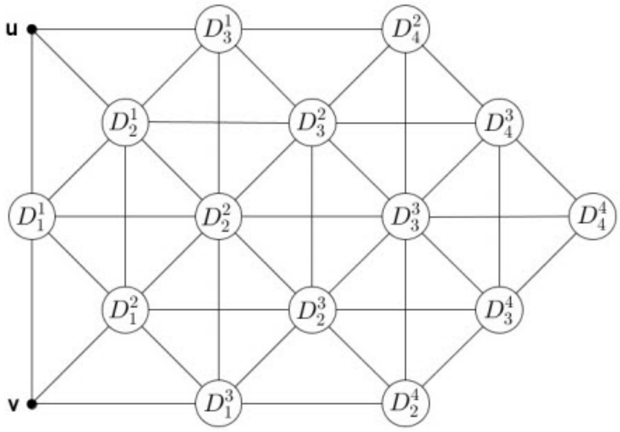

Figure 2. A distance partition of a graph with diameter 4 with respect to vertices u and v at distance 2.

Example 1. Some graphs such as P_2 (a path with length 2), star graphs S_k (with $k > 1$), Friendship graphs F_n (with $n > 1$) and the wheel graphs W_n (with $n > 4$), diamond graph, butter y graph are non-regular 2-distance-balanced graphs but without distance-balanced property. In Figure 3 we give an example for non-regular graph with both distance-balanced and 2-distance-balanced properties.

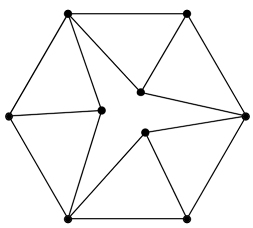

Figure 3. Distance-balanced and 2-distance-balanced non-regular graph.

Now we use graph joint to obtain next results.

Definition 4. Suppose that G is an arbitrary graph and $K_1 = \{v\}$ an external vertex and not belonging to G. Then graph joint $G \vee K_1$ (or denoted by $G + K_1$) of graphs G and K_1 is a graph with

$$\begin{cases} C(G \vee K_1) = V(G) \cup \{v\}, \\ E(G \vee K_1) = E(G) \cup \{uv \mid u \in V(G)\}. \end{cases}$$

Obviously, $G \vee K_1$ is connected and its diameter is at most 2. Moreover, G is connected if and only if $G \vee K_1$ is 2-connected, i.e., it remains connected whenever any arbitrary vertex is removed.

Lemma 1. *Suppose that G is a non-complete regular graph, then $G \vee K_1$ is 2-DB.*

Proof. Let G be a regular graph with valency k and $G \vee K_1$ be a graph constructed as above where $K_1 = \{v\}$ for an arbitrary fixed vertex $v \not\in V(G)$. Assume $G_1, G_2, \dots, G_n$ be all the connected components of G for some $n \geq 1$. All essentially different types of vertices a, b with $d(a,b) = 2$ in $G \vee K_1$ are either both from $V(G_i)$ or one from $V(G_i)$, the other from $V(G_j)$ for some $1 \leq i \neq j \leq n$. First, suppose that $a, b \in V(G_i)$ such that $d(a,b) = 2$ and $1 \leq i \leq n$. Then the fact that $\operatorname{diam}(G \vee K_1) \leq 2$ yields

$$W_{a\underline{2}b}^{G \vee K_1} = \{a\} \cup (N_{G_i}(a) \setminus (N_{G_i}(a) \cap N_{G_i}(b))). \tag{2.1}$$

We remind that $N_G(u)$ stands for the set of all neighbors of vertex u in graph G. According to (2.1) we observe that

$$|W_{a\underline{2}b}^{G \vee K_1}| = 1 + |N_{G_i}(a)| - |N_{G_i}(a) \cap N_{G_i}(b)|.$$

Similarly

$$|W_{b\underline{2}a}^{G \vee K_1}| = 1 + |N_{G_i}(b)| - |N_{G_i}(a) \cap N_{G_i}(b)|.$$

Now regularity of G shows that $|W_{a\underline{2}b}^{G \vee K_1}| = |W_{b\underline{2}a}^{G \vee K_1}|$ and the claim is proven for the first case. For the second case, let us to pick arbitrary $a \in V(G_i)$ and $b \in V(G_j)$ for some $1 \leq i \neq j \leq n$. Then we have

$$W_{a\underline{2}b}^{G \vee K_1} = \{a\} \cup N_{G_i}(a), \qquad W_{b\underline{2}a}^{G \vee K_1} = \{b\} \cup N_{G_j}(b) \tag{2.2}$$

Again the regularity of G together with (2.2) implies that $|W_{a\underline{2}b}^{G \vee K_1}| = |W_{b\underline{2}a}^{G \vee K_1}| = k + 1$. Therefore, $G \vee K_1$ is 2-DB and this completes the proof. □

Lemma 2. *If G is a connected 2-DB graph that is not 2-connected, then $G \cong H \vee K_1$ for some regular graph H which is not connected and graph K_1 with an existent vertex $v \in V(G)$.*

Proof. Consider G as a connected 2-distance-balanced graph that is not 2-connected and v a cut vertex in G. If we remove the vertex v, we get some subgraph H with connected components $H_1, H_2, \dots, H_n$ for some $n \geq 2$. We claim that v is adjacent to every vertex in H_k for at least one k, $1 \leq k \leq n$. To show this, suppose this statement is not true. Then for arbitrary H_i and H_j there exists $a_i \in V(H_i)$ such that $d(a_i, v) = 2$. From the connectivity of H_i we can find

$b_i \in V(H_i)$ with $d(b_i,v) = d(a_i,b_i) = 1$. Similarly, there are $a_j, b_j \in V(H_j)$ such that $d(a_j,v) = 2$ and $d(b_j,v) = d(a_j,b_j) = 1$. On the other hand, we have

$$\begin{cases} W^G_{v\underline{2}a_i} \supseteq \{v\} \cup V(H_j) \Longrightarrow 1 + |V(H_j)| \le |W^G_{v\underline{2}a_i}|, \\ W^G_{a_i\underline{2}v} \subseteq V(H_i) \setminus \{b_i\} \Longrightarrow |W^G_{a_i\underline{2}v}| \le |V(H_i)| - 1 \end{cases}$$

which together with the fact that G is 2-DB implies that

$$|V(H_j)| \le |V(H_i)| - 2. \tag{2.3}$$

Similar to this process we obtain the same relation for a_j and b_j as follows.

$$\begin{cases} W^G_{v\underline{2}a_j} \supseteq \{v\} \cup V(H_i) \Longrightarrow 1 + |V(H_i)| \le |W^G_{v\underline{2}a_j}|, \\ W^G_{a_j\underline{2}v} \subseteq V(H_j) \setminus \{b_j\} \Longrightarrow |W^G_{a_j\underline{2}v}| \le |V(H_j)| - 1 \end{cases}$$

which implies that

$$|V(H_i)| \le |V(H_j)| - 2.$$

The recent inequality together with (2.3) implies that

$$|V(H_j)| + 2 \le |V(H_i)| \le |V(H_j)| - 2$$

which is a contradiction. Hence, v is adjacent to every vertex in H_k for at least one k, $1 \le k \le n$. From now on, without loss of generality, we may assume that v is adjacent to every vertex in $V(H_1)$. Now we show that the induced subgraph H_1 is regular. Let us pick arbitrary $u \in V(H) \setminus V(H_1)$ adjacent to v. Obviously, $d(u,w) = 2$ for every $w \in V(H_1)$. On the other hand, for arbitrary $x, y \in V(H_1)$ if $w \in W^G_{u\underline{2}x} \cap V(H_1)$ then

$$d(u,w) = d(u,v) + d(v,w) = 2 < d(w,x)$$

which is a contradiction, hence we get $W^G_{u\underline{2}x} \subseteq V(G) \setminus V(H_1)$. If we pick $w \in W^G_{u\underline{2}x} \cap (V(G) \setminus V(H_1))$, then

$$d(x,w) = d(w,v) + d(v,x) = d(w,v) + 1 = d(w,v) + d(v,y) = d(y,w).$$

This shows that $|W^G_{u\underline{2}x}| = |W^G_{u\underline{2}y}|$. We note that since $W^G_{x\underline{2}u} \subseteq V(H_1)$, so $W^G_{x\underline{2}u} = \{x\} \cup (N_G(x) \setminus \{v\})$ for all $x \in V(H_1)$. Therefore,

$$|W^G_{x\underline{2}u}| = 1 + |N_G(x)| - 1 = |N_{H_1}(x)| + 1 \quad \text{for every } x \in V(H_1),$$

$$\Rightarrow \quad |N_{H_1}(x)| + 1 = |N_G(x)| = |W^G_{x\underline{2}u}| = |W^G_{u\underline{2}x}| = |W^G_{u\underline{2}y}| = |W^G_{y\underline{2}u}| = |N_G(y)| = |N_{H_1}(y)| + 1$$

for arbitrary $x, y \in V(H_1)$. Thus, induced subgraph H_1 is regular. Assume that H_1 has a valency k. We claim that v is adjacent to every vertex in $V(H)$. On the contrary suppose that this statement is not true. Then there exists $u_2 \in V(H_2)$ such that $d(u_2,v) = 2$. Hence we can find $u_1 \in V(H_2)$ such that $d(u_1,v) = d(u_1,u_2) = 1$. As we know that $|N_G(w)| = k + 1$ for arbitrary $w \in V(H_1)$, then we have

$$W^G_{w\underline{2}u_1} = \{w\} \cup (N_G(w) \setminus \{v\}) \quad \Longrightarrow \quad |W^G_{w\underline{2}u_1}| = k + 1,$$

$$W^G_{v\underline{2}u_2} \supseteq V(H_1) \cup \{v\} \quad \Longrightarrow \quad |W^G_{v\underline{2}u_2}| \ge |V(H_1)| + 1 \ge k + 2.$$

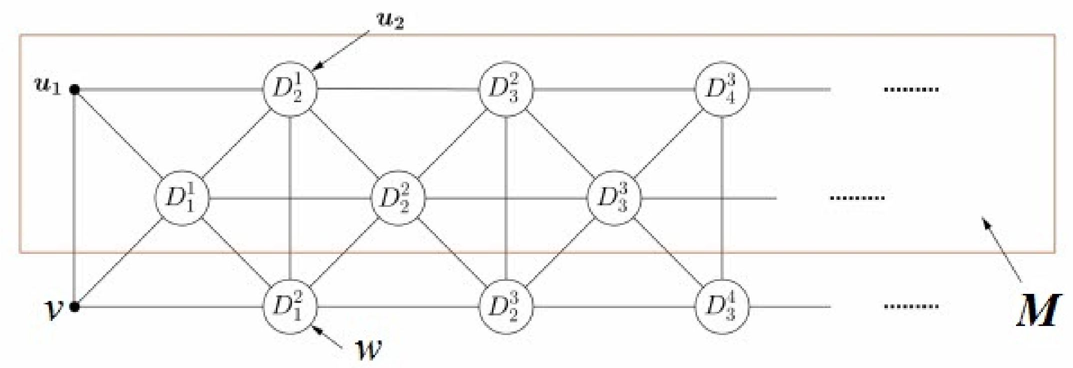

Figure 4. A distance partition of G with representation of M, cut vertex v and existent vertices u_1, u_2.

Now define $M = \bigcup_{i=1}^{d} (D_i^{i-1} \cup D_i^i)$. We easily see that $W_{u_2\underline{2}v}^G \subseteq M \subseteq W_{u_1\underline{2}w}^G$ which means that

$$|W_{u_2\underline{2}v}^G| \le |W_{u_1\underline{2}w}^G| = k+1.$$

Consequently,

$$k+2 \le |W_{v\underline{2}u_2}^G| = |W_{u_2\underline{2}v}^G| \le k+1,$$

which is a contradiction. So v is adjacent to every vertex in $V(H)$. Finally, we prove that graph H is regular with valency k. To do this, we only need to prove that H_2 is regular with valency k. Let us choose arbitrary $u \in V(H_2)$ and $w \in V(H_1)$ (we already found that $d(u,w) = 2$). Then

$$W_{u\underline{2}w}^G = \{u\} \cup (N_G(u) \setminus \{v\}) = \{u\} \cup N_{H_2}(u),$$

$$W_{w\underline{2}u}^G = \{w\} \cup (N_G(w) \setminus \{v\}) = \{w\} \cup N_{H_1}(w).$$

Hence we get $|W_{u\underline{2}w}^G| = 1 + |N_{H_2}(u)|$ which together with the fact that $|W_{w\underline{2}u}^G| = 1+k$ implies that

$$|N_{H_2}(u)| = k \quad \text{for all } u \in V(H_2),$$

which completes the proof. □

Theorem 1. *Graph G is connected 2-DB graph that is not 2-connected if and only if $G \cong H \vee K_1$ for some regular graph H which is not connected.*

Proof. Following Lemmas 1 and 2 the consequence is obtained. □

3. Strongly 2-distance-balanced graphs

As we mentioned before relation (1.2) shows a characterization for DB graphs. Moreover, if $|D_{k-1}^k(u,v)| = |D_k^{k-1}(u,v)|$ holds for $1 \le k \le d$ and for every edge $uv \in E(G)$, then G is distance-balanced. However, the converse is not necessarily true. For instance, in the generalized Petersen graphs GP(24,4), GP(35,8), and GP(35,13), one can find two adjacent vertices u,v and an integer k, such that $|D_{k-1}^k(u,v)| \ne |D_k^{k-1}(u,v)|$. But it is easy to observe that these graphs are distance-balanced. Obviously, by taking $u,v \in V(G)$ with $d(u,v) = 2$ relation (1.2) together with

Figure 2 inspires us a characterization for 2-DB graphs as follows.

$$\sum_{i=1}^{2}\sum_{k=i}^{d}|D^{k}_{k-i}(u,v)|=\sum_{i=1}^{2}\sum_{k=i}^{d}|D^{k-i}_{k}(u,v)|.$$

Definition 5. We say G is *strongly* 2*-distance-balanced* (or 2-SDB for short), if $|D^{k}_{k-i}(u,v)|=|D^{k-i}_{k}(u,v)|$ for every integer $k\in\{i,\dots,d\}$, $i=1,2$ and any vertices $u,v\in V(G)$ with $d(u,v)=2$.

Obviously, each 2-SDB graph is 2-DB graph but the converse is not necessarily true. For a graph G, a vertex u of G and an integer i, let $S_i(u)=\{x\in V(G)\mid d(x,u)=i\}$ denotes the set of vertices of G which are at distance i from u. Let $u,v\in V(G)$ be two arbitrary vertices at distance 2. Observe that $S_i(u)$ is a disjoint union of the sets $D^{i}_{i-2}(u,v)$, $D^{i}_{i-1}(u,v)$, $D^{i}_{i}(u,v)$, $D^{i}_{i+1}(u,v)$ and $D^{i}_{i+2}(u,v)$. Similarly, $S_i(v)$ is a disjoint union of the sets $D^{i-2}_{i}(u,v)$, $D^{i-1}_{i}(u,v)$, $D^{i}_{i}(u,v)$, $D^{i+1}_{i}(u,v)$ and $D^{i+2}_{i}(u,v)$. In the following we give a necessary condition for 2-SDB graphs.

Proposition 1. *Let G be a graph with diameter $d\ge 2$. If G is 2-SDB then $|S_i(u)|=|S_i(v)|$ holds for every vertices $u,v\in V(G)$ with $d(u,v)=2$ and every $i\in\{0,1,\dots,d\}$.*

Proof. Suppose that G is 2-strongly distance-balanced and let $u,v\in V(G)$ with $d(u,v)=2$. Following the definition, we have $|D^{k}_{k+i}(u,v)|=|D^{k+i}_{k}(u,v)|$ for every integer $k\in\{0,1,\dots,d-i\}$, $i=1,2$. However, since

$$S_i(u)=D^{i}_{i-2}(u,v)\cup D^{i}_{i-1}(u,v)\cup D^{i}_{i}(u,v)\cup D^{i}_{i+1}(u,v)\cup D^{i}_{i+2}(u,v),$$

$$S_i(v)=D^{i-2}_{i}(u,v)\cup D^{i-1}_{i}(u,v)\cup D^{i}_{i}(u,v),D^{i+1}_{i}(u,v)\cup D^{i+2}_{i}(u,v),$$

we have also $|S_i(u)|=|S_i(v)|$ for every $i\in\{0,1,\dots,d\}$. □

We notice that opposite to the SDB graphs, for 2-SDB graphs the converse of the previous statement is still open and the question arises is it possible to find some conditions for the converse to be satisfied. Observe that connectedness implies that $|S_k(u)|=|S_k(v)|$ holds for any pair of vertices $u,v\in V(G)$ and every $k\in\{0,1,\dots,d\}$. It is worth mentioning that graphs with this property are also called *distance-degree regular*. Distance-degree regular graphs were studied in [5].

In the following we study conditions under which the lexicographic and Cartesian products give rise to a strongly 2-distance-balanced graph. Note that such graph products, constructed from two graphs G and H, have vertex set $V(G)\times V(H)$. Let (a,u) and (b,v) be two distinct vertices in $V(G)\times V(H)$. Then (a,u) and (b,v) are adjacent in the lexicographic product $G[H]$, if $ab\in E(G)$ or if $a=b$ and $uv\in E(H)$. They are also adjacent in the Cartesian product $G\Box H$ if they coincide in one of the two coordinates and are adjacent in the other coordinate.

Theorem 2. *Let G and H are connected graphs such that $G[H]$ is connected. Then G and H are 2-SDB and regular graphs, respectively, if and only if $G[H]$ is 2-SDB.*

Proof. Let us choose a pair of vertices from $V(G)\times V(H)$. Consider first the case where the vertices are as form of (a,x), (a,y) such that $d((a,x),(a,y))=2$ (which we mean $d_{G[H]}((a,x),(a,y))=2$), then $d(x,y)\ge 2$. Let (u,v) be a vertex not incident with (a,x) or

(a,y). We consider two case on the pair (a,u): if $d(a,u) \geq 2$, then following the definition of distance in lexicographic product we have $d((a,x),(u,v)) = d((a,y),(u,v)) = d(a,u)$, which implies $(u,v) \in D_i^i((a,x),(a,y))$, where $i = d(a,u)$. We recall that

$$d_{G[H]}((g,h),(\acute{g},\acute{h})) = \begin{cases} d_G(g,\acute{g}); & \text{if } g \neq \acute{g}, \\ 1; & \text{if } g = \acute{g} \text{ and } h\acute{h} \in E(H), \\ 2; & \text{if } g = \acute{g} \text{ and } h\acute{h} \notin E(H). \end{cases}$$

Moreover, by the above definition, if $d(a,u) \leq 1$, then $d((a,x),(u,v)) \leq 2$ and $d((a,y),(u,v)) \leq 2$. Therefore, among the sets $D_i^{i+1}((a,x),(a,y))$, $D_{i+1}^i((a,x),(a,y))$, $D_{i+2}^i((a,x),(a,y))$ and $D_i^{i+2}((a,x),(a,y))$, $i \in \{0,1,\dots,d\}$, only the following sets may be nonempty:

$$D_1^2((a,x),(a,y)) = \{(a,v) \mid v \in S_1(y) \setminus S_1(x)\},$$
$$D_2^1((a,x),(a,y)) = \{(a,v) \mid v \in S_1(x) \setminus S_1(y)\}. \quad (3.1)$$

Since H is regular we obtain $|D_2^1((a,x),(a,y))| = |D_1^2((a,x),(a,y))|$.

Now assume that $(a,x),(b,y) \in V(G) \times V(H)$, where $a \neq b$ and $d((a,x),(b,y)) = 2$. Clearly, we have $d(a,b) = 2$. For such case we have

$$D_1^2((a,x),(b,y)) = (D_1^2(a,b) \times V(H)) \cup \{(b,v) \mid v \in S_1(y)\} = [D_1^2(a,b) \times V(H)] \cup [\{b\} \times S_1(y)],$$
$$D_2^1((a,x),(b,y)) = (D_2^1(a,b) \times V(H)) \cup \{(a,v) \mid v \in S_1(x)\} = [D_2^1(a,b) \times V(H)] \cup [\{a\} \times S_1(x)]$$
$$D_3^2((a,x),(b,y)) = \{(u,v) \mid u \in S_2(a) \cap S_3(b)\} = [S_2(a) \cap S_3(b)] \times V(H) = D_3^2(a,b) \times V(H),$$
$$D_2^3((a,x),(b,y)) = \{(u,v) \mid u \in S_3(a) \cap S_2(b)\} = [S_3(a) \cap S_2(b)] \times V(H) = D_2^3(a,b) \times V(H),$$
$$D_1^3((a,x),(b,y)) = \{(u,v) \mid u \in S_1(b) \cap S_3(a)\} = [S_1(b) \cap S_3(a)] \times V(H) = D_1^3(a,b) \times V(H),$$
$$D_3^1((a,x),(b,y)) = \{(u,v) \mid u \in S_3(b) \cap S_1(a)\} = [S_3(b) \cap S_1(a)] \times V(H) = D_3^1(a,b) \times V(H).$$

In general, we observe that for $i \geq 2$

$$D_i^{i+1}((a,x),(b,y)) = D_i^{i+1}(a,b) \times V(H), \quad D_{i+1}^i((a,x),(b,y)) = D_{i+1}^i(a,b) \times V(H),$$
$$D_i^{i+2}((a,x),(b,y)) = D_i^{i+2}(a,b) \times V(H), \quad D_{i+2}^i((a,x),(b,y)) = D_{i+2}^i(a,b) \times V(H). \quad (3.2)$$

Therefore, using all relations in (3.2) together with previous case, since H is regular and G is 2-SDB, then also $G[H]$ is 2-SDB and the proof is complete. For the converse first let us to prove that H is regular. If $G[H]$ is 2-SDB then (3.1) is satisfied for all $x,y \in V(H)$ and that is possible if H is regular. In order to show that G is 2-SDB one can use the obtained sets in the last case as above where the vertices $(a,x),(b,y) \in V(G) \times V(H)$ with $a \neq b$ have been chosen and the consequence is easily followed. □

We now investigate the strongly distance-balanced property of Cartesian graph product.

Theorem 3. *Let G be a graph with diameter less than 4 and H be connected graph. Then $G \square H$ is 2-SDB if and only if both G and H are 2-SDB.*

Proof. Suppose that both G and H are 2-SDB. Consider first the case where the vertices are as form of $(a,x),(a,y) \in V(G) \times V(H)$ such that $d((a,x),(a,y)) = 2$, then $d(x,y) = 2$. Let (u,v) be a vertex in $D_1^2((a,x),(a,y))$. So

$$d((a,x),(u,v)) = d(a,u) + d(x,v) = 2.$$

Now we get

$$d(a,u) = 2 \implies (u,v) \in S_2(a) \times \{x\},$$

$$d(x,v) = 2 \implies (u,v) \in \{a\} \times S_2(x),$$

$$d(a,u) = 1,\ d(x,v) = 1 \implies (u,v) \in S_1(a) \times S_1(x).$$

Similarly, since $d((a,y),(u,v)) = d(a,u) + d(y,v) = 1$, we have

$$d(a,u) = 1 \implies (u,v) \in S_1(a) \times \{y\},$$

$$d(y,v) = 1 \implies (u,v) \in \{a\} \times S_1(y).$$

Suppose that

$$A := (S_2(a) \times \{x\}) \cup (\{a\} \times S_2(x)) \cup (S_1(a) \times S_1(x)),$$

$$B := (S_1(a) \times \{y\}) \cup (\{a\} \times S_1(y)).$$

Then

$$D_1^2((a,x),(a,y)) = A \cap B = \{a\} \times (S_1(y) \cap S_2(x)) = \{a\} \times D_1^2(x,y),$$

and similarly we get $D_2^1((a,x),(a,y)) = \{a\} \times D_2^1(x,y)$. On the other hand, assume that $(u,v) \in D_1^3((a,x),(a,y))$ and thus

$$d((a,x),(u,v)) = 3 \implies (u,v) \in \bigcup_{i=0}^{3} S_i(a) \times S_{3-i}(x),$$

$$d((a,y),(u,v)) = 1 \implies (u,v) \in \bigcup_{i=0}^{1} S_i(a) \times S_{1-i}(y),$$

which shows that

$$D_1^3((a,x),(a,y)) = (\{a\} \times D_1^3(x,y)) \cup (S_1(a) \times \{y\}).$$

Hence we also have

$$D_3^1((a,x),(a,y)) = (\{a\} \times D_3^1(x,y)) \cup (S_1(a) \times \{x\}).$$

Following this process we obtain for $i \geq 2$ and $l = 1,2$

$$D_i^{i+l}((a,x),(a,y)) = \bigcup_{j=0}^{i} S_j(a) \times D_{i-j}^{i-j+l}(x,y),$$

$$D_{i+l}^{i}((a,x),(a,y)) = \bigcup_{j=0}^{i} S_j(a) \times D_{i-j+l}^{i-j}(x,y), \tag{3.3}$$

For the case $(a,x),(b,y) \in V(G) \times V(H)$ with $a \neq b$ we have

$$D_i^{i+1}((a,x),(b,y)) = \bigcup_{j=0}^{i} \left(D_j^j(a,b) \times D_{i-j}^{i-j+1}(x,y)\right) \cup \left(D_j^{i+1}(a,b) \times \left\{x \mid d(x,y) = i-j\right\}\right)$$

$$= \bigcup_{j=0}^{i} \left(D_j^j(a,b) \times D_{i-j}^{i-j+1}(x,y)\right) \cup \left(D_i^{i+1}(a,b) \times \{x\}\right) \cup \left(D_{i-1}^{i+1}(a,b) \times \{x\}\right). \tag{3.4}$$

Similarly

$$D_{i+1}^i((a,x),(b,y)) = \bigcup_{j=0}^{i} \left(D_j^j(a,b) \times D_{i-j+1}^{i-j}(x,y)\right) \cup \left(D_{i+1}^i(a,b) \times \{y\}\right) \cup \left(D_{i+1}^{i-1}(a,b) \times \{y\}\right). \tag{3.5}$$

Also,

$$D_1^3((a,x),(b,y)) = \bigcup_{j=0}^{1} \left((D_j^j(a,b) \times D_{1-j}^{3-j}(x,y)) \cup (D_j^2(a,b) \times D_{1-j}^1(x,y)) \cup (D_j^3(a,b) \times D_{1-j}^0(x,y))\right)$$

$$= \bigcup_{j=0}^{1} \left((D_j^j(a,b) \times D_{1-j}^{3-j}(x,y)) \cup (D_j^2(a,b) \times D_{1-j}^1(x,y))\right)$$

$$\cup \left(\{b\} \times \{x \mid d(x,y)=1\}\right) \cup \left(D_1^3(a,b) \times \{x=y\}\right), \tag{3.6}$$

$$D_3^1((a,x),(b,y)) = \bigcup_{j=0}^{1} \left((D_j^j(a,b) \times D_{3-j}^{1-j}(x,y)) \cup (D_2^j(a,b) \times D_1^{1-j}(x,y)) \cup (D_3^j(a,b) \times D_0^{1-j}(x,y))\right)$$

$$= \bigcup_{j=0}^{1} \left((D_j^j(a,b) \times D_{3-j}^{1-j}(x,y)) \cup (D_2^j(a,b) \times D_1^{1-j}(x,y))\right)$$

$$\cup \left(\{a\} \times \{y \mid d(x,y)=1\}\right) \cup \left(D_3^1(a,b) \times \{x=y\}\right), \tag{3.7}$$

and for all $i \geq 2$

$$D_i^{i+2}((a,x),(b,y)) = \bigcup_{j=0}^{i} \left((D_j^j(a,b) \times D_{i-j}^{i-j+2}(x,y)\right),$$

$$D_{i+2}^i((a,x),(b,y)) = \bigcup_{j=0}^{i} \left((D_j^j(a,b) \times D_{i-j+2}^{i-j}(x,y)\right). \tag{3.8}$$

Now equalities (3.3)-(3.8) show that $G\square H$ is 2-SDB. For the converse, we only need to show that G is 2-SDB. Let $G\square H$ is 2-SDB and $d(a,b)=2$ for given $a,b \in V(G)$. For a fixed vertex $x \in V(H)$ we obtain

$$(u,v) \in D_{k-i}^k((a,x),(b,x)) \implies (u,v) \in A_a \cap A_b := \left(\bigcup_{j=0}^{k} S_j(a) \times S_{k-j}(x)\right) \cap \left(\bigcup_{l=0}^{k-i} S_l(b) \times S_{k-l-i}(x)\right).$$

On the other hand, $A_a \cap A_b \neq \emptyset$ implies that $j-l=i=1,2$ and therefore, we have

$$D_{k-i}^k((a,x),(b,x)) = \bigcup_{l=0}^{k-i} \left(D_l^{l+i}(a,b) \times S_{k-l-i}(x)\right),$$

$$D_k^{k-i}((a,x),(b,x)) = \bigcup_{l=0}^{k-i} \left(D_{l+i}^l(a,b) \times S_{k-l-i}(x)\right),$$

which shows that $|D_l^{l+i}(a,b)| = |D_{l+i}^l(a,b)|$ and particularly $|D_{k-i}^k(a,b)| = |D_k^{k-i}(a,b)|$. Hence, G is 2-SDB. One can follow the process as above for vertices $(a,x),(a,y) \in V(G)\times V(H)$ and conclude the same result for H to be 2-SDB.

Remark 1. Note that in previous theorem if $\mathrm{diam}(G) \geq 4$ then we can not conclude (3.8) and the proof would be complicated. This condition is needed here while it does not need for Cartesian product of SDB graphs. We remind that the Cartesian product of two graphs is SDB iff both of graphs are SDB (see [9]).

4. Conclusions

In this paper, inspired by the structure of distance-balanced graphs, 2-distance-balanced graphs has been introduced and the relation of such graphs to the property of connectivity of graph has been investigated. Besides, some examples of this class have been presented and the related results for these graphs have been proved. Moreover, a subclass of such graphs so-called strongly 2-distance-balanced graphs has been introduced and a characterization of these graphs has been revealed based on Cartesian and lexicographic products of two graphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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