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g-Bessel Sequences and Operators

Research Article

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Abstract. Let H be a separable Hilbert space, and let $g\mathfrak{B}$ be the set of all g-Bessel sequences for H. We show that $g\mathfrak{B}$ is a C^*-algebra isometrically isomorphic to $L(H)$ (the algebra of all bounded linear operators of H). Also, we classify g-Bessel sequences in H in terms of different kinds of operators in $L(H)$. Using operator theory tools, we investigate geometry of g-Bessel sequences.

Keywords. g-Bessel sequence; g-Orthonormal basis; g-Riesz basis; Bounded linear operator

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1. Introduction and preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer in 1952 to study some deep questions in non-harmonic Fourier series. Today, frame theory is a central tool in many areas such as function space and signal analysis. For an introduction to frame theory (see [4, 12]).

g-frame, introduced by W. Sun in [18], is a generalization of frame which covers many extensions of frames, e.g. pseudo-frames, outer frames, oblique frames, continuous frames, and fusion frames. Recently, g-frames in Hilbert spaces have been studied intensively.

Since the investigation of Bessel sequences (g-Bessel sequences) is essential for study of frames (g-frames), in literature there are many studies about Bessel sequences (g-Bessel sequences). For instance, the authors in [3], have studied finite extensions of Bessel sequences in infinite dimensional Hilbert spaces. The authors in [16], have investigated some equalities and inequalities for g-Bessel sequences in Hilbert spaces with pseudo-inverse operators. In the references [2, 6], the authors showed that the set of all Bessel sequences for H is a Banach space and obtained important results. Also, in [7], the authors gave a C^*-algebra structure to the set of all Bessel sequences for H.

Let H be a separable Hilbert space. Let $g\mathfrak{B}$, $g\mathscr{F}$, $g\mathscr{R}$, and $g\mathscr{O}$ be the set of all g-Bessel sequences, g-frames, g-Riesz bases, g-orthonormal bases for H, respectively. In this paper, we show that $g\mathfrak{B}$ is a C^*-algebra isometrically isomorphic to $L(H)$ (the algebra of all bounded linear operators of H). Also, we classify g-Bessel sequences for H in terms of different kinds of operators in $L(H)$. Using a bijection between the set $g\mathfrak{B}$ and $L(H)$ we obtain interesting results in g-frames. Using operator theory tools, we investigate geometry of g-Bessel sequences. More precisely, we show that $g\mathscr{F}$ is a disconnected open subset of $g\mathfrak{B}$, and we characterize the connected components of $g\mathscr{F}$. Also, we show that $g\mathscr{R}$ is arcwise connected and open in $g\mathfrak{B}$, and the set $g\mathscr{O}$ is a both open and closed and arcwise connected.

By using this geometric properties, we get some results, e.g. since $g\mathfrak{B}$, $g\mathscr{F}$ are open sets, we can say that they are stable under small perturbations and since $g\mathscr{R}$ is arcwise connected, we can get some results for the sum of two g-Riesz bases and so on.

Throughout this paper H denotes a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\{H_i : i \in \mathbb{N}\}$ is a sequence of separable Hilbert spaces. Also, for every $i \in \mathbb{N}$, $L(H,H_i)$ is the set of all bounded linear operators from H to H_i and $L(H)$ is the algebra of all bounded linear operators on H. For an operator $T \in L(H)$ we write T^* for its adjoint, $ker(T)$ for its kernel, and $R(T)$ for its range.

In the rest of this section we review several well-known definitions and results. The new results are stated in Section 2.

For every sequence $\{H_i\}_{i\in\mathbb{N}}$, the space

$$\left(\sum_{i\in\mathbb{N}} \bigoplus H_i\right)_{\ell^2} = \left\{(f_i)_{i\in\mathbb{N}} : f_i \in H_i, i \in \mathbb{N}, \sum_{i\in\mathbb{N}} \|f_i\|^2 < \infty\right\}$$

with pointwise operations and the following inner product is a Hilbert space

$$\langle (f_i)_{i\in\mathbb{N}}, (g_i)_{i\in\mathbb{N}}\rangle = \sum_{i\in\mathbb{N}} \langle f_i, g_i\rangle.$$

A sequence $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ is called a *g-frame* for H with respect to $\{H_i : i \in \mathbb{N}\}$ if there exist $A, B > 0$ such that for every $f \in H$

$$A\|f\|^2 \le \sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 \le B\|f\|^2,$$

A, B are called g-frame bounds. We call Λ a *tight g-frame* if $A = B$ and a *Parseval g-frame* if $A = B = 1$. If only the right hand side inequality is required, Λ is called a *g-Bessel sequence*.

If Λ is a g-Bessel sequence, then *the synthesis operator* for Λ is the linear operator,

$$T_\Lambda : \left(\sum_{i\in\mathbb{N}} \bigoplus H_i\right)_{\ell^2} \mapsto H \qquad T_\Lambda (f_i)_{i\in\mathbb{N}} = \sum_{i\in\mathbb{N}} \Lambda_i^* f_i.$$

We call the adjoint of the synthesis operator, *the analysis operator*. The analysis operator is the linear operator,

$$T_\Lambda^* : H \mapsto \left(\sum_{i\in\mathbb{N}} \bigoplus H_i\right)_{\ell^2} \qquad T_\Lambda^* f = (\Lambda_i f)_{i\in\mathbb{N}}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the *g-frame operator* of Λ and $S_\Lambda f = \sum_{i\in\mathbb{N}} \Lambda_i^* \Lambda_i f \quad (f \in H)$.

If $\Lambda = (\Lambda_i)_{i\in\mathbb{N}}$ is a g-frame with lower and upper g-frame bounds A, B, respectively, then the g-frame operator of Λ is a bounded, positive and invertible operator on H and

$$A\langle f, f\rangle \le \langle S_\Lambda f, f\rangle \le B\langle f, f\rangle \quad (f \in H)$$

so

$$A \cdot I \le S_\Lambda \le B \cdot I.$$

The canonical dual g-frame for $\Lambda = (\Lambda_i)_{i\in\mathbb{N}}$ is defined by $\widetilde{\Lambda} = (\widetilde{\Lambda}_i)_{i\in\mathbb{N}}$, where $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$ which is also a g-frame for H with lower and upper g-frame bounds $\frac{1}{B}$ and $\frac{1}{A}$, respectively. Also for every $f \in H$, we have

$$f = \sum_{i\in\mathbb{N}} \Lambda_i^* \widetilde{\Lambda}_i f = \sum_{i\in\mathbb{N}} \widetilde{\Lambda}_i^* \Lambda_i f.$$

We say that $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ is a *g-frame sequence* if it is a g-frame for $\overline{\operatorname{span}\{\Lambda_i^*(H_i)\}_{i\in\mathbb{N}}}$. A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ is *g-complete* if $\{f : \Lambda_i f = 0$, for all $i \in \mathbb{N}\} = \{0\}$. We note that the g-Bessel sequence Λ is g-complete if and only if T_Λ^* is injective. We say Λ is a *g-orthonormal basis* for H, if

$$\langle \Lambda_i^* f_i, \Lambda_j^* f_j\rangle = \delta_{i,j}\langle f_i, f_j\rangle, \quad \forall\ f_i \in H_i,\ f_j \in H_j,\ i, j \in \mathbb{N}$$

and

$$\sum_{i\in\mathbb{N}} \|\Lambda_i f\|^2 = \|f\|^2 \quad (f \in H).$$

A sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ is a *g-Riesz sequence* if there exist $A, B > 0$ such that for every finite subset $F \subset \mathbb{N}$, $g_i \in H_i$, and $i \in F$

$$A \sum_{i\in F} \|g_i\|^2 \le \left\| \sum_{i\in F} \Lambda_i^* g_i \right\|^2 \le B \sum_{i\in F} \|g_i\|^2. \tag{1.1}$$

g-Riesz sequence $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ is called a *g-Riesz basis* if it is g-complete, too. Clearly, every g-orthonormal basis is a g-Riesz basis.

Let $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in \mathbb{N}\}$ be g-Bessel sequences with g-Bessel bounds B and C, respectively. The operator $S_{\Lambda\Theta} : H \mapsto H$ defined by

$$S_{\Lambda\Theta} f = \sum_{i\in\mathbb{N}} \Lambda_i^* \Theta_i f, \quad (f \in H)$$

is a bounded operator, $\|S_{\Lambda\Theta}\| \le \sqrt{BC}$, $S_{\Lambda\Theta}^* = S_{\Theta\Lambda}$ and $S_{\Lambda\Lambda} = S_\Lambda$.

Two g-Bessel sequences $\Lambda = \{\Lambda_i \in L(H, H_i) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in L(H, H_i) : i \in \mathbb{N}\}$ are called dual g-frames if

$$f = \sum_{i\in\mathbb{N}} \Lambda_i^* \Theta_i f = \sum_{i\in\mathbb{N}} \Theta_i^* \Lambda_i f, \quad (f \in H).$$

For more details about g-frames, see [1, 15, 18].

2. Main results

Let $g\mathfrak{B}$ be the set of all g-Bessel sequences for H with respect to $(H_i)_{i\in\mathbb{N}}$. In the following theorem, we show that $g\mathfrak{B}$ is a C^*-algebra isometrically isomorphic to $L(H)$.

Theorem 2.1. *Let $g\mathfrak{B}$ be the set of all g-Bessel sequences for H with respect to $(H_i)_{i\in\mathbb{N}}$. For every $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\} \in g\mathfrak{B}$, set $\|\Lambda\|_{g\mathfrak{B}} = \|T_\Lambda^*\| = \|T_\Lambda\|$, then $(g\mathfrak{B}, \|\cdot\|_{g\mathfrak{B}})$ is a C^*-algebra isometrically isomorphic to $L(H)$.*

Proof. It is easy to check that $g\mathfrak{B}$ is a vector space, and $\|\cdot\|_{g\mathfrak{B}}$ is a norm. Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a fixed g-orthonormal basis for H. Consider the mapping

$$A_\Upsilon : g\mathfrak{B} \to L(H) \tag{2.1}$$

$$\Lambda = (\Lambda_i)_{i=1}^\infty \to S_{\Upsilon\Lambda}. \tag{2.2}$$

Clearly, $S_{\Upsilon\Lambda} \in L(H)$ and A_Υ is a well-defined bounded linear operator from $g\mathfrak{B}$ to $L(H)$. Consider the mapping $B_\Upsilon : L(H) \to g\mathfrak{B}$ by $B_\Upsilon(T) = (\Upsilon_i T)_{i=1}^\infty$. For every $f \in H$

$$\sum_{i=1}^\infty \|\Upsilon_i T f\|^2 = \|Tf\|^2 \le \|T\|^2 \|f\|^2.$$

Therefore $(\Upsilon_i T)_{i=1}^\infty \in g\mathfrak{B}$ and B_Υ is a well-defined linear operator. Clearly B_Υ is the inverse of A_Υ. Also, $\|A_\Upsilon(\Lambda)\| = \|S_{\Upsilon\Lambda}\| = \|T_\Lambda^*\| = \|\Lambda\|_{g\mathfrak{B}}$.

Therefore $g\mathfrak{B}$ is isometrically isomorphic to $L(H)$. Completeness of $L(H)$ implies that $(g\mathfrak{B}, \|\cdot\|_{g\mathfrak{B}})$ is a Banach space.

Now, we define a multiplication on the Banach space $g\mathfrak{B}$. Define $\cdot_{g\mathfrak{B}} : g\mathfrak{B} \times g\mathfrak{B} \to g\mathfrak{B}$ by $((\Lambda_i)_{i=1}^\infty, (\Theta_i)_{i=1}^\infty) \to (\Upsilon_i S_{\Upsilon\Lambda} S_{\Upsilon\Theta})_{i=1}^\infty$, which is well-defined, since for every $f \in H$

$$\begin{aligned}\sum_{i=1}^\infty \|\Upsilon_i S_{\Upsilon\Lambda} S_{\Upsilon\Theta} f\|^2 &= \|S_{\Upsilon\Lambda} S_{\Upsilon\Theta} f\|^2 \\ &\le \|(\Lambda_i)_{i=1}^\infty\|_{g\mathfrak{B}}\ \|S_{\Upsilon\Theta} f\|^2 \\ &\le \|(\Lambda_i)_{i=1}^\infty\|_{g\mathfrak{B}}\ \|(\Theta_i)_{i=1}^\infty\|_{g\mathfrak{B}}\ \|f\|^2.\end{aligned}$$

Therefore $(\Upsilon_i S_{\Upsilon\Lambda} S_{\Upsilon\Theta})_{i=1}^\infty \in g\mathfrak{B}$. The above relation implies that

$$\|(\Lambda_i)_{i=1}^\infty \cdot_{g\mathfrak{B}} (\Theta_i)_{i=1}^\infty)\|_{g\mathfrak{B}} \le \|(\Theta_i)_{i=1}^\infty\|_{g\mathfrak{B}}\ \|(\Lambda_i)_{i=1}^\infty\|_{g\mathfrak{B}}.$$

Also, $(g\mathfrak{B}, \cdot_{g\mathfrak{B}}, \|\cdot\|_{g\mathfrak{B}})$ satisfies the usual algebraic rules. Thus $(g\mathfrak{B}, \cdot_{g\mathfrak{B}}, \|\cdot\|_{g\mathfrak{B}})$ is a Banach algebra.

Consider the mapping $*_{g\mathfrak{B}} : g\mathfrak{B} \to g\mathfrak{B}$ by $(\Lambda_i)_{i=1}^\infty \to (\Upsilon_i S_{\Lambda\Upsilon})_{i=1}^\infty$. It is easy to check that $*_{g\mathfrak{B}}$ is an involution on $g\mathfrak{B}$, and for every $(\Lambda_i)_{i=1}^\infty \in g\mathfrak{B}$, $\|(\Lambda_i)_{i=1}^\infty{}^{*_{g\mathfrak{B}}} \cdot_{g\mathfrak{B}} (\Lambda_i)_{i=1}^\infty\|_{g\mathfrak{B}} = \|(\Lambda_i)_{i=1}^\infty\|^2$. Moreover, A_Υ is an algebraic isomorphism. Since for $(\Theta_i)_{i=1}^\infty, (\Lambda_i)_{i=1}^\infty \in g\mathfrak{B}$ we have

$$\begin{aligned}A_\Upsilon((\Lambda_i)_{i=1}^\infty \cdot_{g\mathfrak{B}} (\Theta_i)_{i=1}^\infty) &= A_\Upsilon((\Upsilon_i S_{\Upsilon\Lambda} S_{\Upsilon\Theta})_{i=1}^\infty) \\ &= S_{\Upsilon(\Upsilon_i S_{\Upsilon\Lambda} S_{\Upsilon\Theta})_{i=1}^\infty} \\ &= S_{\Upsilon\Lambda} S_{\Upsilon\Theta} \\ &= A_\Upsilon((\Lambda_i)_{i=1}^\infty) A_\Upsilon((\Theta_i)_{i=1}^\infty),\end{aligned}$$

and

$$\begin{aligned}A_\Upsilon((\Lambda_i)_{i=1}^\infty{}^{*_{g\mathfrak{B}}}) = A_\Upsilon(\Upsilon_i S_{\Lambda\Upsilon})_{i=1}^\infty &= S_{\Upsilon(\Upsilon_i S_{\Lambda\Upsilon})_{i=1}^\infty} \\ &= S_{\Lambda\Upsilon} = (A_\Upsilon(\Lambda_i)_{i=1}^\infty)^{*_{g\mathfrak{B}}}.\end{aligned}$$

Therefore, $g\mathfrak{B}$ is a C^*-algebra isometrically isomorphic to $L(H)$. □

In the following proposition, we give a characterization of g-Riesz bases.

Proposition 2.1. *Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Bessel sequence and $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis. Then $S_{\Lambda\Upsilon}$ is an invertible linear operator on H if and only if Λ is a g-Riesz basis. Moreover, ${S_{\Lambda\Upsilon}}^{-1} = S_{\widetilde{\Upsilon}\widetilde{\Lambda}}$, where $\widetilde{\Lambda}$ and $\widetilde{\Upsilon}$ are the dual g-Riesz bases of Λ and Υ, respectively.*

Proof. We know that $S_{\Lambda\Upsilon} = T_\Lambda T_\Upsilon^*$ since Υ is a g-Riesz bases, then T_Υ^* is invertible (see [20]). Therefore $S_{\Lambda\Upsilon}$ is invertible if and only if T_Λ is invertible and this is equivalent to Λ is a g-Riesz basis.

Now, we claim that ${S_{\Lambda\Upsilon}}^{-1} = S_{\widetilde{\Upsilon}\widetilde{\Lambda}}$. For every $f, g \in H$,

$$\begin{aligned}\langle S_{\Lambda\Upsilon} \circ S_{\widetilde{\Upsilon}\widetilde{\Lambda}} f, g\rangle = \langle S_{\widetilde{\Upsilon}\widetilde{\Lambda}} f, S_{\Upsilon\Lambda} g\rangle &= \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} \langle \widetilde{\Upsilon}_k^* \widetilde{\Lambda}_k f, \Upsilon_i^* \Lambda_i g\rangle \\ &= \sum_{i\in\mathbb{N}} \langle \widetilde{\Lambda}_i f, \Lambda_i g\rangle = \langle S_{\Lambda\widetilde{\Lambda}} f, g\rangle \\ &= \langle f, g\rangle.\end{aligned}$$

Thus $S_{\Lambda\Upsilon} \circ S_{\widetilde{\Upsilon}\widetilde{\Lambda}} f = f$. Similarly, $S_{\widetilde{\Upsilon}\widetilde{\Lambda}} \circ S_{\Lambda\Upsilon} f = f$. □

The following result has been proved in [18]. By Proposition 2.1, we give another proof for it.

Proposition 2.2. *A sequence $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ is a g-Riesz basis for H if and only if there exists a g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in \mathbb{N}\}$ for H and an invertible operator T on H such that $\Lambda_i = \Gamma_i T$, for every $i \in \mathbb{N}$.*

Proof. Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis for H, and let $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a fixed g-orthonormal basis for H. By Proposition 2.1, $T = S_{\Gamma\Lambda} \in L(H)$ is invertible and $\Gamma_i T = \Lambda_i$, for every $i \in \mathbb{N}$.

Conversely, suppose that there exist a g-orthonormal basis $\Gamma = \{\Gamma_i \in L(H,H_i) : i \in \mathbb{N}\}$ for H and an invertible operator T on H such that $\Lambda_i = \Gamma_i T$, for every $i \in \mathbb{N}$. Clearly, $(\Gamma_i T)_{i\in\mathbb{N}}$ is a g-Bessel sequence. Since Γ is a g-orthonormal basis, $S_{(\Gamma_i T)_{i\in\mathbb{N}}\Gamma} = T^*$. But T^* is invertible, then by Proposition 2.1, $(\Gamma_i T)_{i\in\mathbb{N}} = (\Lambda_i)_{i\in\mathbb{N}}$ is a g-Riesz basis for H. □

In the following proposition, we give characterizations of g-frames and g-frame sequences.

Proposition 2.3. *Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis for H and $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\} \in g\mathfrak{B}$. Then the following statements hold:*

(1) *Λ is a g-frame if and only if $S_{\Lambda\Upsilon}$ is surjective.*

(2) *Λ is a g-frame sequence if and only if $S_{\Lambda\Upsilon}$ is a closed range operator.*

Proof. (1) Clearly, $S_{\Lambda\Upsilon} = T_\Lambda T_\Upsilon^*$. Since Υ is a g-Riesz basis, then T_Υ^* is invertible. This easily implies that $R(S_{\Lambda\Upsilon}) = R(T_\Lambda T_\Upsilon^*) = R(T_\Lambda)$. We know that Λ is a g-frame if and only if T_Λ is surjective and this is equivalent to $S_{\Lambda\Upsilon}$ is a surjective operator.

(2) Λ is a g-frame sequence if and only if T_Λ is a closed range operator (see [19]). Now, a proof similar to the proof of (1) proves the claim. □

In the following proposition, we give a characterization of g-Riesz sequences.

Proposition 2.4. *Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Bessel sequence and $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis for H. Then the following statements are equivalent:*

(1) *Λ is a g-Riesz sequence.*

(2) *T_Λ is an injective with closed range operator.*

(3) *$S_{\Lambda\Upsilon}$ is an injective with closed range operator.*

Proof. (1)$\Longrightarrow$(2) Suppose that Λ is a g-Riesz sequence, then there exist constants $A,B>0$ such that for every $(g_i)_{i=1}^\infty \in \left(\sum_{i=1}^\infty \oplus H_i\right)_{\ell^2}$

$$A\|(g_i)_{i=1}^\infty\|^2 \le \|T_\Lambda((g_i)_{i=1}^\infty)\|^2 \le B\|(g_i)_{i=1}^\infty\|^2.$$

This implies that T_Λ is an injective operator with closed range.

(2)$\Longrightarrow$(1) Suppose that T_Λ is an injective operator with closed range, then there exists a constant $A>0$ such that for every $(g_i)_{i=1}^\infty \in \left(\sum_{i=1}^\infty \oplus H_i\right)_{\ell^2}$

$$A\|(g_i)_{i=1}^\infty\|^2 \le \|T_\Lambda((g_i)_{i=1}^\infty)\|^2.$$

On the other hand, Λ is a g-Bessel sequence, then the upper bound condition is fulfilled, too. Therefore, Λ is a g-Riesz sequence.

(2)$\Longleftrightarrow$(3) Since Υ is a g-Riesz basis, then $R(S_{\Lambda\Upsilon}) = R(T_\Lambda T_\Upsilon^*) = R(T_\Lambda)$. Also, it is easy to check that $S_{\Lambda\Upsilon}$ is injective if and only if T_Λ is injective. Now, by (1)$\Longleftrightarrow$(2) the claim is obvious. □

We recall the following definition of g-Besselian frames.

Definition 2.1 ([8]). *A g-frame $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ is called a g-Besselian frame for H if the convergence of $\sum_{i=1}^\infty \Lambda_j^* g_j$ implies that $(g_j)_{j=1}^\infty \in (\sum_{i=1}^\infty \oplus H_i)_{\ell^2}$.*

Proposition 2.5. *Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis for H. Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\} \in g\mathfrak{B}$ and $\dim H_i < \infty$, for every $i \in \mathbb{N}$. Then Λ is a g-Besselian frame if and only if $S_{\Lambda\Upsilon}$ is a surjective Fredholm operator.*

Proof. Since Υ is a g-Riesz basis, easily we can see that $\dim\ker T_\Lambda = \dim\ker S_{\Lambda\Upsilon}$. By [8, Theorem 2.3], Λ is a g-Besselian frame if and only if Λ is a g-frame and $\dim\ker T_\Lambda < \infty$. By Proposition 2.3, this is equivalent to $S_{\Lambda\Upsilon}$ is surjective, and $\dim\ker S_{\Lambda\Upsilon} = \dim\ker T_\Lambda < \infty$. Hence, $S_{\Lambda\Upsilon}$ is a surjective Fredholm operator.

Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathscr{I}\}$ be a g-orthonormal basis for H. Consider the mapping

$$O_\Upsilon : g\mathfrak{B} \to L(H) \tag{2.3}$$

$$\Lambda = (\Lambda_i)_{i=1}^\infty \to S_{\Lambda\Upsilon}. \tag{2.4}$$

Clearly, $S_{\Lambda\Upsilon} \in L(H)$ and O_Υ is a well-defined bounded anti-linear operator from $g\mathfrak{B}$ to $L(H)$.

Consider the mapping $Q_\Upsilon : L(H) \to g\mathfrak{B}$ by $Q_\Upsilon(T) = (\Upsilon_i T^*)_{i=1}^\infty$. For every $f \in H$

$$\sum_{i=1}^\infty \|\Upsilon_i T^* f\|^2 = \|T^* f\|^2 \le \|T^*\|^2 \|f\|^2.$$

Therefore $(\Upsilon_i T^*)_{i=1}^{\infty} \in g\mathfrak{B}$ and Q_Υ is a well-defined anti-linear operator. Clearly Q_Υ is the inverse of O_Υ. Therefore O_Υ is a bijection between $g\mathfrak{B}$ and $L(H)$. Also, $\|O_\Upsilon(\Lambda)\| = \|S_{\Lambda\Upsilon}\| = \|T_\Lambda\| = \|T_\Lambda^*\| = \|\Lambda\|_{g\mathfrak{B}}$ and this implies that O_Υ is continuous. □

In the following proposition, by using the bijection O_Υ, we characterize several classes of bounded linear operators on H in terms of their corresponding g-Bessel sequences.

Proposition 2.6. *Let $\Upsilon = \{\Upsilon_i \in L(H, H_i) : i \in \mathscr{I}\}$ be a g-orthonormal basis for H, and O_Υ be defied as* (2.3)*. Then the following statements hold:*

(1) *Let $\mathscr{A}$ be the set of all unitary operators in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-orthonormal basis}\}.$$
(2) *Let $\mathscr{A}$ be the set of all invertible operators in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-Riesz basis}\}.$$
(3) *Let $\mathscr{A}$ be the set of all surjective operators in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-frame}\}.$$
(4) *Let $\mathscr{A}$ be the set of all closed range operators in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-frame sequence}\}.$$
(5) *Let $\mathscr{A}$ be the set of all injective and closed range operators in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-Riesz sequence}\}.$$
(6) *Let $\mathscr{A}$ be the set of all partial isometries in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a Parseval } g\textit{-frame sequence}\}.$$
(7) *Let $\mathscr{A}$ be the set of all co-isometries in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a Parseval } g\textit{-frame}\}.$$
(8) *If* $\dim(H_i) < \infty$, *for every $i \in \mathbb{N}$, and $\mathscr{A}$ is the set of all surjective Fredholm operators in $L(H)$, then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-Besselian frame}\}.$$
(9) *Let $\mathscr{A}$ be the set of all operators with dense range in $L(H)$. Then*
$$O_\Upsilon^{-1}(\mathscr{A}) = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \Lambda \textit{ is a } g\textit{-complete}\}.$$

Proof. (1) It is enough to prove that T is a unitary operator if and only if $(\Theta_i)_{i=1}^{\infty} = (\Upsilon_i T^*)_{i=1}^{\infty}$ is a g-orthonormal basis. Let T be a unitary operator. Then easily we can see $(\Theta_i)_{i=1}^{\infty} = (\Upsilon_i T^*)_{i=1}^{\infty}$ is a g-orthonormal basis for H.

Conversely, suppose that $(\Upsilon_i T^*)_{i=1}^{\infty}$ is a g-orthonormal basis for H. For every $f \in H$ we have
$$\|T^* f\|^2 = \sum_{i=1}^{\infty} \|\Upsilon_i T^* f\|^2 = \|f\|^2.$$
Therefore T is a co-isometry operator. On the other hand, $S_{(\Upsilon_i T^*)_{i=1}^{\infty}\Upsilon} = T$. Since $(\Upsilon_i T^*)_{i=1}^{\infty}$ is a g-orthonormal basis, then it is a g-Riesz basis, too. By Proposition 2.1, T is invertible. Subsequently, T is a unitary operator.

Similarly, we can see that the parts of (2), (3), (4), (5), (8) are consequences of 2.1, 2.3, 2.3, 2.4, 2.5, respectively. We omit details of the proofs.

(6) It is enough to prove that T is a partial isometry if and only if $(\Theta_i)_{i=1}^{\infty} = (\Upsilon_i T^*)_{i=1}^{\infty}$ is a Parseval g-frame sequence. Let T be a partial isometry, then $T^*T = P_{\ker T^{\perp}}$ (see [5]). Note that $R(T) = \overline{\operatorname{span}_{i=1}^{\infty}(\Theta_i^* H_i)}$. If $y \in R(T)$, then there exists a $x \in \ker T^{\perp}$ such that $y = Tx$. Hence

$$\begin{aligned}\|y\|^2 &= \|Tx\|^2 = \|x\|^2 = \|T^*Tx\|^2\\ &= \sum_{i=1}^{\infty} \|\Upsilon_i T^* Tx\|^2 = \sum_{i=1}^{\infty} \|\Upsilon_i T^* y\|^2.\end{aligned}$$

Therefore, $(\Theta_i)_{i=1}^{\infty} = (\Upsilon_i T^*)_{i=1}^{\infty}$ is a Parseval g-frame sequence.

Conversely, Suppose that $(\Upsilon_i T^*)_{i=1}^{\infty}$ is a Parseval g-frame sequence. Then for every $x \in H$

$$\|Tx\|^2 = \sum_{i=1}^{\infty} \|\Upsilon_i T^* Tx\|^2 = \|T^*Tx\|^2.$$

Therefore, T^*T is Projection. Subsequently, T is a partial isometry, see [9, Proposition 4.38].

(7) $T \in L(H)$ is a co-isometry if and only if for every $x \in H$

$$\|x\|^2 = \|T^*x\|^2 = \sum_{i=1}^{\infty} \|\Upsilon_i T^* x\|^2.$$

Therefore T is a co-isometry operator if and only if $(\Upsilon_i T^*)_{i=1}^{\infty}$ is a Parseval g-frame, and this proves the claim.

(9) By the definition of g-completeness, $\Lambda \in g\mathfrak{B}$ is g-complete if and only if T_Λ^* is injective and this equivalent to T_Λ is an operator with dense range. Since Υ is a g-orthonormal basis, then $R(T_\Lambda) = R(S_{\Lambda\Upsilon})$. Therefore $\Lambda \in g\mathfrak{B}$ is g-complete if and only if $S_{\Lambda\Upsilon}$ is an operator with dense range. Now, a proof similar to the proof of (1) proves the claim. □

In the following proposition, using operator theory tools, we investigate geometry of g-Bessel sequences.

Proposition 2.7. *Let $g\mathscr{F}$ be the set of all g-frames, $g\mathscr{O}$ be the set of all g-orthonormal bases, $g\mathscr{R}$ be the set of all g-Riesz bases, $g\mathscr{P}$ be the set of Parseval g-frames, and $g\mathscr{R}S$ be the set of all g-Riesz sequences in $g\mathfrak{B}$. Then the following statements hold:*

1. *The sets $g\mathscr{F}$ and $g\mathscr{R}$ are open subsets of $g\mathfrak{B}$.*
2. *If H is an infinite dimensional Hilbert space, then the set $g\mathscr{R}$ is both open and closed in $g\mathscr{F}$, consequently $g\mathscr{F}$ is disconnected.*
3. *The set $g\mathscr{R}$ is arcwise connected.*
4. *The set of connected components of $g\mathscr{F}$ are, precisely $\{\delta_n : n \in \mathbb{N} \cup \{\infty\}\}$, where*
$$\delta_n = \{\Lambda = (\Lambda_i)_{i=1}^{\infty} \in g\mathfrak{B} : \dim\ker T_\Lambda = n\}.$$
5. *The set $g\mathscr{O}$ is both open and closed, and arcwise connected in $g\mathfrak{B}$.*
6. *The set $g\mathscr{P}$ is closed in $g\mathscr{F}$ and $g\mathfrak{B}$.*
7. *The sets $g\mathscr{F} \setminus g\mathscr{R}$ and $g\mathscr{R}S$ are open subsets of $g\mathfrak{B}$.*

Proof. Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathscr{I}\}$ be a fixed g-orthonormal basis for H.

(1) Let $GL(H)$ be the set of all invertible operators in $L(H)$, and let ε be the set of all surjective operators in $L(H)$. By Proposition 2.6, $g\mathscr{R} = O_\Upsilon^{-1}(GL(H))$, and $g\mathscr{F} = O_\Upsilon^{-1}(\varepsilon)$. We know that $GL(H)$ is open in $L(H)$ and by [6, Corollary 2.3], ε is open in $L(H)$. Now, by the continuity of O_Υ, the sets $g\mathscr{F}$ and $g\mathscr{R}$ are open subsets of $g\mathfrak{B}$.

(2) By [14, Proposition 7.8], $GL(H)$ is both open and closed in ε. Since O_Υ is continuous, then $O_\Upsilon^{-1}(GL(H)) = g\mathscr{R}$ is both open and closed in $O_\Upsilon^{-1}(\varepsilon) = g\mathscr{F}$. Hence, $g\mathscr{F}$ is disconnected.

(3) By [9, Corollary 5.30], $GL(H)$ is arcwise connected. Since O_Υ^{-1} is continuous, then $O_\Upsilon^{-1}(GL(H))$ is arcwise connected in $g\mathfrak{B}$.

(4) By [6, Proposition 2.8], for every $n \in \mathbb{N} \cup \{\infty\}$, $\varepsilon_n = \{T \in L(H) : \dim\ker(T) = n\}$ is the connected component of ε. Since O_Υ^{-1} is continuous, then $O_\Upsilon^{-1}(\varepsilon_n)$ is the connected component of $g\mathscr{F} = O_\Upsilon^{-1}(\varepsilon)$. But for every $n \in \mathbb{N} \cup \{\infty\}$,

$$\delta_n = O_\Upsilon^{-1}(\varepsilon_n) = \{\Lambda = (\Lambda_i)_{i=1}^\infty \in g\mathfrak{B} : \dim\ker S_{\Lambda\Upsilon} = \dim\ker T_\Lambda = n\}.$$

(5) The set of all unitary operators are both open and closed and arcwise connected in $L(H)$, see [10]. Now a proof similar to the proofs of (1) and (4), proves the claims.

(6) Let ε_0 be the set of co-isometries in $L(H)$. By [6, Corollary 2.3], ε_0 is closed in ε. By Proposition 2.6, $O_\Upsilon^{-1}(\varepsilon) = g\mathscr{F}$ and $O_\Upsilon^{-1}(\varepsilon_0) = g\mathscr{P}$. Since O_Υ is continuous, then $g\mathscr{P}$ is closed in $g\mathscr{F}$. On the other hand, the mapping $A : L(H) \to L(H)$ defined by $A(T) = TT^*$, is continuous. Since $\{I_H\}$ is closed in $L(H)$, then $A^{-1}\{I_H\} = \varepsilon_o$ is a closed subset of $L(H)$. Now, by continuity of O_Υ, $g\mathscr{P}$ is closed in $g\mathfrak{B}$.

(7) The set of surjective and non-invertible operators, and the set of all bounded below operators (injective and closed range operators) are open subsets of $L(H)$. Now, a proof similar to the proof of (1) proves the claim. □

By using geometric properties, we can get some new results, e.g. by using Proposition 2.6 the following result can be obtained, easily, see also [8, 9].

Proposition 2.8. (1) *Let $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in L(H,H_i) : i \in \mathbb{N}\}$ be Parseval g-frames in H, and let T be a bounded operator which satisfies $\Lambda_i T = \Theta_i$, for every $i \in \mathbb{N}$. Then T is an isometry operator on H. Moreover, if T is invertible, then it is unitary.*

(2) *If $\Lambda = \{\Lambda_i \in L(H,H_i) : i \in \mathbb{N}\}$ is both a g-Riesz basis and a Parseval g-frame, then it is must be a g-orthonormal basis.*

Proof. (1) For every $f \in H$ we have

$$\|f\|^2 = \sum_{i=1}^\infty \|\Theta_i f\|^2 = \sum_{i=1}^\infty \|\Lambda_i(Tf)\|^2 = \|T(f)\|^2.$$

So T is an isometry, hence $T^*T = I$. If T is invertible, then $T^* = T^{-1}$ and T is a unitary operator.

(2) Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a fixed g-orthonormal basis in H. Since Λ is a g-Riesz basis and a Parseval g-frame, by Proposition 2.6, there exists an invertible co-isometry T_1 such that $O_\Upsilon(\Lambda) = T_1$. Since T_1 is an invertible co-isometry, then T_1 is a unitary operator in $L(H)$. Now, by Proposition 2.6, $O_\Upsilon^{-1}(T_1) = \Lambda$ is a g-orthonormal basis for H. □

Proposition 2.9. *Let $\Upsilon = \{\Upsilon_i \in L(H,H_i) : i \in \mathbb{N}\}$ be a g-Riesz basis for H, and let T be a compact operator in $L(H)$. Then the following statements hold:*

(1) *$(\Upsilon_i T^*)_{i\in\mathbb{N}}$ is the limit of a sequence of frame sequences in $g\mathfrak{B}$.*

(2) *If H is an infinite dimensional Hilbert space, then $(\Upsilon_i T^*)_{i\in\mathbb{N}}$ is never a g-frame.*

Proof. (1) Define the mapping $A_{\widetilde{\Upsilon}} : g\mathfrak{B} \to L(H)$ by $A_{\widetilde{\Upsilon}}(\Lambda) = S_{\Lambda\widetilde{\Upsilon}}$, where $\widetilde{\Upsilon}$ is the dual g-Riesz basis of Υ. It is easy to check that $A_{\widetilde{\Upsilon}}$ is a well-defined anti-linear bounded operator. Consider the mapping $B_{\widetilde{\Upsilon}} : L(H) \to g\mathfrak{B}$ by $B_{\widetilde{\Upsilon}}(T) = (\Upsilon_i T^*)_{i=1}^{\infty}$. Clearly, $B_{\widetilde{\Upsilon}}$ is a well-defined anti-linear bounded operator and $B_{\widetilde{\Upsilon}}$ is the inverse of $A_{\widetilde{\Upsilon}}$. Since T is a compact operator in $L(H)$, there exists a sequence $(T_n)_{n=1}^{\infty}$ of finite rank operators in $L(H)$ such that $T_n \to T$. Since $B_{\widetilde{\Upsilon}} = A_{\widetilde{\Upsilon}}^{-1}$ is continuous, then $(\Upsilon_i T_n^*)_{i=1}^{\infty} = A_{\widetilde{\Upsilon}}^{-1}(T_n) \to A_{\widetilde{\Upsilon}}^{-1}(T) = (\Upsilon_i T^*)_{i=1}^{\infty}$. For every $n \in \mathbb{N}$, $S_{(\Upsilon_i T_n^*)_{i=1}^{\infty}\widetilde{\Upsilon}} = T_n$, then $\dim R(S_{(\Upsilon_i T_n^*)_{i=1}^{\infty}\widetilde{\Upsilon}}) = \dim T_n < \infty$, and consequently $R(S_{(\Upsilon_i T_n^*)_{i=1}^{\infty}\widetilde{\Upsilon}})$ is closed. Therefore by proposition 2.3, $(\Upsilon_i T_n^*)_{i=1}^{\infty} = A_{\widetilde{\Upsilon}}^{-1}(T_n)$ is a g-frame sequence in $g\mathfrak{B}$, for $n \in \mathbb{N}$. Hence $(\Upsilon_i T^*)_{i\in\mathbb{N}}$ is the limit of a sequence of frame sequences in $g\mathfrak{B}$.

(2) If H is an infinite dimensional Hilbert space, then the compact operator T is not surjective, see [17]. By Proposition 2.3, $(\Upsilon_i T^*)_{i\in\mathbb{N}}$ is a g-frame if and only if $S_{(\Upsilon_i T^*)_{i=1}^{\infty}\widetilde{\Upsilon}} = T$ is a surjective operator. Therefore if T is compact operator, then $(\Upsilon_i T^*)_{i\in\mathbb{N}}$ is never a g-frame. □

3. Conclusions

We note that since Riesz bases, frames, oblique frames and fusion frames are special cases of g-Riesz bases and g-frames, our results hold for all of them. with our results in geometry of $g\mathfrak{B}$ many problems can be investigated.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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