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A Novel Approach for the Stability Analysis of State Dependent Differential Equation

Research Article

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Abstract. In this paper, we investigate the stability of a differential equation with state-dependent delay under some conditions on delay term. New necessary and sufficient criterions are elaborated for the asymptotic stability of the differential equations with state dependent delay. Moreover, the asymptotic stability of it is illustrated for a special delay function.

Keywords. Asymptotic stability; State depended delay; Delay differential equation

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1. Introduction

State-Dependent delay Differential Equations (SDDE) have a number of applications in various research arias ranging from population dynamics to control theory, see Section 2 in [14] as a review. SDDE is used to make more realistic modelling in the systems whose delay varies according to the internal effects of the system. Therefore they are generating increasing interest from engineers and scientist in recent years. It is shown that the length of time to maturity of Antarctic whales and seals alter according to the state of the population in [13] and it is analyzed by using a mathematical model with SDDE in [1].

SDDE have been investigated for the last five decades. The fundamental theory for local existence and uniqueness theorem for SDDE having Lipschitz continuous initial functions was developed by Drive [8, 9] and Driver and Norris [10]. Winston [37] showed that SDDE has a unique solution under some conditions in addition to continuos initial function. There exist some of the earliest studies on SDDE in [2, 6, 32]. Moreover, lots of theoretical and numerical analysis of SDDE have been done so far, [1, 3–5, 11, 14–16, 18, 20, 22–30, 33–35]. Especially [36] can be seen as a review in order to have much more detail about DDE and SDDE and researches on them.

In this study, we consider the following type of SDDE

$$u'(t) = -A_0u(t) - A_1u(t - \tau(t,u(t))), \tag{1}$$

where A_0, $A_1 \in \mathbb{R}$ and $\tau(t,u(t)) > 0$, $\tau(t,0) \neq 0$ for all $t \in \mathbb{R}^+$. Since the variation of solution can play an important role in the variation of delay, hence its stability, we use the following characteristic equation

$$g(\lambda) = \lambda + A_0 + A_1e^{-\lambda\tau(t,e^{\lambda t})} = 0 \tag{2}$$

in order to analyze stability of solution of equation (1).

In the general case, the characteristic roots λ_j, $j = 1,2,\dots,$ of equation (1) are obtained by solving the characteristic equation (2) where λ_j is a complex number. If the characteristic roots have negative real parts, i.e., $\mathrm{Re}(\lambda_j) < 0$ for all $j = 1,2,\dots$ then the solution of (1) is asymptotically stable and if at least one of the characteristic roots have positive real parts, i.e., $\mathrm{Re}(\lambda_j) > 0$ for some $j = 1,2,\dots$ then the solution of (1) is unstable.

It is attempted to determine the stability and instability regions of the system in parameter space (A_0, A_1) by using D-partition method. The method is originated from paper [31]. Moreover, the analysis by this method are conducted in [12,17,19,21] and [7]. We consider the characteristic equation $g(\lambda, A_0, A_1)$ in two parameters for equation (1). D-partition method is based on fact that the roots of the characteristic equation are continuos functions of the parameters A_0 and A_1. Varying the parameters, λ_j change continuously in complex plane and at least one λ_j crosses the imaginary axis at the point where the stability changes. In this method parameter space is divided into subregions by the hypersurfaces called the D-curves. The points of the D-curves correspond to pure imaginary roots or zero root of the characteristic equation. Moreover, the characteristic equation has the same number of roots with positive real part in each subregion in the parameter space determined by the D-curves. Thus, finding specific point at which the solution of equation (1) is stable, it is enough to find the stability region, including this point.

In order to obtain D-curves, pure imaginary number $\lambda = i\omega$ is substituted in characteristic equation $g(\lambda, A_0, A_1)$. Equating to zero the real and imaginary parts, we have

$$U(\omega, A_0, A_1) = \mathrm{Re}(g(i\omega, A_0, A_1)) = 0, \tag{3}$$

$$V(\omega, A_0, A_1) = \mathrm{Im}(g(i\omega, A_0, A_1)) = 0. \tag{4}$$

Hence, by making use of (3) and (4), parametric equations can be rewritten in the following form

$$A_0 = A_0(\omega), \quad A_1 = A_1(\omega)$$

where ω is a parameter, ranging from $-\infty$ to ∞. These curves and singular solutions of equations (3) and (4) constitute D-curves.

In Section 2, we establish necessary and sufficient conditions for the stability of the solution of equation (1) by using D-partition method. We illustrate these results in Section 3.

2. Necessary and Sufficient Conditions for the Stability of SDDE

It is clear that if $\tau(t,u(t))$ is a linear function with respect to $u(t)$, then $\mathrm{Re}(\tau(t,e^{i\omega t}))$ is an even function and $\mathrm{Im}(\tau(t,e^{i\omega t}))$ is an odd function with respect to ω. In this section, we consider equation (1) for $\tau(t,e^{i\omega t})=\tau_1(t,\omega)+i\tau_2(t,\omega)$ such that $\tau_1(t,0)\neq 0$, $\tau_1(t,\omega)$ is an even function and $\tau_2(t,\omega)$ is an odd function with respect to ω.

The characteristic equation

$$g(\lambda)=\lambda+A_0+A_1e^{-\lambda\tau(t,e^{\lambda t})}=0. \tag{5}$$

As a part of the D-partition method, we have

$$C_*: A_0+A_1=0, \quad \text{for } \lambda=0 \tag{6}$$

this straight line is a line forming the boundary of the D-partition and is denoted by C_*. Substituting $\lambda=i\omega$ and equating to zero the real and imaginary parts in characteristic equation (5), we find following equations

$$A_0+A_1e^{\omega\tau_2(t,\omega)}\cos(\omega\tau_1(t,\omega))=0, \tag{7}$$

$$\omega+A_1e^{\omega\tau_2(t,\omega)}\sin(\omega\tau_1(t,\omega))=0. \tag{8}$$

Solving the above equations for A_0 and A_1, following parametric curve equations are obtained

$$A_0(\omega)=-\frac{\omega\cos(\omega\tau_1(t,\omega))}{\sin(\omega\tau_1(t,\omega))}, \tag{9}$$

$$A_1(\omega)=\frac{\omega}{e^{\omega\tau_2(t,\omega)}\sin(\omega\tau_1(t,\omega))}. \tag{10}$$

Since $A_0(\omega)$ and $A_1(\omega)$ are an even with respect to ω, it is sufficient to take $\omega\in(0,\infty)$. When ω ranges from 0 to ∞, equations (9)-(10) define the D-curves for each $t\in\mathbb{R}^+$. The equations (9)-(10) have singularity for $\omega\tau_1(t,\omega)=k\pi$, $k=0,1,2,\dots$. Thus, we introduce intervals $J_k=(k\pi,(k+1)\pi)$ and denote by C_k the curve in the parameter space (A_0,A_1) for $\omega\tau_1(t,\omega)\in J_k$.

To analyze the curves C_k under assumptions $\tau_1(t,\omega)\le h_1$ and $\omega\tau_2(t,\omega)\le h_2$ where h_1 and h_2 are non-negative real numbers, we firstly consider the following equation

$$\widetilde{A}_1(\omega,T_2)=\frac{\omega}{T_2\sin(\omega\tau_1(t,\omega))} \tag{11}$$

where $T_2\in(0,e^{h_2}]$ is a real number. Equations (9)-(11) define a family of curves since T_2 is not a constant. Holding T_2 fixed, these define $\widetilde{A}_1(\omega,T_2)$ as function of ω, providing a parametric representation of a curve. Different values of T_2 give different curves in the family. We denote the family of curves by $\widetilde{C}_k(T_2)$ for $\omega\tau_1(t,\omega)\in J_k$.

Proposition 1. *If $\omega\tau_2(t,\omega)\le h_2$, the curve $\widetilde{C}_{2k}(e^{h_2})$ lies below the curve C_{2k} in parameter space (A_0,A_1) for $k=0,1,2,\dots$.*

Proof. For every $\omega\tau_1(t,\omega)\in J_{2k}$, equations (9)-(10) give a point $L(A_0,A_1)$ on the curve C_{2k} and equations (9)-(11) give a point $\widetilde{L}(A_0,\widetilde{A}_1)$ on the curve $\widetilde{C}_{2k}(e^{h_2})$. Since $e^{\omega\tau_2(t,\omega)}\le e^{h_2}$ when $\omega\tau_2(t,\omega)\le h_2$ and $A_1(\omega)>0$, $\widetilde{A}_1(\omega,T_2)>0$ for each $\omega\tau_1(t,\omega)\in J_{2k}$, we have $A_1>\widetilde{A}_1$. □

Proposition 2. *If $\omega\tau_2(t,\omega)\le h_2$, the curve $\widetilde{C}_{2k+1}(e^{h_2})$ lies above the curve C_{2k+1} in parameter space (A_0,A_1) for $k=0,1,2,\dots$.*

Proof. It is similar to the proof of Proposition 1. □

Lemma 1. *The curves $\widetilde{C}_k(T_2)$ do not intersect each other for $k = 0,1,2,\dots$.*

Proof. Suppose that there exist an intersection point. It means that, there exist $\omega_1 \neq \omega_2 \in \mathbb{R}^+$ such that $A_0(\omega_1,T_2) = A_0(\omega_2,T_2)$ and $\widetilde{A}_1(\omega_1,T_2) = \widetilde{A}_1(\omega_2,T_2)$. These equalities imply that

$$\frac{\omega_1}{T_2\sin(\omega_1\tau_1(t,\omega_1))} = \frac{\omega_2}{T_2\sin(\omega_2\tau_1(t,\omega_2))}, \quad \frac{\omega_1\cos(\omega_1\tau_1(t,\omega_1))}{\sin(\omega_1\tau_1(t,\omega_1))} = \frac{\omega_2\cos(\omega_2\tau_1(t,\omega_2))}{\sin(\omega_2\tau_1(t,\omega_2))} \tag{12}$$

from equation (9) and (11). For $n \in \mathbb{N}$, $\omega_1\tau_1(t,\omega_1) \neq \omega_2\tau_1(t,\omega_2) + 2n\pi$ is obtained from the left equality in (12) because of $\omega_1 \neq \omega_2$. In addition, left and right equalities in (12) lead to $\cos(\omega_1\tau_1(t,\omega_1)) = \cos(\omega_2\tau_1(t,\omega_2))$ which is a contradiction. □

Lemma 2. *The following limits are satisfied for $k = 1,2,\dots$*

$$\lim_{\omega\tau_1(t,\omega)\to(2k\pi)^+} A_0(\omega) = \lim_{\omega\tau_1(t,\omega)\to(2k\pi)^+} \widetilde{A}_1(\omega) = \lim_{\omega\tau_1(t,\omega)\to((2k-1)\pi)^-} \widetilde{A}_1(\omega,h) = +\infty,$$
$$\lim_{\omega\tau_1(t,\omega)\to((2k-1)\pi)^-} A_0(\omega) = -\infty \tag{13}$$

and

$$\lim_{\omega\tau_1(t,\omega)\to(2k\pi)^-} A_0(\omega) = \lim_{\omega\tau_1(t,\omega)\to(2k\pi)^-} \widetilde{A}_1(\omega) = \lim_{\omega\tau_1(t,\omega)\to((2k-1)\pi)^+} \widetilde{A}_1(\omega,h) = -\infty,$$
$$\lim_{\omega\tau_1(t,\omega)\to((2k-1)\pi)^+} A_0(\omega) = +\infty. \tag{14}$$

Then, we consider the following family of curves denoted by $\overline{C}_k(T_1)$

$$\overline{C}_k(T_1): \begin{cases} \overline{A}_0(\omega,T_1) = -\frac{\omega\cos(\omega T_1)}{\sin(\omega T_1)}, & T_1 \in (-\infty,h_1], \\ \overline{A}_1(\omega,T_1) = \frac{\omega}{e^{h_2}\sin(\omega T_1)}, & \omega T_1 \in J_k. \end{cases} \tag{15}$$

Proposition 3. *If $\tau_1(t,\omega) \le h_1$ and $\omega\tau_2(t,\omega) \le h_2$, the curve $\overline{C}_{2k}(h_1)$ lies below the curve C_{2k} in parameter space (A_0,A_1) for $\omega h_1 \in J_{2k}$, $k = 0,1,2,\dots$.*

Proof. $A_0(\omega) < \overline{A}_0(\omega,h_1)$ for all $\omega h_1 \in J_k$, since the following partial derivative of $\overline{A}_0(\omega,T_1)$ with respect to T_1

$$\frac{\partial\overline{A}_0}{\partial T_1} = \frac{\omega^2}{\sin^2(\omega T_1)} > 0, \quad \forall\ \omega T_1 \in J_k.$$

Moreover, taking the derivative of $\overline{A}_1(\omega,T_1)$ with respect to T_1, we obtain

$$\frac{\partial A_1}{\partial h} = \frac{-\omega^2\cos(\omega T_1)}{e^{h_2}\sin^2(\omega T_1)}$$

and $\overline{A}_1(\omega,T_1)$ is a monotone decreasing function for $\omega T_1 \in \left(2k\pi, \frac{(2k+1)\pi}{2}\right)$. Therefore, the point $\overline{L}(\overline{A}_0(\omega,h_1),\overline{A}_1(\omega,h_1))$ lies below the point $\widetilde{L}(A_0(\omega),\widetilde{A}_1(\omega,e^{h_2}))$ for $\omega h_1 \in \left(2k\pi, \frac{(2k+1)\pi}{2}\right)$. Because of Lemma 1 and limits (13), $\overline{C}_{2k}(h_1)$ lies below the curve $\widetilde{C}_{2k}(e^{h_2})$ for $\omega h_1 \in J_{2k}$. Hence, $\overline{C}_{2k}(h_1)$ lies below the curve C_{2k} for $\omega h_1 \in J_{2k}$. □

Proposition 4. *If $\tau_1(t,\omega) \le h_1$ and $\omega\tau_2(t,\omega) \le h_2$, the curve $\overline{C}_{2k+1}(h_1)$ lies above the curve C_{2k+1} in parameter space (A_0,A_1) for $\omega h_1 \in J_{2k+1}$, $k = 0,1,2,\dots$.*

Proof. It is similar to the proof of Proposition 3. □

Lemma 3. *If $e^{h_2} \le 1$, the curve $\overline{C}_0(h_1)$ intersects C_* exactly once at $\left(-\frac{1}{h_1}, \frac{1}{e^{h_2}h_1}\right)$ and $\overline{C}_k(h_1)$ do not intersect C_* for $k = 1,2,\ldots$.*

Proof. Intersection of $\overline{C}_0(h_1)$ and C_* is obvious from the following limit point

$$\left(\lim_{\omega\to 0} \overline{A}_0(\omega,h_1), \lim_{\omega\to 0} \overline{A}_1(\omega,h_1)\right) = \left(-\frac{1}{h_1}, \frac{1}{e^{h_2}h_1}\right).$$

Assume that $\overline{C}_k(h_1)$ and C_* has intersection points, then there exist $\omega h_1 \in J_k$ for equations (15) which satisfies equation (6). By using equations (15) in equation (6), we have

$$\frac{\omega\cos(\omega h_1)}{\sin(\omega h_1)} = \frac{\omega}{e^{h_2}\sin(\omega h_1)}$$

which has no solution $\omega h_1 \in J_k$ for $k = 0,1,2,\ldots$ and this contradicts with our assumption. □

Lemma 4. *If $e^{h_2} > 1$, the curves $\overline{C}_k(h_1)$ intersect C_* at point $(\overline{A}_0(\omega_0,h_1), \overline{A}_1(\omega_0,h_1))$ where ω_0 is the root of $\omega_0 = \frac{1}{h_1}\arccos\left(\frac{1}{e^{h_2}}\right)$ such that $\omega_0 h_1 \in J_k$.*

Proof. A straightforward computation shows that the corresponding point $(\overline{A}_0(\omega_0,h_1), \overline{A}_1(\omega_0,h_1))$ lies on the line C_*. □

Lemma 5. *The curve $\overline{C}_k(h_1)$ intersects the line $A_0 = 0$ exactly once. Moreover, the intersection point $(0,P_k)$ satisfies the following inequalities*

$$P_k < P_{k+2}, \quad \textit{for } k = 2n,\ n \in \mathbb{N},$$
$$P_{k+2} < P_k, \quad \textit{for } k = 2n+1,\ n \in \mathbb{N}.$$

Proof. When $\omega h_1 \in J_k$, the equation $\overline{A}_0(\omega,h_1) = 0$ implies $\omega = \frac{\pi+2k\pi}{2h_1}$. Hence,

$$P_k = \begin{cases} \frac{\pi+2k\pi}{2e^{h_2}h_1} & \text{for } k = 2n,\ n \in \mathbb{N} \\ -\frac{\pi+2k\pi}{2e^{h_2}h_1} & \text{for } k = 2n+1,\ n \in \mathbb{N} \end{cases}$$

is obtained by substituting $\omega = \frac{\pi+2k\pi}{2h_1}$ in $\overline{A}_1(\omega,h_1)$. This completes the proof. □

Lemma 6. *The following limits are satisfied for $k = 1,2,\ldots$*

$$\lim_{\omega h_1\to\left(\frac{(2k-1)\pi}{h}\right)^-} \overline{A}_0(\omega,h_1) = \lim_{\omega h_1\to\left(\frac{(2k-1)\pi}{h}\right)^-} \overline{A}_1(\omega,h_1) = \lim_{\omega h_1\to\left(\frac{2k\pi}{h}\right)^-} \overline{A}_0(\omega,h_1)$$
$$= \lim_{\omega h_1\to\left(\frac{2k\pi}{h}\right)^+} \overline{A}_1(\omega,h_1) = +\infty$$

$$\lim_{\omega h_1\to\left(\frac{(2k-1)\pi}{h}\right)^+} \overline{A}_0(\omega,h_1) = \lim_{\omega h_1\to\left(\frac{(2k-1)\pi}{h}\right)^+} \overline{A}_1(\omega,h_1) = \lim_{\omega h_1\to\left(\frac{2k\pi}{h}\right)^+} \overline{A}_0(\omega,h_1)$$
$$= \lim_{\omega h_1\to\left(\frac{2k\pi}{h}\right)^-} \overline{A}_1(\omega,h_1) = -\infty$$

Theorem 1. *Suppose that $\tau_1(t,0) \neq 0$, $\tau_1(t,\omega)$ is an even function, $\tau_2(t,\omega)$ is an odd function with respect to ω. Moreover, $\tau_1(t,\omega) \leq h_1$, $\omega\tau_2(t,\omega) \leq h_2$ where h_1 and h_2 are non-negative real numbers and $e^{h_2} \leq 1$. The solution of equation (1) is asymptotically stable if the following conditions are satisfied:*

(i) $-\frac{1}{h_1} < A_0$

(ii) $-A_0 < A_1 < \frac{\omega}{e^{h_2}\sin(\omega h_1)}$ *where ω is the root of* $A_0 = -\frac{\omega\cos(\omega h_1)}{\sin(\omega h_1)}$ *such that $\omega h_1 \in J_0$.*

Proof. When $A_0 > 0$ and $A_1 = 0$, the solution of equation (1) is clearly asymptotically stable. The stability region which includes half line $A_0 > 0$ and $A_1 = 0$, lies above C_* and below $\overline{C}_0(h_1)$ as a result of Proposition 3, Proposition 4, Lemma 3, Lemma 5 and Lemma 6. The conditions (i)-(ii) are algebraic representation of this region in parameter space (A_0, A_1). □

Theorem 2. *Suppose that $\tau_1(t,0) \neq 0$, $\tau_1(t,\omega)$ is an even function and $\tau_2(t,\omega)$ is an odd function with respect to ω. Moreover, $\tau_1(t,\omega) \leq h_1$, $\omega\tau_2(t,\omega) \leq h_2$ where h_1 and h_2 are non-negative real numbers and $e^{h_2} > 1$. The solution of equation (1) is asymptotically stable if the following conditions are satisfied:*

(iii) $-\frac{1}{h_1} < A_0$ *or* $\overline{A}_0(\omega, h_1) < A_0$ *where ω is the root of* $\omega = \frac{1}{h_1}\arccos\left(\frac{1}{e^{h_2}}\right)$ *such that $\omega h_1 \in J_0$*

(iv) $-A_0 < A_1 < \frac{\omega}{e^{h_2}\sin(\omega h_1)}$ *where ω is the root of* $A_0 = -\frac{\omega\cos(\omega h_1)}{\sin(\omega h_1)}$ *such that $\omega h_1 \in J_0$*

(v) $\frac{\omega}{e^{h_2}\sin(\omega h_1)} < A_1$ *where ω is the root of* $A_0 = -\frac{\omega\cos(\omega h_1)}{\sin(\omega h_1)}$ *such that $\omega h_1 \in J_1$.*

Proof. The stability region which includes half line $A_0 > 0$ and $A_1 = 0$, lies among C_*, $\overline{C}_1(h_1)$ and $\overline{C}_0(h_1)$ because of Proposition 3, Proposition 4, Lemma 4, Lemma 5 and Lemma 6. The conditions (iii), (iv) and (v) are algebraic representation of this region in parameter space (A_0, A_1). □

Theorem 3. *Suppose that $\tau_1(t,0) \neq 0$, $\tau_1(t,\omega)$ is an even function and $\tau_2(t,\omega)$ is an odd function with respect to ω and $\omega\tau_2(t,\omega) \leq h_2$ where h_2 is a non-negative real number. The solution of equation (1) is asymptotically stable, if the following condition is satisfied:*

(vi) $A_0 \leq \left|A_1 e^{h_2}\right|$.

Proof. It is obvious from (7) that, $A_0 \leq \left|A_1 e^{h_2}\right|$ for all $\omega \in J_k$. Therefore there is no D-curve in the region described by (vi). Moreover, the half line $A_0 > 0$ and $A_1 = 0$ on which the solution equation (1) is asymptotically stable, is in this region. □

Theorem 4. *Suppose that $A_1 \neq 0$, $\tau_1(t,0) \neq 0$, $\tau_1(t,\omega)$ is an even function and $\tau_2(t,\omega)$ is an odd function with respect to ω. If $\omega\tau_2(t,\omega)$ does not have an upper bound, then the solution of equation (1) is not stable.*

Proof. It follows from (10) that we have

$$\lim_{\omega\tau_2(t,\omega)\to\infty} A_1(\omega) = \lim_{\omega\tau_2(t,\omega)\to\infty} \frac{\omega}{e^{\omega\tau_2(t,\omega)}\sin(\omega\tau_1(t,\omega))} = 0.$$

Thus, D-curves tend to half line $A_0 > 0$ and $A_1 = 0$, when ω ranges from 0 to ∞. □

3. The Stability of Equation (1) with Delay Term $\tau(t,u(t)) = \frac{au(t)+b}{cu(t)+d}$

We consider the stability of equation (1) with delay term $\tau(t,u(t)) = \frac{au(t)+b}{cu(t)+d}$ where a, b, c and d are positive real numbers.

The Möbius transformation can be rewritten as follows

$$\tau(z) = \frac{az+b}{cz+d} = \frac{bc-ad}{c}\frac{1}{cz+d} + \frac{a}{c}$$

where $z \in \mathbb{C}$. Hence,

$$\tau_1(t,\omega) = \frac{a}{c}, \quad \tau_2(t,\omega) = 0$$

when $bc - ad = 0$. It follows from Theorem 1 that if $bc - ad = 0$ and the following conditions are satisfied:

(i) $-\frac{c}{a} < A_0$

(ii) $-A_0 < A_1 < \frac{\omega}{\sin\left(\omega\frac{a}{c}\right)}$ where ω is the root of $A_0 = -\frac{\omega\cos\left(\frac{\omega a}{c}\right)}{\sin\left(\frac{\omega a}{c}\right)}$ such that $\frac{\omega a}{c} \in J_0$

then the solution of equation (1) with delay term $\tau(t,u(t)) = \frac{au(t)+b}{cu(t)+d}$ is asymptotically stable.

4. Conclusion

In this study, stability conditions are given in terms of the coefficients in equation (1) under some conditions on the delay function $\tau(t,u(t))$ in Theorem 1, Theorem 2 and Theorem 3. Moreover, it is proved that the condition $\tau_2(t,\omega) \le h_2$ given in Theorem 4, is the necessary condition for the stability of the solution.

In literature, state dependent delays are linearized heuristically by freezing at a constant solution in order to investigate the stability of SDDE. Heuristic linearization is applied taking $\tau(t,u(t)) = \tau(t,0)$ in equation (1) in case of zero solution. It is show that, the stability analysis under the condition $bc - ad = 0$ is the same as the one which is obtained by linearization at zero in Section 3.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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