**Communications in Mathematics and Applications**
Vol. 7, No. 2, pp. 93–103, 2016
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com

# Adjointations of Operator Inequalities and Characterizations of Operator Monotonicity via Operator Means

Research Article

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**Abstract.** We propose adjointations between operator orderings, which convert any operator inequalities/identities associated with certain binary operations to new ones. Then we prove that a continuous function $f:(0,\infty)\to(0,\infty)$ is operator monotone increasing if and only if $f(A\,!_t\,B)\leqslant f(A)\,!_t\,f(B)$ for any positive operators $A,B$ and scalar $t\in[0,1]$. Here, $!_t$ denotes the $t$-weighted harmonic mean. As a counterpart, $f$ is operator monotone decreasing if and only if the reverse of preceding inequality holds. Moreover, we obtain many characterizations of operator monotone increasingness/decreasingness in terms of operator means. These characterizations lead to many operator inequalities involving means

**Keywords.** Operator mean; Operator monotone function; Operator inequality

**MSC.** 15A45; 47A63; 47A64

**Received:** March 15, 2016 **Accepted:** August 7, 2016



## 1. Introduction

Let $\mathscr{B}(\mathbb{H})$ be the algebra of bounded linear operators on a complex Hilbert space $\mathbb{H}$. The cone of positive operators on $\mathbb{H}$ is written by $\mathscr{B}(\mathbb{H})^+$. For selfadjoint operators $A,B\in\mathscr{B}(\mathbb{H})$, the partial order $A\leqslant B$ means that $B-A\in\mathscr{B}(\mathbb{H})^+$, while the strict order $A<B$ indicates that $B-A$ is an invertible positive operator.

Operator monotone functions, introduced by Löwner in a seminal paper [11], are continuous real-valued functions which preserve operator ordering. More precisely, a continuous real-valued function $f$ defined on an interval $\Omega$ is said to be *operator monotone* (*increasing*) if the condition

$$A \leqslant B \implies f(A) \leqslant f(B) \tag{1.1}$$

holds for all operators $A, B \in \mathscr{B}(\mathbb{H})$ whose spectra contained in $\Omega$ and for all Hilbert spaces $\mathbb{H}$. If the reverse inequality in the right hand side of (1.1) holds, then we say that $f$ is *operator monotone decreasing*. It is well known that (see e.g. [7, Example 2.5.9]) the function $f(x) = x^\alpha$ is operator monotone on $[0,\infty)$ if and only if $\alpha \in [0,1]$, and it is operator monotone decreasing if and only if $\alpha \in [-1,0]$. The function $x \mapsto \log(x+1)$ is operator monotone on $(0,\infty)$.

Operator monotony arises naturally in matrix/operator inequalities (e.g. [1, 12]), especially in the so-called Kubo-Ando theory of operator means (e.g. [10]). Characterizations of operator monotonicity in the aspects of differential analysis, Pick functions, integral representations and operator inequalities are provided in [6], [7, Section 2], [8, Chapter 4] and [9]. A closely related concept to operator monotonicity is the concept of operator concavity/convexity. A continuous function $f : (0,\infty) \to \mathbb{R}$ is said to be *operator concave* if

$$f((1-t)A + tB) \geqslant (1-t)f(A) + tf(B) \tag{1.2}$$

for any $A, B > 0$ and $t \in [0,1]$. The continuity of $f$ implies that $f$ is operator concave if and only if $f$ is operator midpoint-concave, in the sense that the condition (1.2) holds for $t = 1/2$. The reverse inequality of (1.2) is equivalent to the *operator convexity* of $f$. It is a fundamental fact that a continuous function $f : (0,\infty) \to (0,\infty)$ is operator monotone if and only if it is operator concave ( [6]).

In the present paper, there are two main objectives. The first one is to propose "adjointation" between operator orderings. These adjointations convert any operator inequalities/identities related to binary operations to new ones. Secondly, we use such adjointations to establish the relationship between the operator monotonicity of functions and operator means. See some related discussions in [2, 3]. Note that the condition (1.2) can be restated in terms of weighted arithmetic means $\triangledown_t$ as follows:

$$f(A \triangledown_t B) \geqslant f(A) \triangledown_t f(B) \tag{1.3}$$

for any $A, B > 0$ and $t \in [0,1]$. We prove an interesting fact about operator monotone functions:

$$f(A \,!_t\, B) \leqslant f(A) \,!_t\, f(B), \quad A, B > 0 \text{ and } t \in [0,1]. \tag{1.4}$$

Here, the symbol $!_t$ stands for the $t$-weighted harmonic mean. Conversely, the above property characterizes the operator monotonicity of $f$. Moreover, $f$ is operator monotone decreasing if and only if the reverse inequality of (1.4) holds. Many characterizations of operator monotone increasing/decreasing functions in this type are established.

In the next section, we provide some preliminaries about operator means. Adjointations of operator orderings are discussed in Section 3. Characterizations of operator monotone increasing/decreasing functions in terms of operator means are established in Section 4. In Section 5, we derive some interesting inequalities concerning operator means by using the previous results.

## 2. Preliminaries on Operator Means

In this section, we review Kubo-Ando theory of operator means (see e.g. [7, Section 3] and [8, Chapter 5]).

An *operator) connection* is a binary operation $\sigma$ on $\mathscr{B}(\mathbb{H})^+$ such that for all positive operators $A, B, C, D$:

(M1) (*joint*) *monotonicity*: $A \leqslant C,\, B \leqslant D \implies A\,\sigma B \leqslant C\,\sigma D$

(M2) *transformer inequality*: $C(A\,\sigma B)C \leqslant (CAC)\,\sigma\,(CBC)$

(M3) (*joint*) *continuity from above*: for $A_n, B_n \in \mathscr{B}(\mathbb{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\,\sigma B_n \downarrow A\,\sigma B$. Here, $X_n \downarrow X$ indicates that $(X_n)$ is a decreasing sequence converging strongly to $X$.

Using (M2), every operator connection $\sigma$ is congruent invariant in the sense that

$$C(A\,\sigma B)C = (CAC)\,\sigma\,(CBC) \tag{2.1}$$

for any $A \geqslant 0, B \geqslant 0$ and $C > 0$.

An *(operator) mean* is an operator connection $\sigma$ with property that $A\,\sigma A = A$ for all $A \geqslant 0$. Classical examples of operator means are the *arithmetic mean* (AM), the *harmonic mean* (HM) and the *geometric mean* (GM). Their weighted versions of AM, HM and GM are defined respectively as follows:

$$\begin{aligned} A \triangledown_t B &= (1-t)A + tB \\ A\,!_t B &= \left[(1-t)A^{-1} + tB^{-1}\right]^{-1} \\ A\,\#_t B &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}. \end{aligned}$$

For simplicity, we abbreviate $\triangledown = \triangledown_{\frac{1}{2}}$, $! = !_{\frac{1}{2}}$ and $\# = \#_{\frac{1}{2}}$.

A major core of Kubo-Ando theory is the one-to-one correspondence between operator connections and operator monotone functions:

**Theorem 1** ( [10, Theorem 3.4])**.** *Given an operator connection $\sigma$, there is a unique operator monotone function $f : [0,\infty) \to [0,\infty)$ such that*

$$f(A) = I\,\sigma A, \quad A \geqslant 0. \tag{2.2}$$

*In fact, the map $\sigma \mapsto f$ is a bijection.*

Theorem 1 serves a simple proof of operator versions of the weighted AM-GM-HM inequalities:

**Proposition 1** (see e.g. [7, Proposition 3.3.2])**.** *For each $A, B \geqslant 0$ and $t \in [0,1]$, we have*

$$A \triangledown_t B \geqslant A\,!_t B \geqslant A\,\#_t B. \tag{2.3}$$

## 3. Adjointations of Operator Orderings Associated with Certain Binary Operations

In this section, we discuss adjointations of operator inequalities/identities associated with binary operations for positive operators.

Let $\sigma$ be a binary operation assigned to each pair of positive operators such that $A\,\sigma B > 0$ for any $A, B > 0$. It is a fact that every nonzero operator connection fulfills this property (see e.g. [5]). We define the *adjoint* of $\sigma$ to be the binary operation

$$\sigma^* : (A,B) \mapsto (A^{-1}\sigma B^{-1})^{-1}.$$

A binary operation $\sigma$ is said to be *symmetric* if $A\,\sigma B = B\,\sigma A$ for all $A, B \geqslant 0$. For a function $f : (0,\infty) \to (0,\infty)$, we define the *adjoint* of $f$ by

$$f^*(x) = \frac{1}{f(1/x)}, \quad x > 0.$$

If a nonzero connection $\sigma$ is associated with an operator monotone function $f$, then $\sigma^*$ is associated with $f^*$ (which is also operator monotone).

Let us start with a simple observation about operator inequalities. Its proof is straightforward and therefore omitted.

**Proposition 2.** *Let $f : (0,\infty) \to (0,\infty)$. Then the following statements are equivalent:*

(1) $f(A) \geqslant f(B)$ *for all* $A \geqslant B > 0$ ;

(2) $f^*(A) \geqslant f^*(B)$ *for all* $A \geqslant B > 0$.

*Similarly, the following statements are equivalent:*

(i) $f(A) \leqslant f(B)$ *for all* $A \geqslant B > 0$ ;

(ii) $f^*(A) \leqslant f^*(B)$ *for all* $A \geqslant B > 0$.

The following result shows that certain operator inequalities are adjointable.

**Theorem 2.** *Let $\sigma$ and $\eta$ be binary operations for invertible positive operators. Let $f, g, h :$ $(0,\infty) \to (0,\infty)$ be continuous functions. Then the following statements are equivalent:*

(1) $f(A\,\sigma B) \leqslant g(A)\,\eta\, h(B)$ *for all* $A, B > 0$ ;

(2) $f^*(A\,\sigma^* B) \geqslant g^*(A)\,\eta^*\, h^*(B)$ *for all* $A, B > 0$.

*Similarly, the following statements are equivalent:*

(i) $f(A\,\sigma B) \geqslant g(A)\,\eta\, h(B)$ *for all* $A, B > 0$ ;

(ii) $f^*(A\,\sigma^* B) \leqslant g^*(A)\,\eta^*\, h^*(B)$ *for all* $A, B > 0$.

*Proof.* Assume (1) and consider $A, B > 0$. By definition of $f^*$ and the operator-monotone decreasingness of the map $t \mapsto t^{-1}$, we have

$$f^*(A\,\sigma^* B) = f^*\left((A^{-1}\sigma B^{-1})^{-1}\right)$$

$$
\begin{aligned}
&= \left[f(A^{-1}\sigma B^{-1})\right]^{-1} \\
&\geqslant \left[g(A^{-1})\eta h(B^{-1})\right]^{-1} \\
&= \left[g^*(A)^{-1}\eta h^*(B)^{-1}\right]^{-1} \\
&= g^*(A)\eta^* h^*(B).
\end{aligned}
$$

To prove (2)$\Longrightarrow$(1), apply (1)$\Longrightarrow$(2) to continuous functions $f^*, g^*, h^*$ and binary operations $\sigma^*, \eta^*$. □

The next corollary is immediate.

**Corollary 1.** *Under the hypothesis of Theorem 2, the following statements are equivalent:*

(1) $f(A\sigma B) = g(A)\eta h(B)$ *for all* $A,B > 0$ ;

(2) $f^*(A\sigma^* B) = g^*(A)\eta^* h^*(B)$ *for all* $A,B > 0$.

**Theorem 3.** *Let* $\sigma$ *and* $\eta$ *be nonzero connections. Then the following statements are equivalent:*

(1) $A\sigma(B\eta C) \leqslant (A\sigma B)\eta(A\sigma C)$ *for all* $A,B,C \geqslant 0$ ;

(2) $A\sigma^*(B\eta^* C) \geqslant (A\sigma^* B)\eta^*(A\sigma^* C)$ *for all* $A,B,C \geqslant 0$.

*Proof.* By Theorem 1, there is an operator monotone function $f : [0,\infty) \to [0,\infty)$ such that $f(A) = I\sigma A$ for any $A \geqslant 0$. Since $\sigma \neq 0$, we have $f(x) > 0$ for any $x > 0$ (see e.g. [5]).

Assume (1). First, consider $A,B,C > 0$. Then

$$f(B\eta C) = I\sigma(B\eta C) \leqslant (I\sigma B)\eta(I\sigma C) = f(B)\eta f(C).$$

By Theorem 2, we have

$$I\sigma^*(B\eta^* C) = f^*(B\eta^* C) \geqslant f^*(B)\eta^* f^*(C) = (I\sigma^* B)\eta^*(I\sigma^* C).$$

Replacing $B$ and $C$ with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ respectively yields

$$I\sigma^*(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\eta^* A^{-\frac{1}{2}}CA^{-\frac{1}{2}}) \geqslant (I\sigma^* A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\eta^*(I\sigma^* A^{-\frac{1}{2}}CA^{-\frac{1}{2}}).$$

It follows from the congruent invariance (2.1) that

$$
\begin{aligned}
A\sigma^*(B\eta^* C) &= A^{\frac{1}{2}}[I\sigma^*(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\eta^* A^{-\frac{1}{2}}CA^{-\frac{1}{2}})]A^{\frac{1}{2}} \\
&\geqslant A^{\frac{1}{2}}[(I\sigma^*(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\eta^*(I\sigma^* A^{-\frac{1}{2}}CA^{-\frac{1}{2}})]A^{\frac{1}{2}} \\
&= (A\sigma^* B)\eta^*(A\sigma^* C).
\end{aligned}
$$

For general $A,B,C \geqslant 0$, perturb these operators with $\epsilon I$ where $\epsilon > 0$ and then use the monotonicity (M1) and the continuity from above (M3).

The proof of (2)$\Longrightarrow$(1) is similar to the previous one and therefore omitted. □

**Corollary 2.** *Let $\sigma$ be a nonzero connection. Then the following statements are equivalent:*

(1) $A\sigma(B\eta C) \leqslant (A\sigma B)\eta(A\sigma C)$ *for all* $A,B,C \geqslant 0$ *and for all nonzero connections* $\eta$ ;

(2) $A\sigma^*(B\eta C) \geqslant (A\sigma^* B)\eta(A\sigma^* C)$ *for all* $A,B,C \geqslant 0$ *and for all nonzero connections* $\eta$.

*Proof.* It follows from Theorem 3 together with the fact that $(\eta^*)^* = \eta$. □

**Corollary 3.** *Let $\eta$ be a nonzero connection. Then the following statements are equivalent:*

(1) $A\sigma(B\eta C) \leqslant (A\sigma B)\eta(A\sigma C)$ *for all* $A,B,C \geqslant 0$ *and for all nonzero connections* $\sigma$ ;

(2) $A\sigma(B\eta^* C) \geqslant (A\sigma B)\eta^*(A\sigma C)$ *for all* $A,B,C \geqslant 0$ *and for all nonzero connections* $\sigma$.

*Proof.* Use Theorem 3 and the fact that $(\sigma^*)^* = \sigma$. □

**Remark 1.** The "equality parts" in Theorem 3, Corollary 2 and Corollary 3 also hold. For example, the following statements are equivalent for nonzero connections $\sigma$ and $\eta$:

(1) $A\sigma(B\eta C) = (A\sigma B)\eta(A\sigma C)$ for all $A,B,C \geqslant 0$ ;

(2) $A\sigma^*(B\eta^* C) = (A\sigma^* B)\eta^*(A\sigma^* C)$ for all $A,B,C \geqslant 0$.

## 4. Characterizations of Operator Monotonicity via Operator Means

In this section, we characterize operator monotone increasing/decreasing functions in terms of operator means. Recall the following result.

**Proposition 3** ([2, Theorem 2.3])**.** *Let $f:(0,\infty)\to(0,\infty)$ be a continuous function. The following statements are equivalent:*

(i) $f$ *is operator monotone* ;

(ii) $f(A\triangledown B) \geqslant f(A)\# f(B)$ *for all* $A,B>0$ ;

(iii) $f(A\triangledown B) \geqslant f(A)\sigma f(B)$ *for all* $A,B>0$ *and for all symmetric means* $\sigma$ ;

(iv) $f(A\triangledown B) \geqslant f(A)\sigma f(B)$ *for all* $A,B>0$ *and for some symmetric mean* $\sigma \neq !$.

The next Theorem further characterizes operator monotonicity.

**Theorem 4.** *Let $f:(0,\infty)\to(0,\infty)$ be a continuous function. Then the following statements are equivalent:*

(I1) $f$ *is operator monotone increasing (i.e.* $f(A\triangledown_t B) \geqslant f(A)\triangledown_t f(B)$ *for all* $A,B>0$ *and for all* $t\in[0,1]$*)* ;

(I2) $f(A\,!\,B) \leqslant f(A)\# f(B)$ *for all* $A,B>0$ ;

(I3) $f(A\,!_t\,B) \leqslant f(A)\#_t f(B)$ *for all* $A,B>0$ *and for all* $t\in[0,1]$ ;

(I4) $f(A\,!\,B) \leqslant f(A)\,!\,f(B)$ *for all* $A,B>0$ ;

(I5) $f(A\,!_t\,B) \leqslant f(A)\,!_t\,f(B)$ *for all* $A,B>0$ *and for all* $t\in[0,1]$ ;

(I6) $f(A\,!\,B) \leqslant f(A)\sigma f(B)$ *for all* $A,B>0$ *and for all symmetric means* $\sigma$ ;

(I7) $f(A\,!\,B) \leqslant f(A)\sigma f(B)$ *for all* $A,B>0$ *and for some symmetric mean* $\sigma \neq \triangledown$.

*Proof.* (I1)$\Longleftrightarrow$(I2). The operator monotonicity of $f$ and $f^*$ are equivalent by Proposition 2. By the equivalent (i)$\Longleftrightarrow$(ii) in Proposition 3, the operator monotonicity of $f^*$ reads

$$f^*(A \triangledown B) \geqslant f^*(A) \# f^*(B) \quad \text{for all } A,B > 0.$$

Theorem 2 says that this condition is equivalent to $f(A\,!\,B) \leqslant f(A) \# f(B)$ for all $A,B > 0$ since $\triangledown^* = !$ and $\#^* = \#$.

(I1)$\Longrightarrow$(I3). Assume that $f$ is operator monotone. Then so is $f^*$ by Proposition 2. The operator monotonicity of $f^*$ assures that for any $A,B > 0$ and $t \in [0,1]$

$$f^*(A \triangledown_t B) \geqslant f^*(A) \triangledown_t f^*(B).$$

Due to the facts that $\triangledown_t^* = !_t$ and $\#_t^* = \#_t$, Theorem 2 converts the above condition to (I3).

(I3)$\Longrightarrow$(I2). It is clear.

(I1)$\Longleftrightarrow$(I4). Note that the operator monotonicity of $f$ and $f^*$ are equivalent by Proposition 2. Since $f$ is continuous, the operator monotonicity of $f^*$ can be expressed as

$$f^*(A \triangledown B) \geqslant f^*(A) \triangledown f^*(B) \quad \text{for all } A,B > 0.$$

Theorem 2 asserts that this condition is equivalent to $f(A\,!\,B) \leqslant f(A)\,!\,f(B)$ for all $A,B > 0$.

(I1)$\Longleftrightarrow$(I5). The proof is similar to that of (I1)$\Longleftrightarrow$(I4).

(I1)$\Longleftrightarrow$(I6). By using (i)$\Longleftrightarrow$(iii) in Proposition 3, the operator monotonicity of $f$ (hence, of $f^*$) is equivalent to the condition that

$$f^*(A \triangledown B) \geqslant f^*(A) \sigma f^*(B)$$

for $A,B > 0$ and for all symmetric means $\sigma$. By Theorem 2, this condition is then equivalent to the following:

$$f(A\,!\,B) \leqslant f(A) \sigma^* f(B)$$

for all symmetric means $\sigma$. Since the map $\sigma \mapsto \sigma^*$ is bijective on the set of symmetric means, we arrive at (I1)$\Longleftrightarrow$(I6).

(I1)$\Longleftrightarrow$(I7). The proof is similar to that of (I1)$\Longleftrightarrow$(I6). Here, we use (i)$\Longleftrightarrow$(iv) in Proposition 3 and the fact that $!^* = \triangledown$. □

Next, we turn to operator monotone decreasingness. Recall the following result:

**Proposition 4** ([2, Theorems 2.1 and 3.1]). *Let $f : (0,\infty) \to (0,\infty)$ be a continuous function. The following statements are equivalent:*

(i) *$f$ is operator monotone decreasing ;*

(ii) *$f(A \triangledown B) \leqslant f(A) \# f(B)$ for all $A,B > 0$ ;*

(iii) *$f(A \triangledown B) \leqslant f(A) \sigma f(B)$ for all $A,B > 0$ and for all symmetric means $\sigma$ ;*

(iv) $f(A \triangledown B) \leqslant f(A) \sigma f(B)$ *for all* $A, B > 0$ *and for some symmetric mean* $\sigma \neq \triangledown$.

(v) $f$ *is operator convex and* $f$ *is decreasing.*

The next Theorem is a counterpart of Theorem 4.

**Theorem 5.** *Let* $f : (0,\infty) \to (0,\infty)$ *be a continuous function. Then the following statements are equivalent:*

(D1) $f$ *is operator monotone decreasing* ;

(D2) $f(A \,!\, B) \geqslant f(A) \# f(B)$ *for all* $A, B > 0$ ;

(D3) $f(A \,!_t\, B) \geqslant f(A) \#_t f(B)$ *for all* $A, B > 0$ *and for all* $t \in [0,1]$ ;

(D4) $f(A \,!\, B) \geqslant f(A) \,!\, f(B)$ *for all* $A, B > 0$ *and* $f$ *is decreasing* ;

(D5) $f(A \,!_t\, B) \geqslant f(A) \,!_t\, f(B)$ *for all* $A, B > 0$ *and for all* $t \in [0,1]$ *and* $f$ *is decreasing* ;

(D6) $f(A \,!\, B) \geqslant f(A) \sigma f(B)$ *for all* $A, B > 0$ *and for all symmetric means* $\sigma$ ;

(D7) $f(A \,!\, B) \geqslant f(A) \sigma f(B)$ *for all* $A, B > 0$ *and for some symmetric mean* $\sigma \neq \,!$.

*Proof.* It is clear that (D3)$\Longrightarrow$(D2), (D5)$\Longrightarrow$(D4) and (D6)$\Longrightarrow$(D7).

(D1)$\Longleftrightarrow$(D2). Note that the operator-monotone decreasingness of $f$ and $f^*$ are equivalent by Proposition 2. According to the equivalent (i)$\Longleftrightarrow$(ii) in Proposition 4, the fact that $f^*$ is operator monotone decreasing reads

$$f^*(A \triangledown B) \leqslant f^*(A) \# f^*(B)$$

for all $A, B > 0$. Theorem 2 states that this condition is equivalent to $f(A \,!\, B) \geqslant f(A) \# f(B)$ for all $A, B > 0$.

(D1)$\Longrightarrow$(D3). Assume that $f$ is operator monotone decreasing. Then so is $f^*$ by Proposition 2. Consider $A, B > 0$ and $t \in [0,1]$. Recall the weighted AM-GM inequality for operators (Proposition 1):

$$A \triangledown_t B \geqslant A \#_t B.$$

It follows that $f^*(A \triangledown_t B) \leqslant f^*(A \#_t B)$. By Theorem 2, we have $f(A \,!_t\, B) \geqslant f(A \#_t B)$.

(D1)$\Longrightarrow$(D4). Assume that $f$ is operator monotone decreasing. By (D2) and the GM-HM inequality for operators (Proposition 1), we have

$$f(A \,!\, B) \geqslant f(A) \# f(B) \geqslant f(A) \,!\, f(B)$$

for all $A, B > 0$. It is trivial that $f$ is decreasing in usual sense.

(D4)$\Longrightarrow$(D1). Suppose that $f(A \,!\, B) \geqslant f(A) \,!\, f(B)$ for all $A, B > 0$ and $f$ is decreasing. Then $f^*$ is also a decreasing function. By Theorem 2, we have $f^*(A \triangledown B) \leqslant f^*(A) \triangledown f^*(B)$ for all $A, B > 0$, i.e. $f^*$ is operator convex. The implication (v)$\Longrightarrow$(i) in Proposition 4 tells us that $f^*$ is operator monotone decreasing. Hence, so is $f$ by Proposition 2.

(D1)$\Longrightarrow$(D5). The proof is similar to that of (D1)$\Longrightarrow$(D4). Here, we use the weighted GM-HM inequality (2.3):

$$A \#_t B \geqslant A \,!_t B$$

for any $A,B > 0$ and $t \in [0,1]$.

(D1)$\Longrightarrow$(D6). Assume that $f$ is operator monotone decreasing. Then so is $f^*$ by Proposition 2. By applying the equivalence (i)$\Longleftrightarrow$(iii) in Proposition 4 to $f^*$, we obtain that $f^*(A \triangledown B) \leqslant f^*(A) \sigma f^*(B)$ for all $A,B > 0$ and for all symmetric means $\sigma$. Now, use Theorem 2.

(D7)$\Longrightarrow$(D1). This is a combination of Theorem 2, the equivalence (i)$\Longleftrightarrow$(iv) in Proposition 4 and Proposition 2. □

## 5. Applications to Operator Inequalities involving Means

Let us derive operator inequalities involving operator means by making use of the previous results.

**Corollary 4.** *For each $A,B \geqslant 0$ and $\alpha, t \in [0,1]$, we have*

$$(A \,!_t B)^\alpha \leqslant A^\alpha \,!_t B^\alpha.$$

*Proof.* By applying Theorem 4 (I1)$\Longrightarrow$(I5) to the operator monotone function $f(x) = x^\alpha$, we get

$$(A \,!_t B)^\alpha \leqslant A^\alpha \,!_t B^\alpha$$

for any $A,B > 0$. For general $A,B \geqslant 0$, consider $A + \epsilon I$, $B + \epsilon I > 0$ for $\epsilon > 0$. By the monotonicity (M1) and the continuity from above (M3) of weighted harmonic means,

$$\begin{aligned}
(A \,!_t B)^\alpha &= \left[\lim_{\epsilon \downarrow 0} (A+\epsilon I) !_t (B+\epsilon I)\right]^\alpha \\
&= \lim_{\epsilon \downarrow 0} \, [(A+\epsilon I) !_t (B+\epsilon I)]^\alpha \\
&\leqslant \lim_{\epsilon \downarrow 0} (A+\epsilon I)^\alpha \,!_t (B+\epsilon I)^\alpha \\
&= A^\alpha \,!_t B^\alpha.
\end{aligned}$$

□

**Corollary 5.** *For each $A,B \geqslant 0$ and $t \in [0,1]$, we have*

$$\log((A \,!_t B) + I) \leqslant \log(A+I) \,!_t \log(B+I). \tag{5.1}$$

*Proof.* First, we prove the inequality (5.1) for $A,B > 0$. This is done by applying Theorem 4 (I1)$\Longrightarrow$(I5) to the operator monotone function $f(x) = \log(x+1)$. For general $A,B \geqslant 0$, use the continuity argument as the proof of Corollary 4. □

**Corollary 6.** *Let $\sigma$ be an operator connection. Then for any $A,B,C \geqslant 0$ and $t \in [0,1]$, we have*

$$A \sigma (B \,!_t C) \leqslant (A \sigma B) !_t (A \sigma C) \tag{5.2}$$

$$A \sigma (B \triangledown_t C) \geqslant (A \sigma B) \triangledown_t (A \sigma C). \tag{5.3}$$

*Proof.* Suppose that $\sigma$ is nonzero. Consider an operator monotone function $f:(0,\infty)\to(0,\infty)$ associated with $\sigma$. The property (2.2) and Theorem 4 (I1)$\Longrightarrow$(I5) together imply that for each $A,B>0$,

$$I\,\sigma\,(A\,!_t\,B)=f(A\,!_t\,B)\leqslant f(A)\,!_t\,f(B)=(I\,\sigma\,A)\,!_t\,(I\,\sigma\,B).$$

It follows from (2.1) that, for $A,B,C>0$,

$$\begin{aligned} A\,\sigma\,(B\,!_t\,C) &= A^{\frac{1}{2}}\left[I\,\sigma\,(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\,!_t\,A^{-\frac{1}{2}}CA^{-\frac{1}{2}})\right]A^{\frac{1}{2}} \\ &\leqslant A^{\frac{1}{2}}\left[(I\,\sigma\,A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\,!_t\,(I\,\sigma\,A^{-\frac{1}{2}}CA^{-\frac{1}{2}})\right]A^{\frac{1}{2}} \\ &= (A\,\sigma\,B)\,!_t\,(A\,\sigma\,C). \end{aligned}$$

For general $A,B,C\geqslant 0$, use a continuity argument as in the proof of Theorem 3. To prove the inequality (5.3), apply Corollary 3 to the inequality (5.2). □

**Corollary 7.** *For each $A,B\geqslant 0$, $r\in[-1,0]$ and $t\in[0,1]$, we have*

$$(A\,!_t\,B)^r\geqslant A^r\,\#_t\,B^r\geqslant A^r\,!_t\,B^r.$$

*Proof.* By continuity, we may assume that $A,B>0$. The left-hand inequality is obtained by applying Theorem 5 (D1)$\Longrightarrow$(D3) to the operator monotone decreasing function $f(x)=x^r$. Another inequality comes from the weighted GM-HM inequality (2.3). □

Our final result is a symmetric counterpart of Corollary 7.

**Corollary 8.** *Let $\sigma$ be a symmetric operator mean. For each $A,B\geqslant 0$ and $r\in[-1,0]$, we have*

$$(A\,!\,B)^r\geqslant A^r\,\sigma\,B^r.$$

*Proof.* We may assume that $A,B>0$. The proof is done by applying Theorem 5 (D1)$\Longrightarrow$(D6) to the function $f(x)=x^r$. □

## 6. Conclusion

We show that if we have operator inequalities/identities associated with certain binary operations for positive operators, we can produce new ones by adjointations. This technique leads to many characterizations of operator monotonicity in terms of operator means. These characterizations yield many operator inequalities involving means.

## Acknowledgement

This research was supported by King Mongkut's Institute of Technology Ladkrabang Research Fund grant no. KREF045710).

## Competing Interests

The author declares that he has no competing interests.

## Authors' Contributions

The author wrote, read and approved the final manuscript.

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