Communications in Mathematics and Applications
Vol. 8, No. 3, pp. 333–343, 2017
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

Research Article

On a Generalized Zero-divisor Graph of a Commutative Ring with Respect to an Ideal

Priyanka Pratim Baruah[1,*] and Kuntala Patra[2]

[1]Department of Mathematics, Girijananda Chowdhury Institute of Management and Technology, Guwahati 781017, India
[2]Department of Mathematics, Gauhati University, Guwahati 781014, India
***Corresponding author:** pratimbaruah6@rediffmail.com

Abstract. In this paper, we generalize the notion of the ideal-based zero-divisor graph of a commutative ring. Let R be a commutative ring and let I be an ideal of R. Here, we define a generalized zero-divisor graph of R with respect to I and denote this graph by $\Gamma_I^G(R)$. We show that $\Gamma_I^G(R)$ is connected with diameter at most three. If $\Gamma_I^G(R)$ has a cycle, we show that the girth of $\Gamma_I^G(R)$ is at most four. Also, we investigate the existence of cut vertices of $\Gamma_I^G(R)$. Moreover, we examine certain situations when $\Gamma_I^G(R)$ is a complete bipartite graph.

Keywords. Commutative ring; Ideal; Generalized zero-divisor graph; Diameter; Girth

MSC. Primary 13A15; Secondary 05C25, 05C38, 05C40

Received: January 22, 2016 **Accepted:** July 28, 2016

1. Introduction

Let R be a commutative ring with unity, and $Z(R)$ be its set of all zero-divisors. The concept of a zero-divisor graph of a commutative ring R was first introduced by Beck [5], where all the elements of the ring R were taken as the vertices of the graph. Anderson and Livingston [3] modified this concept by taking the zero-divisor graph $\Gamma(R)$ whose vertices are the non zero zero-divisors of a commutative ring R and two distinct vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graph of a commutative ring has been studied extensively by several authors, e.g. ([1], [2], [4]). Redmond [11] introduced the concept of ideal-based zero-divisor graph of a commutative ring R and he proved some interesting results of this graph. Then the study of ideal-based zero-divisor graph is carried on by Maimani, Pouranki and Yassemi in [10]. Later, Dheena and Elavarsaran studied ideal-based zero-divisor graph of a near ring in [6] and [7]. In this paper, we study a generalized zero-divisor graph of a commutative ring with respect to an ideal.

For the sake of completeness, we state some definitions and notations used throughout this paper. Let R be a commutative ring and I be an ideal of R. For $a \in R$, $\langle a \rangle$ is the ideal of R generated by a. An ideal $I \neq R$ is called a prime ideal if $ab \in I$ implies that $a \in I$ or $b \in I$. An ideal $I \neq R$ is called a semi prime ideal if $a^2 \in I$ implies that $a \in I$. As usual, the ring of integers and ring of integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. Let G be a (simple) undirected graph. We denote the *vertex set* and the *edge set* of G by $V(G)$ and $E(G)$. We say that G is *connected* if there exists a path between any two distinct vertices. Any vertex u of G is called an *end vertex* if degree of u is one. A *subgraph* of G is a graph having all its points and lines in G. For any set S of vertices of G, the *induced subgraph* is the maximal subgraph of G with vertex set S. The *distance* between two vertices x and y of G denoted by $d(x,y)$ is the length of a shortest path connecting them ($d(x,x) = 0$ and if such a path does not exist, then $d(x,y) = \infty$). The *diameter* of G denoted by $\mathrm{diam}(G) = \sup\{d(x,y) \mid x,y$ distinct vertices of $G\}$. A *cycle* of G is a path that begins and ends on the same vertex. The *length* of a cycle is its number of edges (or vertices). The cycle of length n is called a *n-cycle* and denoted by C^n. The *girth* of G denoted by $gr(G)$ is the length of a shortest cycle in G (if G contains no cycle, then $gr(G) = \infty$). A vertex v of a connected graph G is a *cut vertex* of G if $G - \{v\}$ is disconnected. Thus a vertex v of a connected graph G is a cut vertex of G if and only if there exists vertices u and w distinct from v such that v lies on every u—w path of G. A graph G is *complete* if any two distinct vertices are adjacent. A *complete bipartite* graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (we allow m and n to be an infinite cardinal). The *core* of G is the union of all cycles of G. For any vertices x, y in G, if x and y are adjacent, we denote it by x—y.

In this paper, we generalize the notion of ideal-based zero-divisor graph of a commutative ring. Throughout this paper, all rings are commutative (non-trivial), not necessarily with unity unless otherwise stated. In our discussion, we assume $I \neq R$ for an ideal I of R.

In section 2, we give some definitions and preliminary results. In section 3 , we discuss connectedness and cut vertices of $\Gamma_I^G(R)$. Moreover, in section 4, we discuss when $\Gamma_I^G(R)$ is a complete bipartite graph and in section 5, we give the conclusion of the paper.

2. Definitions and Preliminaries

Redmond [11] introduced the definition of the ideal-based zero-divisor graph of a commutative ring R as follows:

Definition 2.1 ([11])**.** Let R be a commutative ring with unity and let I be an ideal of R. Then the ideal-based zero-divisor graph of R, denoted by $\Gamma_I(R)$, is the (simple) undirected graph whose vertex set is $\{a \in R - I \mid ab \in I \text{ for some } b \in R - I\}$, and two distinct vertices a and b are adjacent if and only if $ab \in I$. If $I = \{0\}$, then $\Gamma_I(R)$ is the zero-divisor graph $\Gamma(R)$ which is defined by Anderson and Livingston in [3].

Here, we define a generalized zero-divisor graph of a commutative ring with respect to an ideal as follows:

Definition 2.2. Let R be a commutative ring and let I be an ideal of R. We define a generalized zero-divisor graph of R with respect to I, denoted by $\Gamma_I^G(R)$, as the (simple) undirected graph whose vertex set is $\{a \in R - I \mid \text{there exists } b \in R - I \text{ such that } a_1 b_1 \in I \text{ for some } a_1 \in \langle a \rangle - I \text{ and for some } b_1 \in \langle b \rangle - I\}$, and two distinct vertices a and b are adjacent if and only if $a_1 b_1 \in I$ for some $a_1 \in \langle a \rangle - I$ and for some $b_1 \in \langle b \rangle - I$. If $I = \{0\}$, then $\Gamma_I^G(R)$ is denoted by $\Gamma^G(R)$.

By definitions it follows that every vertex and every edge of $\Gamma_I(R)$ is a vertex and an edge of $\Gamma_I^G(R)$, respectively. But converse is not true, which can be shown by the following examples. Thus $\Gamma_I(R)$ is a subgraph of $\Gamma_I^G(R)$.

Example 2.1. (i) Let $R = \mathbb{Z}_8$ and $I = \{0\}$. We have $2 \in \langle 1 \rangle$ and $2 \cdot 4 = 0$. Thus 1 is a vertex of $\Gamma^G(R)$; but 1 is not a vertex of $\Gamma(R)$, as $1 \cdot a \neq 0$ for all $a \in R - \{0\}$. Also, we have $2 \in \langle 2 \rangle$, $4 \in \langle 6 \rangle$ and $2 \cdot 4 = 0$. Thus 2—6 is an edge of $\Gamma^G(R)$; but 2—6 is not an edge of $\Gamma(R)$, as $2 \cdot 6 \neq 0$.

(ii) Let $R = \mathbb{Z}_8$ and $I = \{0, 4\}$. Then, we have $2 \in \langle 2 \rangle$, $2 \in \langle 3 \rangle$ and $2 \cdot 2 = 4 \in I$. Thus 2—3 is an edge of $\Gamma_I^G(R)$, but 2—3 is not an edge of $\Gamma_I(R)$ as $2 \cdot 3 = 6 \notin I$.

Remark 2.1. Suppose that I is an ideal of a commutative ring R such that $ab \in I$ for all $a, b \in R - I$. Then the generalized zero-divisor graph $\Gamma_I^G(R)$ with respect to I and ideal-based zero-divisor graph $\Gamma_I(R)$ will coincide. In support of this remark, we consider the following example.

Example 2.2. Let us take the commutative ring $R = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| a \in \mathbb{Z}_3 \right\}$ and $I = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$. We have $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in R - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the graphs $\Gamma^G(R)$ and $\Gamma(R)$ will coincide. The graphs $\Gamma^G(R)$ and $\Gamma(R)$ are shown in Figure 1, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

(a) $\Gamma^G(R)$ **(b)** $\Gamma(R)$

Figure 1

Theorem 2.1. *Let I be a nonzero ideal of a commutative ring R. Then $\Gamma_I^G(R)$ is an empty graph if and only if I is a prime ideal of R.*

Proof. Suppose that $\Gamma_I^G(R)$ is an empty graph. If possible assume that I is not a prime ideal of R. Then there exists two elements $a,b \in R-I$ such that $ab \in I$. So the vertex set of $\Gamma_I^G(R)$ is non-empty, a contradiction. Hence I is a prime ideal of R.

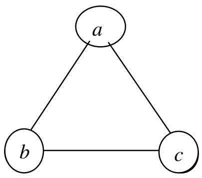
Conversely, suppose that I is a prime ideal of R. Then $ab \in I$ implies $a \in I$ or $b \in I$. So the vertex set of $\Gamma_I^G(R)$ is empty. Hence $\Gamma_I^G(R)$ is an empty graph. □

Remark 2.2. Theorem 2.1 is equivalent to saying that $\Gamma_I^G(R)$ is an empty graph if and only if R/I is an integral domain.

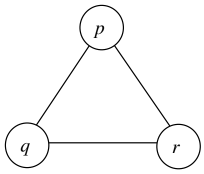
Theorem 2.2 ([11]). *Let R be a commutative ring with unity and let I be an ideal of R. Then $\Gamma_I(R)$ is connected and* $\mathrm{diam}(\Gamma_I(R)) \le 3$.

The following example shows that non-isomorphic commutative rings may have isomorphic generalized zero-divisor graph.

Example 2.3. Let $R_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I_1 = \{(0,0)\}$ and $R_2 = \mathbb{Z}_2[X]/\langle X^2\rangle$ and $I_2 = \{\langle X^2\rangle\}$. Then the graphs $\Gamma^G(R_1)$ and $\Gamma^G(R_2)$ are as follows, where $a=(0,1)$, $b=(1,0)$, $c=(1,1)$, $p = 1+\langle X^2\rangle$, $q = x+\langle X^2\rangle$ and $r = 1+x+\langle X^2\rangle$.



(a) $\Gamma^G(R_1)$

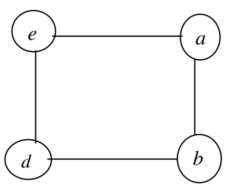


(b) $\Gamma^G(R_2)$

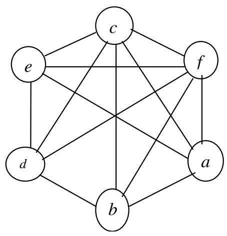
Figure 2

The next example shows that the graph structures $\Gamma_I(R)$ and $\Gamma_I^G(R)$ are not isomorphic.

Example 2.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $I = \langle (0,2)\rangle$. Then the graphs $\Gamma_I(R)$ and $\Gamma_I^G(R)$ are as follows, where $a=(0,1)$, $b=(1,0)$, $c=(1,1)$, $d=(0,3)$, $e=(1,2)$ and $f=(1,3)$.



(a) $\Gamma_1(R)$



(b) $\Gamma_1^G(R)$

Figure 3

In this paper, we show that $\Gamma_I^G(R)$ is connected with diameter at most three. If $\Gamma_I^G(R)$ has a cycle, we show that the girth of $\Gamma_I^G(R)$ is at most four. Also, we investigate the existence of cut vertices of $\Gamma_I^G(R)$. Moreover, we examine certain situations when $\Gamma_I^G(R)$ is a complete bipartite graph.

To avoid trivialities when $\Gamma_I^G(R)$ is the empty graph, we will implicitly assume when necessarily that I is not a prime ideal of R. For any subset U and ideal I of a commutative ring R, we define $[I:U] = \{r \in R \mid rU \subseteq I\}$. Then $[I:U]$ is an ideal of R containing I. If $U = \{a\}$, then $[I:\{a\}]$ is simply denoted by $[I:a]$. Any undefined notation or terminology is standard as in [8] or [9].

3. Some Basic Properties of $\Gamma_I^G(R)$

Some characteristics of $\Gamma_I^G(R)$ are studied in this section. We show that $\Gamma_I^G(R)$ is connected with diameter at most 3. If $\Gamma_I^G(R)$ has a cycle, we show that the girth of $\Gamma_I^G(R)$ is at most 4. We also investigate the existence of cut vertices of $\Gamma_I^G(R)$.

Theorem 3.1. *Let I be an ideal of a commutative ring R. If a—b is an edge of $\Gamma_I^G(R)$ for any $a,b \in V(\Gamma_I^G(R))$, then b—c is an edge of $\Gamma_I^G(R)$ for each $c \in R-I$ or a—d is an edge of $\Gamma_I^G(R)$ for some $d \in b-I$.*

Proof. Suppose that a—b is an edge of $\Gamma_I^G(R)$ for any $a,b \in V(\Gamma_I^G(R))$. Suppose that b—c is not an edge of $\Gamma_I^G(R)$ for some $c \in R-I$. Then $a_1b_1 \in I$ for some $a_1 \in a-I$ and for some $b_1 \in b-I$ and $b_1c \notin I$. Let $d = b_1c$. Then $d \in b-I$. Since I is an ideal of R, $(a_1b_1)c \in I$. This implies $a_1(b_1c) \in I$. Thus $a_1d \in I$. Hence a—d is an edge of $\Gamma_I^G(R)$. □

Theorem 3.2. *Let I be an ideal of a commutative ring R. Then $\Gamma_I^G(R)$ is connected and* $\operatorname{diam}(\Gamma_I^G(R)) \le 3$.

Proof. Let a and b be any two distinct vertices of $\Gamma_I^G(R)$. Consider the following cases:

Case 1: If $a_1b_1 \in I$ for some $a_1 \in \langle a\rangle - I$ and for some $b_1 \in \langle b\rangle - I$, then a—b is an edge of $\Gamma_I^G(R)$.

Case 2: Let $a_1b_1 \notin I$ for all $a_1 \in \langle a\rangle - I$ and for all $b_1 \in \langle b\rangle - I$. Then ${a_1}^2 \notin I$ and ${b_1}^2 \notin I$ for all $a_1 \in \langle a\rangle - I$ and for all $b_1 \in \langle b\rangle - I$. Since $a,b \in V(\Gamma_I^G(R))$ there exists $a_2 \in a-I$, $b_2 \in b-I$ and $x_1, y_1 \in R-(I\cup\{a_2,b_2\})$ such that $x_1a_2 \in I$ and $y_1b_2 \in I$. If $x_1 = y_1$, then a—x_1—b is a path of length 2. So assume that $x_1 \ne y_1$. If $x_1y_1 \in I$, then a—x_1—y_1—b is a path of length 3. If $x_1y_1 \notin I$, then $\langle x_1\rangle \cap \langle y_1\rangle \nsubseteq I$. Now for each $c \in \langle x_1\rangle \cap \langle y_1\rangle - (I\cup\{a_2,b_2\})$, we have $ca_2 \in \langle c\rangle\langle a_2\rangle \subseteq \langle x_1\rangle\langle a_2\rangle \subseteq I$ and $cb_2 \in \langle c\rangle\langle b_2\rangle \subseteq \langle y_1\rangle\langle b_2\rangle \subseteq I$. Hence a—c—b is a path of length 2.

Thus we conclude that $\Gamma_I^G(R)$ is connected and $\operatorname{diam}(\Gamma_I^G(R)) \le 3$. □

Theorem 3.3. *Let I be an ideal of a commutative ring R. If x—a—y is a path in $\Gamma_I^G(R)$, then $I\cup\{a_1\}$ is an ideal of R for some $a_1 \in \langle a\rangle - I$ or x—a—y lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \le 4$.*

Proof. Suppose that x—a—y is a path in $\Gamma_I^G(R)$. Then there exists $a_1, a_2 \in \langle a\rangle - I$ and $x_1 \in \langle x\rangle - I$, $y_1 \in \langle y\rangle - I$ such that $x_1a_1 \in I$ and $y_1a_2 \in I$. Consider the following cases:

Case 1: If $x'y' \in I$ for some $x' \in \langle x\rangle - I$ and for some $y' \in \langle y\rangle - I$, then $C^3 : x—a—y—x$ is a cycle of length 3 and hence $x—a—y$ lies on the cycle C^3 of length 3.

Case 2: Let $x'y' \notin I$ for all $x' \in \langle x\rangle - I$ and for all $y' \in \langle y\rangle - I$. Suppose that $a_1 = a_2$. Then $I \cup \{a_1\} \subseteq [I : x_1] \cap [I : y_1]$. If $[I : x_1] \cap [I : y_1] = I \cup \{a_1\}$, then $I \cup \{a_1\}$ is an ideal of R. Otherwise, there exists a $z \in [I : x_1] \cap [I : y_1]$ such that $z \notin I \cup \{a_1\}$. Then $zx_1 \in I$ and $zy_1 \in I$. Thus $C^4 : x—a—y—z—x$ is a cycle of length 4 and hence $x—a—y$ lies on the cycle C^4 of length 4. Suppose that $a_1 \neq a_2$. Then, we have $\langle x_1\rangle \cap \langle y_1\rangle \nsubseteq I$. Then for each $z \in \langle x_1\rangle \cap \langle y_1\rangle - I$, we have $za_1 \in \langle x_1\rangle\langle a_1\rangle \subseteq I$ and $za_2 \in \langle y_1\rangle\langle a_2\rangle \subseteq I$. Clearly, either $a_1 \neq a$ or $a_2 \neq a$. Without loss of generality assume that $a_1 \neq a$. Then $x—a_1—y$ is a path and $C^4 : x—a—y—a_1—x$ is a cycle of length 4, and hence $x—a—y$ lies on the cycle C^4 of length 4.

Thus, we conclude that $I \cup \{a_1\}$ is an ideal of R for some $a_1 \in a - I$ or $x—a—y$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$. □

The bound for the length of the cycle is sharp in Theorem 3.3, as the following example shows.

Example 3.1. Consider the commutative ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $I = \langle(0,2)\rangle$. Then the graph $\Gamma_I^G(R)$ is as follows where $a = (0,1)$, $b = (1,0)$, $c = (1,1)$, $d = (0,3)$, $e = (1,2)$ and $f = (1,3)$.

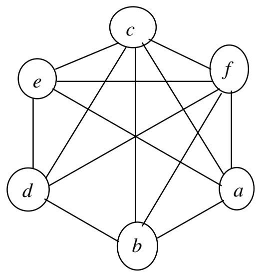


Figure 4. $\Gamma_1^G(R)$

We have $I \cup \{b_1\}$ is not an ideal of R for any $b_1 \in b - I$ and $d—b—a$ does not lie on any cycle of length 3.

Corollary 3.1. *Let I be an ideal of a commutative ring R with $|V(\Gamma_I^G(R))| \geq 3$. If $I \cup \{a\}$ is not an ideal of R for any $a \in R - I$, then every edge of $\Gamma_I^G(R)$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$ and $\Gamma_I^G(R)$ is a union of 3-cycles or 4-cycles.*

Proof. Suppose that $I \cup \{a\}$ is not an ideal of R for any $a \in R - I$. Since $|V(\Gamma_I^G(R))| \geq 3$, the graph $\Gamma_I^G(R)$ contains at least three vertices. Since $\Gamma_I^G(R)$ connected, every path of $\Gamma_I^G(R)$ of length 2 lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$ by Theorem 3.3. Thus every edge of $\Gamma_I^G(R)$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$ and hence $\Gamma_I^G(R)$ is a union of 3-cycles or 4-cycles. □

Theorem 3.4. *Let I be an ideal of a commutative ring R with $|V(\Gamma_I^G(R))| \geq 3$. If $I \cup \{a\}$ is not an ideal of R for any $a \in R - I$, then every pair of vertices in $\Gamma_I^G(R)$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 6$.*

Proof. Suppose that $I \cup \{a\}$ is not an ideal of R for any $a \in R - I$. Since $|V(\Gamma_I^G(R))| \geq 3$, the graph $\Gamma_I^G(R)$ contains at least three vertices. Let x, y be any two distinct vertices of $\Gamma_I^G(R)$. If $x—y$ is an edge of $\Gamma_I^G(R)$, then $x—y$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$ by Corollary 3.1. If $x—a—y$ is a path in $\Gamma_I^G(R)$, then $x—a—y$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$ by Theorem 3.3. If $x—a—b—y$ is a path in $\Gamma_I^G(R)$, then we have the cycles $x—a—b—c—x$ and $y—b—a—d—y$, where $c \neq a$ and $d \neq b$ by Theorem 3.3. This implies $C^6 : x—a—d—y—b—c—x$ is a cycle of length 6 in $\Gamma_I^G(R)$. Thus every pair of vertices in $\Gamma_I^G(R)$ lies on a cycle $C^i$of $\Gamma_I^G(R)$ with length $i \leq 6$. □

Theorem 3.5. *Let I be an ideal of a commutative ring R. If $\Gamma_I^G(R)$ has a cycle, then any cycle C^i of length $i \geq 5$ is not an induced subgraph of $\Gamma_I^G(R)$ and $gr(\Gamma_I^G(R)) \leq 4$.*

Proof. Suppose that $\Gamma_I^G(R)$ has a cycle $C^i : x_1—x_2—x_3—x_4—x_5— \cdots —x_i—x_1$ of length $i \geq 5$ which is an induced subgraph of $\Gamma_I^G(R)$. Then $x_1—x_2—x_3$ is a path which lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \geq 5$. Thus $I \cup \{x_2'\}$ is an ideal of R for some $x_2' \in x_2 - I$ by Theorem 3.3. Since $x_4—x_5$ is an edge there exists $x_4' \in \langle x_4 \rangle - I$ and $x_5' \in \langle x_5 \rangle - I$ such that $x_4' x_5' \in I$. Since $I \cup \{x_2'\}$ is an ideal, we have $x_2' x_4' = x_2' = x_2' x_5'$. Thus $x_2'(x_4' x_5') \in I$. This implies $(x_2' x_4') x_5' \in I$. This implies $x_2' x_5' \in I$. This implies $x_2' \in I$, which is a contradiction. Thus any cycle C^i of length $i \geq 5$ is not an induced subgraph of $\Gamma_I^G(R)$, and hence $gr(\Gamma_I^G(R)) \leq 4$. □

Remark 3.1. Let I be an ideal of a commutative ring R. Then $\Gamma_I^G(R)$ cannot be realized as a cycle C^i of length $i \geq 5$ by Theorem 3.5.

Theorem 3.6. *Let I be an ideal of a commutative ring R. Then the following results hold:*

(i) *if R is a commutative ring with unity, then $\Gamma_I^G(R)$ has no cut vertices;*

(ii) *if R is a commutative ring without unity and I is a nonzero ideal of R, then $\Gamma_I^G(R)$ has no cut vertices.*

Proof. Suppose that the vertex u of $\Gamma_I^G(R)$ is a cut vertex. Let $a—u—b$ be a path in $\Gamma_I^G(R)$. Since u is a cut vertex, u lies in every path connecting a and b.

(i) Suppose that R is a commutative ring with unity. Then for any $a, c \in V(\Gamma_I^G(R))$, there exists a path $a—1—b$ in $\Gamma_I^G(R)$. Thus $u (\neq 1)$ is not a cut vertex of $\Gamma_I^G(R)$. Suppose that $u = 1$. Then there exists $a_1 \in \langle a \rangle - I$, $b_1 \in \langle b \rangle - I$ and $x_1, x_2 \in R - I$ such that $a_1 x_1 \in I$ and $b_1 x_1 \in I$, which shows that $a_1, b_1 \in V(\Gamma_I(R))$. Since $\Gamma_I(R)$ is connected, there exists $y_1, y_2 \in R - (I \cup \{u\})$ such that $a_1—y_1—b_1$ or $a_1—y_1—y_2—b_1$ is a path in $\Gamma_I(R)$ by Theorem 2.2. This implies $a—y_1—b—1—a$ or $a—y_1—y_2—b—1—a$ is a cycle in $\Gamma_I^G(R)$, which contradicts that $u = 1$ is a cut vertex.

(ii) Suppose that R is a commutative ring without unity and I is a nonzero ideal of R. Since $a—u—b$ is a path from a to b, there exists $a_1 \in \langle a \rangle - I$, $b_1 \in \langle b \rangle - I$ and $u_1, u_2 \in \langle u \rangle - I$ such that $a_1 u_1 \in I$ and $b_1 u_2 \in I$.

Case 1: Suppose that $u_1 = u_2$. If $a_1 + I = u_1 + I$, then $a_1 b_1 \in I$. This implies a and b are adjacent. If $u_2 + I = b_1 + I$, then $a_1 b_1 \in I$. This implies a and b are adjacent. So assume that $a_1 + I \neq u_1 + I$ and $u_2 + I \neq b_1 + I$. Since I is nonzero, there is a $i \in I$ such that $i \neq 0$.

Since $a_1u_1 \in I$ and $b_1u_2 \in I$, then it implies that $a_1(u_1+i), b_1(u_1+i) \in I$. If $u = u_1+i$, then $u \neq u_1$. Thus we have $a—u_1—b$ is a path in $\Gamma_I^G(R)$. Otherwise $a—(u_1+i)—b$ is a path in $\Gamma_I^G(R)$. Therefore, there is a path from a to b that is not passing throw u, which is a contradiction.

Case 2: Suppose that either $u_1 = u$ or $u_2 = u$. Without loss of generality assume that $u_1 = u$ and $u_2 \neq u$. Then $a_1u \in I$ and $b_1u_2 \in I$. This implies that $a_1u_2 \in I$ and $b_1u_2 \in I$. Thus $a—u_2—b$ is a path in $\Gamma_I^G(R)$. Therefore, there is a path from a to b that is not passing through u, which is a contradiction.

Case 3: Suppose that $u_1 \neq u$ and $u_2 \neq u$ such that $u_1 \neq u_2$. If $u_1u_2 \in I$, then $a—u_1—u_2—b$ is a path in $\Gamma_I^G(R)$. Therefore, there is a path from a to b that is not passing throw u, which is a contradiction. Otherwise, we have $u_1u_2 \notin I$. If $u_1u_2 = u$, then $a_1u \in I$ and $b_1u \in I$. If $a_1 + I = u + I$, then $a_1b_1 \in I$. This implies a and b are adjacent. If $u + I = b_1 + I$, then $a_1b_1 \in I$. This implies a and b are adjacent. So assume that $a_1 + I \neq u + I$ and $u + I \neq b_1 + I$. Since I is nonzero, there is a $i \in I$ such that $i \neq 0$. Since $a_1u \in I$ and $b_1u \in I$, then it implies that $a_1(u+i)$, $b_1(u+i) \in I$. Then $u+i \notin I$ and $u \neq u+i$. Thus we have $a—(u+i)—b$ is a path in $\Gamma_I^G(R)$. Therefore there is a path from a to b that is not passing through u, which is a contradiction. If $u_1u_2 \neq u$. Then $a—u_1u_2—b$ is a path in $\Gamma_I^G(R)$. Therefore, there is a path from a to b that is not passing through u, which is a contradiction.

Thus u can not be a cut vertex of $\Gamma_I^G(R)$. □

Recall that, the core of a graph G is the union of all cycles of G.

Theorem 3.7. *Let I be an ideal of a commutative ring R. If $\Gamma_I^G(R)$ has a cycle, then the core K of $\Gamma_I^G(R)$ is a union of 3-cycles or 4-cycles. Moreover, any vertex in $\Gamma_I^G(R)$ is either a vertex of the core K of $\Gamma_I^G(R)$ or is an end vertex of $\Gamma_I^G(R)$.*

Proof. Suppose that $\Gamma_I^G(R)$ has a cycle. Then any cycle C^i of length $i \geq 5$ is not an induced subgraph of $\Gamma_I^G(R)$ and $gr(\Gamma_I^G(R)) \leq 4$ by Theorem 3.5. Thus the core K of $\Gamma_I^G(R)$ is a union of 3-cycles or 4-cycles.

For the second statement we assume that $|V(\Gamma_I^G(R))| \geq 3$. Let u be any vertex of $\Gamma_I^G(R)$. Then we have the followings and one of them is true.

Case 1: u is in the core K of $\Gamma_I^G(R)$.

Case 2: u is an end vertex of $\Gamma_I^G(R)$.

Case 3: $a—u—b$ is a path in $\Gamma_I^G(R)$, where a is an end vertex, $u \notin K$ and $b \in K$.

Case 4: $a—u—v—b$ or $a—v—u—b$ is a path in $\Gamma_I^G(R)$, where a is an end vertex, $u, v \notin K$ and $b \in K$.

In *Cases* 1 and 2, there is nothing to prove.

Suppose that *Case* 3 holds. Assume that $a—u—b$ is a path in $\Gamma_I^G(R)$, where a is an end vertex, $u \notin K$ and $b \in K$. Then, $I \cup \{u_1\}$ is an ideal of R for some $u_1 \in \langle u \rangle - I$ by Theorem 3.3. Since $b \in K$ we have $u—b—c—d—b$ or $u—b—c—d—e—b$ is a path in $\Gamma_I^G(R)$. Then $c_1d_1 \in I$ for some $c_1 \in \langle c \rangle - I$ and for some $d_1 \in \langle d \rangle - I$. Since $I \cup \{u_1\}$ is an ideal of R, we have $u_1c_1 = u_1$.

Thus $u_1(c_1d_1) \in I$. This implies $(u_1c_1)d_1 \in I$. This implies $u_1d_1 \in I$. Thus u is a vertex of the cycle $u—b—c—d—u$, which is a contradiction.

Suppose that *Case* 4 holds. Without loss of generality assume that $a—u—v—b$ is a path in $\Gamma_I^G(R)$, where a is an end vertex, $u,v \notin K$ and $b \in K$. Since $b \in K$, there is some $c \in K$ such that $c \neq b$ and $b—c$ lies on a cycle C^i of $\Gamma_I^G(R)$ with length $i \leq 4$. Then we have $a—u—v—b—c$ is a path in $\Gamma_I^G(R)$. Since $\mathrm{diam}(\Gamma_I^G(R)) \leq 3$, we have $v—c$ or $u—c$ is an edge. If $v—c$ is an edge, then $v \in K$. Then $u \in K$ by *Case* 3. Thus we get a contradiction. Again if $u—c$ is an edge, then $u—v—b—c—u$ is a cycle. Thus $u,v \in K$, a contradiction.
Hence any vertex in $\Gamma_I^G(R)$ is either a vertex of the core K of $\Gamma_I^G(R)$ or is an end vertex of $\Gamma_I^G(R)$. □

Corollary 3.2. *Let I be an ideal of a commutative ring R. If R has unity with $|V(\Gamma_I^G(R))| \geq 3$ or if R has no unity and I is a nonzero ideal of R with $|V(\Gamma_I^G(R))| \geq 3$, then $\Gamma_I^G(R) = K$, where K is the core of $\Gamma_I^G(R)$.*

Proof. Suppose that R has unity with $|V(\Gamma_I^G(R))| \geq 3$. Then $\Gamma_I^G(R)$ has no cut vertices by Theorem 3.6(i), and hence $\Gamma_I^G(R)$ has no end vertex. Then every vertex of $\Gamma_I^G(R)$ is a vertex of the core K of $\Gamma_I^G(R)$ by Theorem 3.7. Thus $\Gamma_I^G(R) = K$. Next, suppose that R has no unity and I is a nonzero ideal of R with $|V(\Gamma_I^G(R))| \geq 3$. Then $\Gamma_I^G(R)$ has no cut vertices by Theorem 3.6(ii), and hence $\Gamma_I^G(R)$ has no end vertex. Then every vertex of $\Gamma_I^G(R)$ is a vertex of the core K of $\Gamma_I^G(R)$ by Theorem 3.7. Thus $\Gamma_I^G(R) = K$. □

4. Complete Bipartite Graph

We have $\Gamma_I^G(R)$ is an empty graph if and only if I is a nonzero prime ideal of R. In this section, we examine certain situations when $\Gamma_I^G(R)$ is a complete bipartite graph.

Theorem 4.1. *Let I be a semi prime ideal of a commutative ring R. If $\Gamma_I^G(R)$ is a complete bipartite graph then there exists prime ideals P and Q of R such that $I = P \cap Q$.*

Proof. Suppose that $\Gamma_I^G(R)$ is a complete bipartite graph with partitions C and D. Suppose that $V_1 = \{x \in C$ is vertex of $\Gamma_I^G(R) \mid xy \in I$ for each $y \in D\}$ and $V_2 = \{y \in D$ is vertex of $\Gamma_I^G(R) \mid xy \in I$ for each $x \in C\}$. Clearly, V_1 and V_2 are non-empty sets and $V_1 \cap I = V_2 \cap I = \emptyset$. Let $P = V_1 \cup I$ and $Q = V_2 \cup I$. Then $I = P \cap Q$. We need to show that P and Q are prime ideals of R.

First we show P is a prime ideal of R. Let $x, y \in P$.

Case 1: If $x, y \in I$, then $x - y \in I$ and hence $x - y \in P$.

Case 2: If $x, y \in V_1$, then for any $z \in V_2$, we have $xz \in I$ and $yz \in I$. Clearly, $z \notin I$. Then $(x-y)z \in I$. If $x - y \in I$, then $x - y \in P$. Suppose that $x - y \notin I$. We have $x - y \notin V_2$ and $x - y \neq z$. Otherwise, if $x - y \in V_2$ or $x - y = z$, then due to $(x-y)z \in I$ we get a contradiction. Now $(x-y)z \in I$ with $x - y \notin I$ and $z \notin I$. Thus $x - y \in C$ and $z \in D$. Therefore, $x - y \in V_1$, and hence $x - y \in P$.

Case 3: Suppose that $x \in V_1$ and $y \in I$. Then $x - y \notin I$ and for any $z \in V_2$ we have $xz \in I$. Clearly, $z \notin I$. Then $(x - yz) \in I$. We have $x - y \notin V_2$ and $x - y \neq z$. Now $(x-y)z \in I$ with $x - y \notin I$ and $z \notin I$. Thus $x - y \in C$ and $z \in D$. Therefore $x - y \in V_1$, and hence $x - y \in P$.

Now suppose that $r \in R$ and $x \in P$.

Case 1: If $x \in I$, then $rx \in I$ and hence $rx \in P$.

Case 2: If $x \in V_1$, then for any $z \in V_2$ we have $zx \in I$. So $(rx)z \in I$. If $rx \in I$, then $rx \in P$. Suppose that $rx \notin I$. Clearly, $z \neq rx$. Then $(rx)z \in I$ with $rx \notin I$ and $z \notin I$. Thus $rx \in C$ and $z \in D$. Therefore $rx \in V_1$ and hence $rx \in P$.

Thus P is an ideal of R.

We now prove that P is a prime ideal of R. Let $x, y \in R$ with $xy \in P$ and $x, y \notin P$. Since $P = V_1 \cup I$, we have $xy \in V_1$ or $xy \in I$. So in both the cases, for any $z \in V_2$ we have $(xy)z \in I$. Thus $x(yz) \in I$. If $yz \in I$, then by the definition of $\Gamma_I^G(R)$, $y \in V_1$, which is a contradiction. Hence $yz \notin I$ and so $yz \in V_1$. Therefore $(yz)z \in I$. This implies $yz^2 \in I$. Since I is semi prime, we have $z^2 \notin I$. Hence $z^2 \in V_2$. So $y \in V_1$, and hence $y \in P$, a contradiction. Therefore P is a prime ideal of R. Proceeding in the same manner we can show that Q is also a prime ideal of R. □

The following example shows that in Theorem 4.1, the converse is not true in general.

Example 4.1. Consider the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $I = \langle(0,2)\rangle$ is a semi prime ideal of R. Consider the prime ideals $P = \langle(0,1)\rangle$ and $Q = \langle(1,2)\rangle$. Then, we have $I = P \cap Q$. But the graph $\Gamma_I^G(R)$ is not complete bipartite, where $a = (0,1)$, $b = (1,0)$, $c = (1,1)$, $d = (0,3)$, $e = (1,2)$ and $f = (1,3)$.

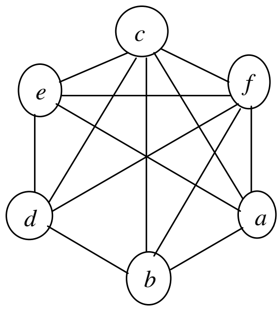


Figure 5. $\Gamma_1^G(R)$

The following example shows that the Theorem 4.1 will fail if I is not a semi prime ideal.

Example 4.2. Consider the commutative ring $R = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \middle| a \in \mathbb{Z}_4 \right\}$. Then $I = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \right\}$ is an ideal of R and I is not semi prime. Then $\Gamma_I^G(R)$ is $K^{1,1}$, but I cannot be written as the intersection of two prime ideals. The graph $\Gamma_I^G(R)$ is as follows, where $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$.

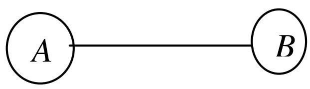


Figure 6. $\Gamma_1^G(R)$

5. Conclusion

In this paper, we have defined a generalized zero-divisor graph $\Gamma_I^G(R)$ of a commutative ring R with respect to an ideal I and have discussed some basic properties of $\Gamma_I^G(R)$. This chapter is just an opening for making a bridge of generalization of zero-divisor graph. The study of connectedness, cut vertices of this generalized graph will develop many ring-theoretic concepts. We have also investigated some properties of prime and semi-prime ideals when $\Gamma_I^G(R)$ is a complete bipartite graph.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** (2004), 847 – 855.

[2] D.D. Anderson and M. Naseer, Beck's coloring of a commutative ring, *J. Algebra* **159** (1993), 500 – 514.

[3] D.F. Anderson and P.S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434 – 447.

[4] D.F. Anderson, A. Frazier, A. Laave and P.S. Livingston, The zero-divisor graph of a commutative ring II, *Lecture Notes in Pure and Appl. Math.*, **220** (2001), Marcel Dekker, New York—Basel, 61 – 72.

[5] I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208 – 226.

[6] P. Dheena and B. Elavarasan, An ideal-based zero-divisor graph of 2-primal near-rings, *Bull. Commun. Korean Math. Soc.* **46** (6) (2009), 1051 – 1060.

[7] P. Dheena and B. Elavarasan, A generalized ideal-based zero-divisor graph of a near-ring, *Commun. Korean Math. Soc.* **24** (2) (2009), 161 – 169.

[8] R. Diestel, *Graph Theory*, Springer-Verlag, New York (1997).

[9] N. Herstein, *Topics in Algebra*, 2nd edition, John Wiley & Sons (Asia) Pte. Ltd. (1999).

[10] H. Maimani, M.R. Pouranki and S. Yassemi, *Zero-divisor graph with respect to an ideal*, *Communications in Algebra* **34** (3) (2006), 923 – 929.

[11] S.P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, *Comm. Algebra* **31** (9) (2003), 4425 – 4443.