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Optimal Approximate Solution for Generalized Contraction Mappings

Research Article

Somayya Komal[1], Nazra Sultana[2], Azhar Hussain[2] and Poom Kumam[3]

[1] *Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan; Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand*
[2] *Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan*
[3] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand*
***Corresponding author:** poom.kum@kmutt.ac.th

Abstract. In this paper, we obtain the best proximity point theorems for α-Geraghty contractions in the setting of complete metric spaces. We present some examples to prove the validity of our results. Our results extend and unify many existing results in the literature.

Keywords. Best proximity point; P-property; Triangular α-admissible.

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1. Introduction

The Banach contraction principle [11], which is a useful tool in the study of many branches of mathematics and mathematical sciences, is one of the earlier and fundamental result in fixed point theory. Because of its importance in nonlinear analysis, a number of authors have improved, generalized and extended this basic result either by defining a new contractive mapping in the context of a complete metric space or by investigating the existing contractive mappings in various abstract spaces; see, e.g., [2, 5, 14–17, 19, 21, 24] and references therein.

Best proximity point theory involves an intertwining of approximation and global optimization. Indeed, it explores the existence and computation of an optimal approximate solution of non-linear equations of the form $fx = x$, where f is a non-self mapping in some framework. Such equations are confronted when we attempt the mathematical formulation of several problems. Given a non-self mapping $f : A \to B$, where A and B are non-empty subsets of a metric space, the equation $fx = x$ does not necessarily have a solution because of the fact that a solution of the preceding equation constrains the equality between an element in the domain and an element in the range of the mapping. In such circumstances a natural question arises: "*Is it possible to find an optimal approximate solution with the least possible error*?"

Best proximity point theory is an outgrowth of attempts in many directions to answer previously posed question for various families of non-self mappings. In fact, a best proximity point theorem furnishes sufficient conditions for the existence and computation of an approximate solution x^* that is optimal in the sense that the error $d(x^*, fx^*)$ assumes the global minimum value $d(A,B)$. Such an optimal approximate solution is known as a best proximity point of the mapping f. It is straightforward to observe that a best proximity point becomes a solution of the equation in the special case that the domain of the mapping intersects the co-domain of the mapping. Best proximity point theorems for several types of non-self mappings have been derived in [1,3,4,6–10,23].

Recently, Geraghty [15] obtained a generalization of the Banach contraction principle in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami [5] characterized the result of Geraghty in the context of a partially ordered complete metric space. This result is of particular interest since many real world problems can be identified in a partially ordered complete metric space. Cabellero *et al.* [12] discussed the existence of a best proximity point of Geraghty contraction.

In this article, we get the best proximity point theorems for generalized Geraghty contractions for non self mappings in the framework of complete metric spaces. We establish few examples in the favor of our results.

2. Preliminaries

Definition 2.1 ([20])**.** Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{ there exists some } b \in B \text{ such that } d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : \text{ there exists some } a \in A \text{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

In [8], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Definition 2.2 ([22])**.** Let $T : A \to B$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then f is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.

Definition 2.3 ([18])**.** An α-admissible map f is said to be triangular α-admissible if $\alpha(x,z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$.

We denote by $\mathscr{F}$ the class of all functions $\beta : [0,\infty) \to [0,1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 2.4 ([15]). Let (X,d) be a metric space. A map $f : X \to X$ is called Geraghty contraction if there exists $\beta \in \mathscr{F}$ such that for all $x,y \in X$,

$$d(fx,fy) \le \beta(d(x,y))d(x,y).$$

By using such maps Geraghty [15] proved the following fixed point result:

Theorem 2.1. *Let (X,d) be a complete metric space. Mapping $f : X \to X$ is Geraghty contraction. Then f has a fixed point $x \in X$, and $\{f^n x\}$ converges to x.*

Cho *et al.* [14] generalized the concept of Geraghty contraction to α-Geraghty contraction and prove the fixed point theorem for such contraction.

Definition 2.5 ([14]). Let (X,d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $f : X \to X$ is called α-Geraghty contraction if there exists $\beta \in \mathscr{F}$ such that for all $x,y \in X$,

$$\alpha(x,y)d(fx,fy) \le \beta(d(x,y))d(x,y).$$

Theorem 2.2 ([14]). *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : X \to X$ satisfying the following conditions:*

1. *f is continuous α-Geraghty contraction;*
2. *f be a triangular α-admissible;*
3. *there exists $x_1 \in X$ such that $\alpha(x_1, fx_1) \ge 1$;*

Then f has a fixed point $x \in X$, and $\{f^n x_1\}$ converges to x.

Definition 2.6 ([20]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any x_1,x_2,x_3, $x_4 \in A_0$,

$$\left.\begin{aligned} d(x_1,fx_3) &= d(A,B) \\ d(x_2,fx_4) &= d(A,B) \end{aligned}\right\} \Rightarrow \quad d(x_1,x_2) = d(fx_3,fx_4).$$

Definition 2.7 ([20]). Let (X,d) be a metric space and $f : A \to B$, then a point $x \in A$ is called best proximity point of the mapping f if

$$d(x,fx) = d(A,B).$$

We denote the set of best proximity points for given mapping f as BPf.

3. Optimal Approximate Solution

In this section, we introduced the new concept of the existence of best proximity point for α-Geraghty contraction in the framework of Cauchy metric spaces. Our first notion is about α-Geraghty contraction for non self mapping, instead of self mapping in [14].

Definition 3.1. Let (X,d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $f : A \to B$ is called α-Geraghty contraction if there exists $\beta \in \mathscr{F}$ such that for all $x,y \in A$,

$$\alpha(x,y)d(fx,fy) \le \beta(d(x,y))d(x,y),$$

where $A,B \subseteq X$.

Now, we are in a position to prove our main result.

Theorem 3.1. *Let A,B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : A \to B$ satisfying the following conditions:*

(1) *f is continuous α-Geraghty contraction with $f(A_0) \subseteq B_0$;*

(2) *f be a triangular α-admissible;*

(3) *there exists $x_0, x_1 \in A_0$ such that $d(x_1, fx_0) = d(A,B)$ and $\alpha(x_0, x_1) \geq 1$;*

(4) *the pair (A,B) has the P-property.*

Then there exists x^* in A such that $d(x^*, fx^*) = d(A,B)$.

Proof. Let $x_0 \in A_0$, since $f(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, fx_0) = d(A,B) \text{ with } \alpha(x_0, x_1) \geq 1. \tag{3.1}$$

Again, since $f(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, fx_1) = d(A,B). \tag{3.2}$$

Repeating this process, we get a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, fx_n) = d(A,B)$$

for any $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $d(x_{n_0}, fx_{n_0}) = d(x_{n_0+1}, fx_{n_0}) = d(A,B)$ implies that x_{n_0} is a best proximity point of f. If we define $x_m = x_{n_0}$ for all $m \geq n_0$, then $\{x_n\}$ converges to a best proximity point of f. The proof is complete. Assume that

$$d(x_n, x_{n+1}) > 0, \text{for all } n \geq 0. \tag{3.3}$$

Note that $x_n, x_{n+1} \in A_0$ and $fx_n \in B_0$ for all $n \geq 0$. We claim that

$$\alpha(x_n, x_{n+1}) \geq 1, \tag{3.4}$$

for all $n \geq 0$. If $n = 0$, then $\alpha(x_n, x_{n+1}) \geq 1$ holds by given hypothesis. Suppose that $\alpha(x_n, x_{n+1}) \geq 1$ for some $n > 0$. As f is α-admissible, for $x_n, x_{n+1}, x_{n+2} \in A_0$, $\alpha(x_n, x_{n+1}) \geq 1$, we have $\alpha(x_{n+1}, x_{n+2}) \geq 1$. Thus (3.4) holds.

Since (A,B) has the P-property, we have that

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \quad \text{for any } n \in \mathbb{N}.$$

Taking into account that f is α-Geraghty contraction, so for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \\ &\leq \beta(d(x_{n-1}, x_n)) d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n), \end{aligned}$$

where $\beta(d(x_{n-1},x_n)<1$ and $\alpha(x_{n-1},x_n)\geq 1$.

$$\Rightarrow \qquad d(x_n,x_{n+1})<d(x_{n-1},x_n),$$

so $\{d(x_n,x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers.

Suppose that there exists $n_0\in\mathbb{N}$ such that $d(x_{n_0},x_{n_0+1})=0$. In this case,

$$0=d(x_{n_0},x_{n_0+1})=d(fx_{n_0-1},fx_{n_0}),$$

implies that

$$d(fx_{n_0-1},fx_{n_0})=0$$

and consequently

$$fx_{n_0-1}=fx_{n_0}.$$

Therefore,

$$d(A,B)=d(x_{n_0},fx_{n_0-1})=d(x_{n_0},fx_{n_0}).$$

Thus in this case, there exists best proximity point, i.e. there exists x^* in A such that $d(x^*,fx^*)=d(A,B)$.

In the contrary case, suppose that $d(x_n,x_{n+1})>0$ for any $n\in N$. Since $\{d(x_n,x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers and hence there exists $r\geq 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=r.$$

We have to show that $r=0$. Let $r\neq 0$ and $r>0$, then

$$1<\frac{d(x_n,x_{n+1})}{d(x_{n-1},x_n)}\leq \beta d((x_{n-1},x_n))<1, \quad \text{for any } n\in\mathbb{N}.$$

Which yields that

$$\lim_{n\to\infty}\beta(d(x_{n-1},x_n))=1,$$

since $\beta\in\mathscr{F}$, the above equation implies that

$$\lim_{n\to\infty} d(x_{n-1},x_n)=0.$$

Hence $r=0$ and this contradicts our assumption that $r>0$. Therefore,

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

Since $d(x_{n+1},fx_n)=d(A,B)$ for any $n\in\mathbb{N}$, for fixed $p,q\in\mathbb{N}$, we have

$$d(x_p,fx_{p-1})=d(x_q,fx_{q-1})=d(A,B)$$

and since (A,B) satisfies P-property, so

$$d(x_p,x_q)=d(fx_{p-1},fx_{q-1}).$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence.

On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon>0$ such that for all $k>0$, there exists $m(k)>n(k)>k$ with (the smallest number satisfying the condition below)

$$d(x_{m(k)},x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1},x_{n(k)}) < \epsilon.$$

Then we have

$$\begin{aligned}\epsilon &\leq d(x_{m(k)},x_{n(k)})\\ &\leq d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)})\\ &< d(x_{m(k},x_{m(k)-1}) + \epsilon.\end{aligned}$$

This implies that $\epsilon < d(x_{m(k)},x_{n(k)}) < d(x_{m(k)},x_{m(k)-1}) + \epsilon$.

Let $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) = \epsilon. \tag{3.5}$$

Now by using Triangular inequality, we have

$$d(x_{m(k)},x_{n(k)}) \leq d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)-1}) + d(x_{n(k)-1},n_k)$$

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) \geq \lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) - \lim_{k\to\infty} d(x_{m(k)},x_{m(k)-1}) - \lim_{k\to\infty} d(x_{n(k)-1},x_{n(k)}).$$

By using (3) and (5), we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) = \epsilon.$$

Since $\alpha(x_{n(k)-1},x_{m(k)-1}) \geq 1$, we have

$$\begin{aligned}d(x_{m(k)},x_{n(k)}) &= d(fx_{m(k)-1},fx_{n(k)-1})\\ &\leq \alpha(x_{n(k)-1},x_{m(k)-1})d(fx_{n(k)-1},fx_{m(k)-1})\\ &\leq \beta(d(x_{n(k)-1},x_{m(k)-1}))d(x_{n(k)-1},x_{m(k)-1})\end{aligned}$$

$$\Rightarrow \quad \frac{d(x_{m(k)},x_{n(k)})}{d(x_{n(k)-1},x_{m(k)-1})} \leq \beta(d(x_{n(k)-1}),x_{m(k)-1}).$$

Letting $m,n \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} \beta(d(x_{n(k)-1},x_{m(k)-1})) = 1,$$

and so

$$\lim_{n\to\infty} d(x_{n(k)-1},x_{m(k)-1}) = 0.$$

Hence $\epsilon = 0$, which contradicts our supposition that $\epsilon > 0$. So we conclude that $\{x_n\}$ is a Cauchy sequence in A. Since $\{x_n\} \subseteq A$ and A is closed subset of a complete metric space (X,d). There is $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, so we have

$$fx_n \to fx^*$$

$$\Rightarrow \quad d(x_{n+1},fx_n) \to d(x^*,fx^*).$$

Taking into account that $\{d(x_{n+1},fx_n)\}$ is a constant sequence with a value $d(A,B)$, we deduce

$$d(x^*,fx^*) = d(A,B),$$

i.e. x^* is best proximity point of f. □

Example 3.1. Consider $X=\mathbb{R}^2$, with the usual metric d. Let $A=\{0\}\times[0,\infty)$ and $B=\{1\}\times[0,\infty)$. Obviously, $d(A,B)=1$ and A,B are nonempty closed subsets of X, take $A_0=A$ and $B_0=B$.

We define $f:A\to B$ as:

$$f(0,x)=(1,\ln(1+x)),$$

where $(0,x)\in A$ and $x\in[0,\infty)$.

Let $\alpha:R^2\times R^2\to[0,\infty)$ defined as:

$$\alpha((x_1,y_1),(x_2,y_2))=\begin{cases}1 & \text{if } 0\le x_1,x_2\le 1 \text{ and } y_1>y_2\ge 0,\\ 0 & \text{elsewhere.}\end{cases}$$

Clearly, f is triangular α-admissible and for $(0,x_1)\in A_0$ one has

$$\alpha((0,x_1),f(0,x_1))=1.$$

f is α-Geraghty contraction as: for $(0,x),(0,y)\in A$ with $x\neq y$ and $x>y$, we have

$$\begin{aligned}\alpha((0,x),(0,y))d(f(0,x),f(0,y)) &= 1\cdot d(f(0,x),f(0,y))\\ &=|\ln(1+x)-ln(1+y)|\\ &=\left|\ln\left(\frac{1+x}{1+y}\right)\right|\\ &=\left|\ln\left(\frac{(1+y)+(x-y)}{1+y}\right)\right|\\ &=\left|\ln\left(1+\frac{x-y}{1+y}\right)\right|\\ &\le\ln(1+|x-y|)\\ &=\frac{\ln(1+|x-y|)}{|x-y|}\cdot|x-y|\\ &=\frac{\ln(1+d((0,x),(0,y)))}{d((0,x),(0,y))}\cdot d((0,x),(0,y)).\end{aligned}$$

Take $\phi(t)=\ln(1+t)$ for $t\ge 0$, we have

$$\alpha((0,x),(0,y))d(f(0,x),f(0,y))\le\frac{\phi(d(0,x),d(0,y))}{d((0,x),(0,y))}\cdot d((0,x),(0,y)).$$

Setting $\beta(t)=\frac{\phi(t)}{t}$ for $t>0$, and $\beta(0)=0$, we have

$$\alpha((0,x),(0,y))d(f(0,x),f(0,y))\le\beta(d((0,x),(0,y)))\cdot d((0,x),(0,y)).$$

Obviously, when $x=y$ the inequality is satisfied. Also $\beta(t)=\frac{\ln(1+t)}{t}\in\mathscr{F}$, by elementary calculus.

The pair (A,B) satisfied the P-property. Indeed if for$(0,x_1),(0,x_2)\in A_0$ and $(1,y_1),(1,y_2)\in B_0$ and let $x_1=y_1,x_2=y_2$

$$d((0,x_1),(1,y_1))=\sqrt{1+(x_1-y_1)^2}=d(A,B)=1,$$

$$d((0,x_1),(1,y_1))=\sqrt{1+(x_1-y_1)^2}=d(A,B)=1,$$

consequently,

$$d((0,x_1),(0,x_2)) = |x_1 - x_2| = |y_1 - y_2| = d((1,y_1),(1,y_2)).$$

Thus, by Theorem 3.1, $(0,0)$ is the unique best proximity point of f.

Example 3.2. Let $X = \mathbb{R}^2$ with the usual metric d. Let $A = \{0\} \times [0,\infty)$ and $B = \{1\} \times [0,\infty)$ be subsets of X. Obviously, $d(A,B) = 1$, and B is not a closed subset of X. Let $A_0 = \{0\} \times [0,\Pi/2)$ and $B_0 = \{1\} \times [0,\Pi/2)$. Define a mapping $f : A \to B$ by

$$f(0,x) = (1,\tan^{-1} x), \quad \text{for any } (0,x) \in A.$$

Let $\alpha : X \times X \to \mathbb{R}^+$ defined by

$$\alpha((x_1,y_1),(x_2,y_2)) = \begin{cases} 1 & \text{if } 0 \le x_1, x_2 \le 1 \text{ and } y_1 > y_2 \ge 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly, f is triangular α-admissible and for $(0,x_1) \in A_0$ one has

$$\alpha((0,x_1), f(0,x_1)) = 1.$$

Map f is α-Geraghty contraction. Indeed for $(0,x),(0,y) \in A$ with $x \ne y; x > y$.

$$\begin{aligned} \alpha((0,x),(0,y))d(f(0,x),f(0,y)) &= 1 \cdot d((1,\tan^{-1}x),(1,\tan^{-1}y)) \\ &= |\tan^{-1}x - \tan^{-1}y|. \end{aligned}$$

Set $\beta_1 = \tan^{-1}x$ and $\beta_2 = \tan^{-1}y$. Since $\beta_1 > \beta_2$. Since the function $\phi(t) = \tan^{-1}t$ for $t \ge 0$ is strictly increasing.

Taking into account that

$$\tan(\beta_1 - \beta_2) = \frac{\tan\beta_1 - \tan\beta_2}{1 + \tan\beta_1 \tan\beta_2}$$

and since $\beta_1, \beta_2 \in [0,\Pi/2)$,we have $\tan\beta_1, \tan\beta_2 \in [0,\infty)$, and consequently from the last inequality it follows that

$$\tan(\beta_1 - \beta_2) \le \tan\beta_1 - \tan\beta_2.$$

Applying ϕ (notice that $\phi(t) = \tan^{-1}t$) to the last inequality and taking into account the increasing character of ϕ, we have

$$\begin{aligned} \phi(\tan(\beta_1 - \beta_2)) &\le \phi(\tan\beta_1 - \tan\beta_2) \\ \tan^{-1}(\tan(\beta_1 - \beta_2)) &\le \tan^{-1}(\tan\beta_1 - \tan\beta_2) \\ \beta_1 - \beta_2 &\le \tan^{-1}(\tan\alpha - \tan\beta). \end{aligned}$$

or equivalently,

$$\tan^{-1}x - \tan^{-1}y = \beta_1 - \beta_2 \le \tan^{-1}(x-y).$$

Thus

$$\begin{aligned} \alpha((0,x),(0,y))d(f(0,x),f(0,y)) &= |\tan^{-1}x - \tan^{-1}y| \\ &\le \tan^{-1}|x-y| \end{aligned}$$

$$
\begin{aligned}
&= \frac{\tan^{-1}|x-y|}{|x-y|}\cdot|x-y| \\
&= \frac{\tan^{-1}(d((0,x),(0,y)))}{d((0,x),(0,y))}\cdot d((0,x),(0,y)) \\
&= \beta(d(0,x),d(0,y))d((0,x),(0,y)),
\end{aligned}
$$

where $\beta(t)=\frac{\tan^{-1}t}{t}$, for $t>0$ and $\beta(0)=0$. Obviously, the inequality is satisfied for $(0,x),(0,y)\in A$ with $x=y$. Also $\beta(t)=\frac{\tan^{-1}(t)}{t}\in\mathscr{F}$. Therefore f is α-Geraghty contraction. Also the pair (A,B) satisfied P-property. Indeed, if for $(0,x_1),(0,x_2)\in A_0$ and $(1,y_1),(1,y_2)\in B_0$ and $x_1=y_1,x_2=y_2$

$$
\begin{aligned}
d((0,x_1),(1,y_1)) &= \sqrt{(1+(x_1-y_1)^2)}=d(A,B)=1,\\
d((0,x_2),(1,y_2)) &= \sqrt{(1+(x_2-y_2)^2)}=d(A,B)=1,
\end{aligned}
$$

consequently

$$d((0,x_1),(0,x_2))=|x_1-x_2|=|y_1-y_2|=d((1,y_1),(1,y_2)).$$

All the conditions of Theorem 3.1 are satisfied. Thus f has unique best proximity point. In this case $(0,0)$ is the unique best proximity point of f.

Notice that in this case B is not closed.

Remark 3.1. The condition A and B are nonempty closed subsets of the metric space (X,d) is not a necessary condition for the existence of the unique best proximity point for α-Geraghty contraction $f:A\to B$, as it is proved in the above example.

Since for any nonempty subset A of X, the pair (A,B) satisfied the P-property, we have the following corollary.

Corollary 3.1. *Let A be a nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty, $\alpha:X\times X\to\mathbb{R}$ be a function. Define a map $f:A\to A$ satisfying the following conditions:*

(1) *f is continuous α-Geraghty contraction;*
(2) *f be a triangular α-admissible;*
(3) *there exists $x_1\in A_0$ such that $\alpha(x_1,fx_1)\geq 1$;*

Then f has a fixed point x^* in A and f is a Picard operator, that is, $f^n(x_1)$ converges to x^*.

Proof. Following Theorem **??** by taking $A=B$, we obtained the desired result. □

Corollary 3.2. *Let A,B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Define a map $f:A\to B$ satisfying the following conditions:*

(1) *f is continuous;*
(2) *f is Geraghty contraction with $f(A_0)\subseteq B_0$;*
(3) *the pair (A,B) has the P-property.*

Then there exists a unique x^* in A such that $d(x^*, fx^*) = d(A,B)$.

Corollary 3.3. *Let A be a nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Define a map $f : A \to A$ such that f is continuous Geraghty contraction. Then f has a fixed point x^* in A and f is a Picard operator, that is, $f^n(x_1)$ converges to x^*.*

Proof. In Corollary 3.1, taking $\alpha(x, y) = 1$ we have the desire result. □

Continuity of the mapping f can be omitted Theorem 3.1. We replace continuity of f with a suitable condition.

Theorem 3.2. *Let A,B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : A \to B$ satisfying the following conditions:*

(1) *f is α-Geraghty contraction with $f(A_0) \subseteq B_0$;*
(2) *f be a triangular α-admissible;*
(3) *there exists $x_1 \in A_0$ such that $\alpha(x_1, fx_1) \ge 1$;*
(4) *the pair (A,B) has the P-property;*
(5) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.*

Then there exists a unique x^* in A such that $d(x^*, fx^*) = d(A,B)$.

Proof. Following Theorem 3.1, we have (x_n) is a Cauchy sequence such that $x_n \to x$ as $n \to \infty$.

Let $x_{m+1}, x_{n+1} \in A_0$ and $fx_m, fx_n \in B_0$, such that

$$d(x_{m+1}, fx_m) = d(A,B),$$
$$d(x_{n+1}, fx_n) = d(A,B),$$

by the P-property, we get

$$d(x_{m+1}, x_{n+1}) = d(fx_m, fx_n).$$

So for $x_n, x_{n+1} \in A_0$, we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) &\le \alpha(x_n, x_{n-1}) d(fx_n, fx_{n-1}) \\ &\le \beta(d(x_n, x_{n-1})) d(x_n, x_{n-1}) \\ &< d(x_n, x_{n-1}) = d(fx_{n-1}, fx_{n-2}), \end{aligned}$$

this implies $\{fx_n\}$ is a Cauchy sequence and $\{fx_n\} \to z$.

Thus $\{d(x_n, x)\} \to 0$ as $n \to \infty$, $\{d(fx_n, fx_{n-1})\} \to 0$ as $n \to \infty$ and

$$d(x_{n+1}, fx_n) = d(A,B).$$

Taking limit as $n \to \infty$, we get

$$d(x, z) = d(A,B).$$

Take a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$, and $\alpha(x_{n(k)},x) \geq 1$.

$$d(fx_{n(k)},fx) \leq \alpha(x_{n(k)},x)d(fx_{n(k)},fx) \leq \beta(x_{n(k)},x)d(x_{n(k)},x).$$

By applying the limit $k \to \infty$

$$d(z,fx) = 0.$$

Thus $d(x,fx) = d(A,B)$. □

4. Uniqueness of best proximity points

In this section, we study sufficient conditions in order to prove the uniqueness of best proximity point.

Definition 4.1. Let $f : A \to B$, $\alpha : X \times X \to [0,\infty)$ be two mappings. A mapping f is called α-regular if for all $x, y \in A_0$ such that $\alpha(x,y) < 1$, there exists $z \in A_0$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Theorem 4.1. *Under the hypothesis of Theorem 2.1, assume that f is α-regular. Then for all best proximity points x and y of f in A_0 we have that $x = y$: In particular, f has a unique best proximity point.*

Proof. Let $x, y \in A_0$ be two best proximity points of f in A_0.Then $d(x,fx) = d(y,fy) = d(A,B)$ and f has P-property, we deduce that

$$d(x,y) = d(fx,fy).$$

We consider two cases:

Case I: If $\alpha(x,y) \geq 1$. Since $x, y \in BPf$, we have

$$\Rightarrow \qquad d(x,fx) = d(A,B) = d(y,fy).$$

By using P-property, we have

$$d(x,y) = d(fx,fy).$$

Using the fact that f is α-Geraghty contraction, we have

$$\begin{aligned} d(x,y) = d(fx,fy) &\leq \alpha(x,y)d(fx,fy) \\ &\leq \beta(d(x,y))d(x,y) \\ &< d(x,y) \end{aligned}$$

$$\Rightarrow \qquad d(x,y) < d(x,y),$$

which is contradiction. So

$$x = y.$$

Case II: If $\alpha(x,y) < 1$, then by the α-regularity of f, there exists $z_0 \in A_0$ such that $\alpha(x,z_0) \geq 1$ and $\alpha(y,z_0) \geq 1$. Based on z_0, we define a sequence $\{z_n\}$ and suppose that z_n converges to x and y, which proves the uniqueness. First, we shall prove that $\{z_n\}$ converges to x.

Indeed, $fz_0 \in fA_0 \subseteq B_0$ implies that $z_1 \in A_0$ such that $d(z_1, fz_0) = d(A,B)$. Following the similar arguments, there exists a sequence $\{z_n\} \subseteq A_0$ such that $d(z_{n+1}, fz_n) = d(A,B)$ for all $n \geq 0$. In particular, $z_{n+1} \in A_0$ and $fz_n \in B_0$. We claim that

$$\alpha(x, z_n) \geq 1, \text{ for all } n \geq 0. \tag{4.1}$$

If $n = 0$, $\alpha(x, z_0) \geq 1$ by the choice of z_0. Suppose that $\alpha(x, z_n) \geq 1$ for some $n \geq 0$. As f is triangular α-admissible, so we have for $x, z_n, z_{n+1} \in A_0$, $\alpha(x, z_n) \geq 1$, $\alpha(z_n, z_{n+1}) \geq 1$ implies $\alpha(x, z_{n+1}) \geq 1$. Hence (4.1) holds for all $n \geq 0$. We have by P-property, $x, z_n, z_{n+1} \in A_0$, $d(x, fx) = d(A,B), d(z_{n+1}, fz_n) = d(A,B)$ imply that $d(x, z_{n+1}) = d(fx, fz_n)$. For all $n \geq 0$, we have

$$\begin{aligned} d(x, z_{n+1}) &= d(fx, fz_n) \\ &\leq \alpha(x, z_n) d(fx, fz_n) \\ &\leq \beta(d(x, z_n)) d(x, z_n) \\ &< d(x, z_n) \end{aligned}$$

which shows that $\{d(x, z_{n+1})\}$ is a decreasing sequence of nonnegative real numbers,and there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x, z_{n+1}) = r$. Assume $r > 0$, then we have

$$0 < \frac{d(x, z_{n+1})}{d(x, z_n)} \leq \beta(d(x, z_n)) < 1, \text{ for any } n \in \mathbb{N}.$$

The last inequality implies that $\lim_{n\to\infty} \beta(d(x, z_n)) = 1$ and since $\beta \in F$, so $r = 0$ and this contradicts our assumption.

Therefore $\lim_{n\to\infty} d(x, z_{n+1}) = 0$, that is $z_{n+1} \to x$ as $n \to \infty$.

Repeating this argument, we have that $z_n \to x$ as $n \to \infty$, which proves that $\{z_n\}$ is a sequence converging to x. Similarly z_n converges to y. By uniqueness of limit we have $x = y$. □

5. Conclusions

Our results formed the best proximity point theorems for α-Geraghty contractions in the setting of complete metric spaces. These theorems extended and cover many existing results in the literature. Moreover, we declared some examples to prove the validity of our results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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