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On the Phragmén-Lindelöf Principle for Entire Power Series on a Banach Algebra

Research Article

Roberto Contreras J.[1]* and Carlos Palomino J.[2]

[1] *Edificio 104C, Ciudad Universitaria, Jardines de San Manuel, Puebla, Mexico*
[2] *Edificio 104D, Ciudad Universitaria, Jardines de San Manuel, Puebla, Mexico*
***Corresponding author:** rcjuarez@cs.buap.mx

Abstract. Many authors have managed to successfully extend the classical theory of analytic functions to functions defined on more abstract spaces (see for example [3, 8, 9]). The purpose of this paper is to provide a bit in this direction. In this paper an extension of the Phragmén–Lindelöf principles of the classical theory to the functions expressed as power series on a Banach algebra not necessarily commutative, of course, involving concepts such as harmonic and subharmonic, is introduced.

Keywords. Banach algebra; Phragmén-Lindelöf principle; Power series

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1. Introduction

The theory of analytic functions on commutative Banach algebras can be developed thanks to the definition of differentiability and analyticity introduced by E.R. Lorch [8]. A considerable portion of the classical theory carries over analytic functions in the Lorch sense [1, 3, 7, 9]. For example, an analytic function in the Lorch sense, also known as L-analytic function, may be expanded in a convergent Taylor series about each point of its domain [7].

In this paper, we work with a power series F on a Banach algebra $\mathbb{B}$ but not necessarily commutative and who is defined as

$$F(w) = \sum_{n=0}^{\infty} a_n w^n, \quad w \in \mathbb{B}, \tag{1.1}$$

with $a_n \in \mathbb{B}$ and $\lim\limits_{n\to\infty} \|a_n\|^{\frac{1}{n}} = 0$ [5]. This function will be called entire power series.

A entire power series F has some properties similar to the L-analytic function. For example, F satisfies the maximum module principle [4]. Of course, we can define the order as in the complex case [1], and it has finite order if exist $\mu > 0$ such that

$$M(r,F) \underset{\text{as}}{\leq} e^{r^{\mu}}, \quad r > 0.$$

The infimum values of μ is called *order* of F and it will be denoted by $\rho(F)$ [2]. Moreover,

$$\rho(F) = \limsup_{r\to\infty} \frac{\ln\ln M(r,F)}{\ln r}.$$

And if $\mathbb{B}$ is a commutative Banach algebra, then F is a L-analytic function in whole Banach algebra $\mathbb{B}$ and it's called L-entire function [7].

On the other hand, in the complex field $\mathbb{C}$, all non-zero element z is invertible and $z^{-1} \to 0$ when $z \to \infty$. In a Banach algebra $\mathbb{B}$ this property is not necessarily truth, then a succession $\{w_n\}$ in $\mathbb{B}$ tends to infinite, denoted by $w_n \to \infty$, if for all $n \in \mathbb{N}$, w_n is invertible and $w_n^{-1} \to 0$ when $n \to \infty$. Therefore, a curve $\alpha : [0,1[\to \mathbb{B}$ tend to ∞ and write $\alpha(t) \to \infty$, if there is a succession $\{t_n\}$ such that $t_n \to 1$ and $\alpha(t_n) \to \infty$ when $t_n \to 1$.

2. $\mathbb{B}$-harmonic and $\mathbb{B}$-subharmonic Functions

Based in the fact that the harmonic functions in the complex field $\mathbb{C}$ satisfy the mean-value property [6, 10], in this section are introduced the concepts of $\mathbb{B}$-harmonic and $\mathbb{B}$-subharmonic functions, some properties of these functions and their relationship with the entire powers series will be analyzed.

Definition 2.1. *Let $u : \mathbb{B} \to \bar{\mathbb{R}}$ denote a real function which may assume the value $-\infty$, and let $e^{u(w)}$ be a continuous function in the domain Ω of the Banach algebra $\mathbb{B}$. If for any $w, w_0 \in \Omega$, such that $\{w + e^{i\theta} w_0 : 0 \leq \theta \leq 2\pi\} \subseteq \Omega$ is satisfied*

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + e^{i\theta} w_0) d\theta, \tag{2.1}$$

u is called $\mathbb{B}$-subharmonic function in Ω. If the equality is always satisfied, u will be called $\mathbb{B}$-harmonic in Ω.

Of course, the $\mathbb{B}$-subharmonic functions have a maximum principle, its proof is similar to the complex case [7, 10].

Proposition 2.2 (Maximum Principle)**.** *Let u be $\mathbb{B}$-subharmonic function in the domain Ω and suppose that $M = \sup_{\Omega} u(w)$. If $u(w_0) = M$ with $w_0 \in \Omega$, then $u(w) \equiv M$.*

As a consequence of this proposition, the following corollary is obtained.

Corollary 2.3. *Let u be $\mathbb{B}$-subharmonic function and let v be $\mathbb{B}$-harmonic function in the domain Ω. If for all $w \in \partial\Omega$*

$$u(w) \le v(w),$$

then for all $w \in \Omega$, $u(w) \le v(w)$.

The main interest of the subharmonic functions is because they have important connections with the entire power series. The simplest expressions of this relationship are given in the following propositions.

Proposition 2.4. *The norm of an entire power series on $\mathbb{B}$, is a $\mathbb{B}$-subharmonic function in the whole $\mathbb{B}$.*

Proof. By Proposition 1 of [4],

$$F(w) = \frac{1}{2\pi}\int_0^{2\pi} F(w + re^{i\theta}a)\,d\theta,$$

where $a \in \mathbb{B}$. Taking norm in both sides, the result is followed. □

Proposition 2.5. *If F is an entire power series on $\mathbb{B}$, then the function $\ln\|F(w)\|$ is a $\mathbb{B}$-subharmonic function in the whole $\mathbb{B}$.*

Proof. Let f be the function defined by

$$f(z) = F(w + zw_0),$$

where $z \in \mathbb{C}$ and $w, w_0 \in \mathbb{B}$. Since F is an entire power series, then

$$f(z) = \sum_{n=0}^{\infty} a_n (w + zw_0)^n, \quad a_n, w, w_0 \in \mathbb{B},\ z \in \mathbb{C},$$

and f converges for all $z \in \mathbb{C}$.

Furthermore, considering that $\mathbb{B}$ is not necessarily a commutative Banach algebra, there are elements that depend on w, w_0 and $\{a_n\}$ coefficients of F, namely, $\varphi_k(w, w_0, \{a_n\}) \in \mathbb{B}$, $k \in \mathbb{N}$ such that

$$f(z) = \sum_{k=0}^{\infty} \varphi_k(w, w_0, \{a_n\}) z^k,$$

since f converges for all $z \in \mathbb{C}$, its radius of convergence is infinite so

$$\lim_{k\to\infty} \left\|\varphi_k(w, w_0, \{a_n\})\right\|^{\frac{1}{k}} = 0.$$

Thus, f is an entire function of a complex variable with values in a Banach space, this implies that $\ln\|f(z)\|$ is a subharmonic function in $\mathbb{C}$, i.e.

$$\ln\|f(z)\| \le \frac{1}{2\pi}\int_0^{2\pi} \ln\|f(z+re^{i\theta})\|d\theta.$$

In particular taking $z=0$ and $r=1$,

$$\ln\|f(0)\| \le \frac{1}{2\pi}\int_0^{2\pi} \ln\|f(e^{i\theta})\|d\theta,$$

then

$$\ln\|F(w)\| \le \frac{1}{2\pi}\int_0^{2\pi} \ln\|F(w+e^{i\theta}w_0)\|d\theta,$$

and since w, w_0 were arbitrary, it is shown that $\ln\|F(w)\|$ is a $\mathbb{B}$-subharmonic in the whole $\mathbb{B}$. □

3. PL-domains and PL-measures

At the first glance, Phragmén-Lindelöf theorems seem to be an exclusively analytic and subharmonic functions phenomenon [6, 10]. Even though E.R. Lorch has introduced a definition of differentiability and analyticity for functions defined on Banach algebras [8], we don't have a definition of differentiability for functions on noncommutative algebras.

The picture changes, however, if we adopt a more abstract point of view. First, we may consider entire power series defined on Banach algebras not necessarily commutative. Second, if G is a simply connected domain in $\mathbb{C}$ having the points $z=0$ and $z=\infty$ among its boundary points and $\mu_{\mathbb{C}}$ is the usual measure in $\mathbb{C}$, the function

$$\nu(t) = \frac{\mu_{\mathbb{C}}(G\cap C_t)}{\mu_{\mathbb{C}}(C_1)}, \tag{3.1}$$

where $C_t = \{z\in\mathbb{C} : |t| = t,\ t>1\}$, has some important properties, namely,

(1) For each $t>1$, $\nu(t)>0$.

(2) If the domain G is an angular sector with magnitude $\frac{\pi}{\alpha}$, then $\nu(t)=\frac{t}{\alpha}$.

(3) For each $r>1$, the integral

$$\int_1^r \frac{1}{\nu(t)}dt,$$

is finite.

On the other hand, we can consider the domain Ω in a Banach algebra with identity element $1_{\mathbb{B}}$, which satisfies the following properties.

(a) There is a succession $\{w_n\}$ in Ω, such that, $w_n\to 0$ if $n\to\infty$.

(b) There is a succession $\{u_n\}$ in Ω, such that, $u_n\to\infty$ if $n\to\infty$.

Then, we can say that the domain Ω has "zero" and "infinite" among its boundary points.

Now consider $\Gamma(\Omega)$ the set of all $\mathbb{R}$-subspaces of dimension two in $\mathbb{B}$ such that if $W \in \Gamma(\Omega)$, then

(1) $\Omega_W = W \cap \Omega \neq \emptyset$.

(2) There are two continuous functions $\alpha, \beta : [0,1[\to \mathbb{B}$ that satisfy

  (i) $\alpha(0) = \beta(0) = 0$ and $\alpha(t) \neq \beta(t)$ if $t \in]0,1[$.

  (ii) $\alpha(t) \to \infty$ and $\beta(t) \to \infty$ when $t \to 1$.

  (iii) $\alpha([0,1[) \cup \beta([0,1[) = W \cap \partial\Omega$.

Since W is a two dimensional $\mathbb{R}$-subspace, then W is isomorphic to $\mathbb{C}$ is the field of complex numbers. Let $\gamma_W : W \to \mathbb{C}$ be one isomorphism. Therefore, $D_W = \gamma_W(\Omega_W)$ is a domain in $\mathbb{C}$ containing 0 and ∞ among its boundary points. This analysis motivates the following definition.

Definition 3.1. Let $\mathbb{B}$ be a Banach algebra with identity $1_{\mathbb{B}}$. A PL-domain of $\mathbb{B}$ is a domain $\Omega \subset \mathbb{B}$ for which

(1) There is a succession $\{w_n\}$ in Ω, such that, $w_n \to 0$ if $n \to \infty$.

(2) There is a succession $\{u_n\}$ in Ω, such that, $u_n \to \infty$ if $n \to \infty$.

(3) The set $\Gamma(\Omega)$ containing all $\mathbb{R}$-subspaces of dimension two of $\mathbb{B}$ such that if $W \in \Gamma(\Omega)$,

  (i) $\Omega_W = W \cap \Omega \neq \emptyset$.

  (ii) There are two continuous functions $\alpha, \beta : [0,1[\to \mathbb{B}$ that satisfy:

    (a) $\alpha(0) = \beta(0) = 0$ and $\beta(t) \to \infty$ when $t \to 1$,

    (b) $\alpha(t) \to \infty$ and $\beta(t) \to \infty$ when $t \to 1$,

    (c) $\alpha([0,1[) \cup \beta([0,1[) = W \cap \partial\Omega$,

  is no empty.

The following analysis was taken from [6], it was adapted according to our needs.

Let Ω be a PL-domain in a Banach algebra with identity $1_{\mathbb{B}}$. If $\Delta \subset \partial\Omega$, then, there are some subspaces W in $\Gamma(\Omega)$, such that $W \cap \Delta \neq \emptyset$. Let us take

$$\Gamma_\Omega(\Delta) = \{W \in \Gamma(W) : W \cap \Delta \neq \emptyset\}. \tag{3.2}$$

Suppose that F is an entire power series satisfying

$$\begin{aligned} &\|F(w)\| \le m \quad \text{if } w \in \Delta, \\ &\|F(w)\| \le M \quad \text{if } w \in \partial\Omega \setminus \Delta. \end{aligned} \tag{3.3}$$

Then, for each $W \in \Gamma_\Omega(\Delta)$ and for an arbitrary $w_0 \in \Omega_W$,

$$\ln \|F(w_0)\| \le \varphi_W(\gamma_W(w_0)) \ln m + (1 - \varphi_W(\gamma_W(w_0))) \ln M, \tag{3.4}$$

where $\varphi_W(\gamma_W(w_0))$ is the harmonic measure of $\gamma_W(W \cap \Delta)$ relative to $D_W = \gamma_W(\Omega_W)$ which satisfies

$$\varphi_W(\gamma_W(w))\begin{cases} 1 & \text{if } w \in W \cap \Delta, \\ 0 & \text{if } w \in \partial\Omega \setminus (W \cap \Delta). \end{cases} \tag{3.5}$$

From now on, just write $\varphi_W((\gamma_W(w_0)), \gamma_W(W \cap \Delta), D_W)$ to denote the harmonic measure of $\gamma_W(W \cap \Delta)$ relative to $D_W = \gamma_W(\Omega_W)$ in $\gamma_W(w_0)$.

The proof is similar to the complex case [6]. The left-hand term of the inequality to be proved is a $\mathbb{B}$-subharmonic function in domain Ω and the right-hand term is $\mathbb{B}$-harmonic. The inequality is satisfied for the limiting values on the boundary, since according to (3.5), the function $\varphi_W((\gamma_W(w_0))$ becomes unity at points of the set Δ and vanishes at the remaining points of the boundary of Ω . Applying Corollary 2.3, we obtain the statement.

Now, let $\chi_W(z)$ denote a function that maps the domain D_W conformally onto the half-plan $\mathrm{Re}(\chi_W) > 0$ and that maps the point $z = 0$ and $z = \infty$ into the points $\chi = 0$ and $\chi = \infty$, respectively. Let $z_W(\chi)$ denote the inverse of function $\chi_W(z)$.

We choose any point $z_0 \in D_W$ and a sufficiently large $\rho > 0$. Let $D_W(\rho)$ be that part of the domain D_W contained in th circle $\Gamma_\rho = \{z : |z| < \rho\}$ and containing the point z_0.

For $\rho > |z_0|$ the domain $D_W(\rho)$ is not empty. Let $K_W(\rho)$ denote the image of the domain $D_W(\rho)$ under the mapping $\chi_W = \chi_W(z)$. The domain $K_W(\rho)$ is bounded by segments of the imaginary axis and some curves $L_W(\rho)$, the images of the arcs of the circle contained in $s_W(\rho) = D_W \cap \{z : |z| = \rho\}$. Among the curves $L_W(\rho)$ there must be at least one that joins the positive part with the negative part of the imaginary axis. Let $L^*_W(\rho)$ be such a curve; let $s^*_W(\rho)$ be the arc of $s_W(\rho)$ that is the preimage of $L^*_W(\rho)$ and $\mu_{\mathbb{C}}(s^*_W(\rho))$ denote the length of $s^*_W(\rho)$, here $\mu_{\mathbb{C}}$ is the usual measure of $\mathbb{C}$. The domain bounded by the curve $L^*_W(\rho)$ and the line segment joining its endpoints is denoted by $K^*_W(\rho)$, whose preimage is denoted by $D^*_W(\rho)$.

The domain $K_W(\rho)$ may be imagined as a distorted semicircle. We now determine the radius of the largest semicircle $\{\chi_W : \mathrm{Im}(\chi_W) > 0, |\chi_W| < r(\rho)\}$, contained in domain $K^*_W(\rho)$, i.e., the quantity

$$r(\rho) = \inf_{z \in s^*_W(\rho)} |\chi_W(z)|.$$

We use Ahlfors' inequality for the determination of $r(\rho)$ [6]. To reduce the problem to the same form as that used in this inequality, we denote by G_W th image of the domain D_W under the mapping $x = \ln z$ that is regular in the domain D_W no matter which branch we choose.Then the function $\tau(x) = \ln \chi_W(e^x)$ maps the domain G_W conformally onto the strip $|\mathrm{Im}(\tau)| < \frac{\pi}{2}$ and the intersection of the domain G_W with the lines $\mathrm{Re}(x) = t$ are the images of the intersection of the domain D_W with the circles $|z| = e^t$ under the mapping $x = \ln z$. Here the segment λ_t is the image of the arc $s^*_W(e^t)$ and

$$\lambda(t) = \frac{\mu(s^*_W(e^t))}{e^t}, \quad \ln r(e^t) = \sup_{x \in \lambda_t} \mathrm{Re}(\tau(x)).$$

Thus, Ahlfors' inequality gives us

$$\begin{aligned} r(\rho) &\geq \exp\left\{C+\pi\int_0^{\ln\rho}\frac{dt}{\lambda(t)}\right\} \\ &= e^C\cdot\exp\left\{\pi\int_1^{\rho}\frac{d\tau}{\mu_{\mathbb{C}}(s_W^*(\tau))}\right\} \\ &\geq e^C\cdot\left\{\pi\int_1^{\rho}\frac{d\tau}{\nu(\tau))}\right\} \\ &\geq e^C\sigma_W(\rho). \end{aligned} \tag{3.6}$$

Here, $\nu(\tau)$ is defined like in (3.1).

We showed that the domain $K_W^*(\rho)$ contains the semicircle $\operatorname{Re}(\chi_W)>0$, $|\chi_W|<r(\rho)$, where $r(\rho)>C'\sigma_W(\rho)$. According to the principle of extension of domains [6], the harmonic measure $u(\chi_0,L_W^*(\rho),K_W^*(\rho))$ does not exceed the harmonic measure of the semi-circumference relative to the semicircle $\operatorname{Re}(\chi_W)>0$, $|\chi_W|<r(\rho)$ at this point. This harmonic measure equals

$$\frac{2}{\pi}\operatorname{Im}\left(\ln\frac{r(\rho)-i\chi_0}{r(\rho)+i\chi_0}\right)\sim\frac{1}{r(\rho)}\cdot\frac{4}{\pi}\operatorname{Re}(\chi_0).$$

Therefore, in view of inequality (3.6), if $\rho\to\infty$ and χ_0 fixed, we have

$$\varphi_W(\chi_0,L_W^*(\rho),K_W^*(\rho))\leq C_1\frac{1}{\sigma_W(\rho)}. \tag{3.7}$$

Consider the entire power series $G(\gamma_W^{-1}(\chi_W))=F(\gamma_W^{-1}(z_W(\chi_W)))$, in the domain $K_W^*(\rho)$. If we suppose that $\|F(w)\|\leq A$, for all $w\in\partial\Omega$, since the imaginary axis goes into the boundary of the domain D_W under the mapping $\chi_W=\chi_W(z)$, and D_W is the image of $\Omega\cap W$ under the isomorphism γ_W. So

$$\begin{aligned} \sup_{w\in L_W^*(\rho)}\|G(\gamma_W^{-1}(\chi_W))\| &= \sup_{w\in S_W^*(\rho)}\|F(\gamma_W^{-1}(z_W(\chi_W)))\| \\ &\leq \sup_{w\in s_W(\rho)}\|F(w)\|=M(F,\rho). \end{aligned}$$

By (3.4) and (3.7)

$$\ln\|F(w)\|\leq C_!\frac{\ln M(F,\rho)}{\sigma_W(\rho)}+(1-\varphi(\chi_0,L_W^*(\rho),K_W^*(\rho)))\ln A. \tag{3.8}$$

Definition 3.2. A positive Borel measure μ on the borel sets of $\mathbb{B}$ is a PL-measure if it satisfies:

(1) $\mu(C_r)>0$, where $C_r=\{w\in\mathbb{B}:\|w\|=r,\ r>0\}$

(2) $\mu(C_r)=\mu(C_1)\cdot\beta(r)$, with $\beta(r)$ is a continuous function on $]0,+\infty]$

(3) For all PL-domain $\Omega\subseteq\mathbb{B}$

 (a) $\mu(\Omega\cap C_r)\geq 0$, $r>0$

(b) For each $W \in \Gamma(\Omega)$, $\mu(W \cap C_r) = \mu_{\mathbb{C}}(D_r) = 2\pi r$

(c) For each $W \in \Gamma_{\Omega}(\Delta)$, $\nu_W(t) \le \phi(t)$, $t > 1$ where $\Delta \subset \partial\Omega$ and

$$\nu_W(t) = \frac{\mu_{\mathbb{C}}(D_W \cap C_t)}{\mu_{\mathbb{C}}(D_1)}, \quad \phi(t) = \frac{\mu(\Omega \cap C_t)}{\mu(C_1)}.$$

For $r > 1$, we define the expression

$$\theta(r) = \exp\left\{\frac{1}{2}\int_1^r \frac{dt}{\phi(t)}\right\}.$$

Further, by (3c) of Definition 3.2, for all $W \in \Gamma_{\Omega}(\Delta)$,

$$\theta(r) \le \nu_W(r) \quad \text{for } r > 1. \tag{3.9}$$

4. Phragmen-Lindelöf's Theorems Type

Now, we can enunciate the Phragmen-Lindelöf's theorem to entire power series. These are generalizations of the maximum modulus principle in which we infer the boundedness of a function inside an unbounded domain from the hypothesis that the function is bounded on the boundary and not of too rapid growth inside. The first result is a Lindelöf's theorem type.

Proposition 4.1. *Let $\mathbb{B}$ be a Banach algebra with identity $1_{\mathbb{B}}$. Let F be an entire power series on $\mathbb{B}$ and let Ω be a PL-domain of $\mathbb{B}$. Suppose $F(w) \to \alpha$ if $w \to \infty$ on $\partial\Omega$ and $\|F(w)\| \le M$ on Ω. Then $F(w) \to \alpha$ if $w \to \infty$ on the whole Ω.*

Proof. We take $R > 0$ and consider the entire power series $F(w) - \alpha$ on the domain

$$\Omega_R = \{w \in \Omega : \|w\| > R\}.$$

We define

$$\epsilon(R) = \sup w \in \partial\Omega_R \|F(w) - \alpha\| \quad \text{and} \quad M(F,R) = \sup_{w \in \Omega, \|w\| = R} \|F(w) - \alpha\|,$$

where $\partial\Omega_R = \{w \in \partial\Omega : \|w\| \ge R\}$. Note that both numbers, $\epsilon(R)$ and $M(F,R)$ are finite.

Let $a, a' \in \partial\Omega$. By (3.4), if $W \in \Gamma(a,a')$ and $w \in \Omega_W$

$$\begin{aligned} \ln\|F(w) - \alpha\| &\le \varphi(\gamma_W(w), \partial\Omega_R)\ln\epsilon(R) + (1 - \varphi(\gamma_W(w), \partial\Omega_R))\ln M(F,R) \\ &\le \varphi(\gamma_W(w), \partial\Omega_R)\ln\epsilon(R) + \ln^+ M(F,R), \end{aligned}$$

where

$$\ln^+ M(F,R) = \max\{\ln M(F,R), 0\}.$$

For R sufficiently large, $\epsilon(R) < 1$, so

$$\ln\|F(w) - \alpha\| \le \varphi_W(w)\ln\varepsilon(R).$$

Since $\partial\Omega_{\subseteq}\partial\Omega$ and $\Omega_R \subseteq \Omega$, by the principle of extension of domains,

$$\varphi_W(\gamma_W(w), \partial\Omega_R) \le \varphi_W(\gamma_W(w), \partial\Omega).$$

So

$$\ln \|F(w) - \alpha\| \le \varphi_W(w) \ln \epsilon(R).$$

On the other hand, $\varphi_W(\gamma_W(w), \partial\Omega)$ does not depend on R and $\epsilon(R) \to 0$ when $r \to \infty$, the proposition follows. □

Next result is a generalization of the known fact in the classical theory like fundamental theorem of Phragmen-Lindelöf.

Proposition 4.2. *Let $\mathbb{B}$ be a Banach algebra with unit. Let μ a PL-measure and Ω a PL-domain of $\mathbb{B}$. Suppose that $\|F(w)\| \le 1$ on $\partial\Omega$ and*

$$\lim_{r\to\infty} \frac{\ln \widetilde{M}(F,r)}{\theta(r)} = 0$$

where

$$\widetilde{M}(F,r) = \sup_{w\in\Omega\cap C_r} \|F(w)\|, \quad \theta(r) = \exp \frac{1}{2}\int_1^r \frac{dt}{\phi(t)}$$

and

$$\phi(t) = \frac{\mu(\Omega \cap C_t)}{\mu(C_1)}.$$

Then $\|F(w)\| \le 1$ in whole Ω.

Proof. Consider $\Omega_R = \{w \in \Omega : \|w\| < R\}$ for $R > 0$. Let $\Delta_R = C_R \cap \Omega$ and let $\overline{\partial\Omega_R} = \{w \in \partial\Omega : \|w\| \le R\}$. Then

$$\|F(w)\| \le \widetilde{M}(F,R), \quad \text{if } w \in \Delta_R,$$

$$\|F(w)\| \le 1 \qquad\qquad \text{if } w \in \overline{\partial\Omega_R}.$$

Now, by (3.8) for all $W \in \Gamma_\Omega(\Delta)$

$$\ln \|F(w)\| \le C_1 \frac{\ln \widetilde{M}(F,R)}{\sigma_W(R)},$$

and by (3.9)

$$\ln \|F(w)\| \le C_1 \frac{\ln \widetilde{M}(F,R)}{\theta(R)},$$

for all $w \in \Omega$.

According to the hypothesis, there exists a sequence $R_n \to \infty$ for which

$$\frac{\ln \widetilde{M}(F,R)}{\theta(R)} \to 0.$$

Therefor

$$\ln \|F(w)\| \le 0\,.$$

It follows from this that

$$\|F(w)\| \le 1,$$

for all $w \in \Omega$. □

Finally, the second Lindelöf's Theorem is enunciated.

Proposition 4.3. *Let $\mathbb{B}$ be a Banach algebra with unit and let F a entire power series with finite order. Let Ω_1 and Ω_2 are two PL-domains of $\mathbb{B}$. Suppose that $F(w) \to a$ if $w \to \infty$ on Ω_1 and $F(w) \to b$ if $w \to \infty$ on Ω_2. Let us define*

$$\Omega_{1R} = \{w \in \Omega_1 : \|w\| > R\},$$

$$\Omega_{2R} = \{w \in \Omega_2 : \|w\| > R\}.$$

If for all $R > 0$, $\Omega_{1R} \cap \Omega_{2R} \neq \emptyset$, then $a = b$ and $F(w) \to a$ if $w \to \infty$ on $\Omega_1 \cup \Omega_2$.

Proof. Since $F(w) \to a$ if $w \to \infty$ on Ω_1, for each $\epsilon > 0$ there is a $R_1 > 0$ such that

$$\|F(w) - a\| < \frac{\epsilon}{2} \quad \text{if } \|w\| > R_1\,.$$

In similar way, since $F(w) \to b$ if $w \to \infty$ on Ω_2, for each $\epsilon > 0$ there is a $R_2 > 0$ such that

$$\|F(w) - b\| < \frac{\epsilon}{2}, \quad \text{if } \|w\| > R_2\,.$$

Let $R = \max\{R_1, R_2\}$ and $w \in \Omega_{1R} \cap \Omega_{2R}$, then

$$\begin{aligned} \|a - b\| &= \|a - F(w) + F(w) - b\| \\ &\le \|F(w) - a\| + \|F(w) - b\| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

As ϵ is arbitrary, $a = b$. Other part is obvious. □

5. Conclusion

There is another important aspect of the asymptotic behavior of analytic functions of a complex variable that can be extended to power series defined on a Banach algebra, namely the extension of the Phragmen-Lindelöf's function (see [7]). Our main goal now is to study the possible practical applications that such extensions may have to Banach algebras, and the possible extension of the indicator function its properties and applications [11].

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both the authors contributed equally and significantly in writing this article. Both the authors read and approved the final manuscript.

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