



Quadratic-Phase Hankel Transformation and Calderón's Reproducing Formula

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Abstract. In this paper, we have explored fundamental properties of the quadratic-phase Hankel transformation. Additionally, we have derived Calderón's reproducing formula for quadratic-phase Hankel convolution based on the theory of the quadratic-phase Hankel transformation.

Keywords. Hankel transformation, Quadratic-phase Hankel transformation, Convolution, Calderón's formula

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1. Introduction

The Calderón's reproducing formula is a fundamental construction in harmonic analysis, providing a method for decomposing functions into basic elements that facilitate detailed analysis. This formula is crucial in studying singular integrals, wavelet theory, and various applications in signal processing and partial differential equations. One interesting extension of Calderón's reproducing formula involves its application through Hankel convolution, which is particularly useful for problems exhibiting radial symmetry.

Unlike standard convolution, Hankel convolution is tailored for radial functions and problems with cylindrical symmetry. It is defined via the Hankel transform, an analog of the Fourier transform specifically suited for radial functions in higher dimensions. This transform and the resulting convolution are key tools in analyzing functions with inherent radial structures.

Integrating Calderón's reproducing formula with Hankel convolution allows for the decomposition of functions in radially symmetric contexts. This integration leverages the properties of the Hankel transform, enabling the representation of functions as integrals over their convolutions with specific kernel functions. This formulation not only maintains the foundational principles of the original Calderón formula but also extends its applicability to a broader range of problems, particularly those involving cylindrical coordinates or radial symmetry.

The classical Calderón's formula [1] can be expressed as:

$$f = \int_0^\infty f \star g_k \star h_k \frac{dk}{k}, \quad (1.1)$$

where \star represents the classical convolution operation on the set \mathbb{R} . It was initially applied to singular integral operators in the Calderón-Zygmund theory. However, it was later extended to other fields of practical mathematics, such as wavelet theory (Daubechies [3], Frazier *et al.* [5]). Building upon the work of Frazier *et al.* [5], Pathak and Pandey [6], introduced the Calderón's formula related to Hankel convolution. With Watson convolution, Upadhyay and Tripathi [11] established the reproducing formula of Calderón's further expanding the theory of Frazier *et al.* [5], and Pathak and Pandey [6].

In this work, we have now defined Calderón's formula in the context of quadratic-phase Hankel transformations. The quadratic-phase Hankel transformations $H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e}$, which depend on five real parameters a, b, c, d, e , $b \neq 0$ and four additional real-valued parameters $(\mu, \nu, \alpha, \beta)$, where $\nu\mu + 2\nu - \alpha \geq 1$, for any function f on $I = (0, \infty)$, is defined as follows (Prasad *et al.* [10]):

$$(H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e} f)(\omega) = \int_0^\infty K_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e}(\omega, t) f(t) dt, \quad (1.2)$$

where

$$K_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e}(\omega, t) = \nu\beta \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} t^{-1-2\alpha+2\nu} e^{i\beta(\alpha t^{2\nu} + c\omega^{2\nu} + dt^\nu + e\omega^\nu)} (t\omega)^\alpha J_\mu\left(\frac{\beta}{b}(t\omega)^\nu\right), \quad (1.3)$$

where J_μ represents the Bessel function of first kind with order μ .

The inversion formula of (1.2) is as follows:

$$\begin{aligned} f(t) &= (H_{\mu,\nu,\alpha,\beta}^{-c,-b,-\alpha;-e,-d} (H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e} f)(\omega))(t) \\ &= \int_0^\infty K_{\mu,\nu,\alpha,\beta}^{-c,-b,-\alpha;-e,-d}(t, \omega) (H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e} f)(\omega) d\omega, \end{aligned}$$

and Dirac delta function is given by

$$\delta(\omega - t) = \left(\frac{\nu\beta}{b}\right)^2 \omega^\alpha t^{2\nu-\alpha-1} \int_0^\infty t^{2\nu-1} J_\mu\left(\frac{\beta}{b}(\omega x)^\nu\right) J_\mu\left(\frac{\beta}{b}(tx)^\nu\right) dx. \quad (1.4)$$

The Parseval’s relation is given by

$$\int_0^\infty f(t)\overline{g(t)}t^{-1-2\alpha+2\nu}dx = \int_0^\infty (H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e} f)(\omega)\overline{(H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e} g)(\omega)}\omega^{-1-2\alpha+2\nu}d\omega. \tag{1.5}$$

In this paper, we will be using a specific case of the quadratic-phase Hankel transformation $H_{\mu,\nu,\alpha,\beta}^{a,b,c;d,e}$ for $\nu = \beta = 1, \alpha = -\mu$ and it is denoted by $H_\mu^{a,b,c;d,e}$. Therefore, the Quadratic-Phase Hankel Transformation (QPHT) of a function f with order $\mu \geq -\frac{1}{2}$ can be simplified as follows:

$$(H_\mu^{a,b,c;d,e} f)(\omega) = \tilde{f}(\omega) = \int_0^\infty K_\mu^{a,b,c;d,e}(\omega, t)f(t)dt, \tag{1.6}$$

where

$$K_\mu^{a,b,c;d,e}(\omega, t) = \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}t^{1+2\mu}e^{i(at^2+c\omega^2+dt+c\omega)}(t\omega)^{-\mu}J_\mu\left(\frac{t\omega}{b}\right), \quad b \neq 0.$$

The inversion formula for (1.6) is as follows:

$$f(t) = ((H_\mu^{-c,-b,-a;-e,-d})\tilde{f})(t) = \int_0^\infty K_\mu^{-c,-b,-a;-e,-d}(t, \omega)\tilde{f}(\omega)d\omega. \tag{1.7}$$

For the operator $H_\mu^{a,b,c;d,e}$, the Parseval equality becomes

$$\int_0^\infty f(t)\overline{g(t)}t^{1+2\mu}dt = \int_0^\infty \tilde{f}(\omega)\overline{\tilde{g}(\omega)}\omega^{1+2\mu}d\omega. \tag{1.8}$$

Definition 1.1. As per Prasad and Kumar [9], and Prasad et al. [10], the space $L_\mu^p(I)$, $\mu \in \mathbb{R}$ consists of all real-valued measurable function ϕ on $I = (0, \infty)$ for which the norm

$$\|\phi\|_{L_\mu^p} = \begin{cases} \left(\int_0^\infty |\phi(t)|^p t^{1+2\mu} dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in I} |\phi(t)|, & p = \infty, \end{cases}$$

is finite.

As per Pathak and Dixit [7], Prasad and Kumar [9], and Prasad et al. [10], we have the following quadratic-phase Hankel convolution of the functions ϕ and $\psi \in L_\mu^1(I)$ for $\nu = \beta = 1$ and $\alpha = -\mu$:

$$(\psi \star \phi)(t) = \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \phi(\omega)\tau_t^{a,b;d}\psi(\omega)e^{i(a\omega^2+d\omega)}\omega^{1+2\mu}d\omega, \tag{1.9}$$

where the quadratic-phase Hankel translation $\tau_t^{a,b;d}$ is given as:

$$\begin{aligned} (\tau_t^{a,b;d}\psi)(\omega) &= \psi^{a,b;d}(t, \omega) \\ &= \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \psi(z)D_\mu^{a,b;d}(t, \omega, z)e^{i(az^2+dz)}z^{1+2\mu}dz, \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} D_\mu^{a,b;d}(t, \omega, z) &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty (ts)^{-\mu}J_\mu\left(\frac{ts}{b}\right)(\omega s)^{-\mu}J_\mu\left(\frac{\omega s}{b}\right)(zs)^{-\mu}J_\mu\left(\frac{zs}{b}\right) \\ &\quad \times e^{-i(a(t^2+\omega^2+z^2)+d(t+\omega+z))}s^{1+2\mu}ds. \end{aligned}$$

For any values of t, ω and z such that $0 < t, \omega, z < \infty$, we have

$$\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty (zs)^{-\mu}J_\mu\left(\frac{zs}{b}\right)e^{i(az^2+cs^2+dz+es)}z^{1+2\mu}D_\mu^{a,b;d}(t, \omega, z)dz$$

$$= (ts)^{-\mu} J_{\mu} \left(\frac{ts}{b} \right) (\omega s)^{-\mu} J_{\mu} \left(\frac{\omega s}{b} \right) e^{-i(at^2+dt+a\omega^2+d\omega)} e^{i(cs^2+es)},$$

and

$$\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^{\infty} D_{\mu}^{a,b;d}(t, \omega, z) e^{i(az^2+dz)} z^{1+2\mu} dz = \frac{e^{-i(at^2+dt+a\omega^2+d\omega)}}{(2b)^{\mu} \Gamma(\mu+1)}. \quad (1.11)$$

Moreover, as per Chui [2], Debnath and Shah [4], Pathak and Dixit [7], and Prasad and Kumar [9] for $\nu = \beta = 1$ and $\alpha = -\mu$ the quadratic-phase Hankel wavelet $\psi_{n,m}^{a,b;d}$ of $\psi \in L_{\mu}^2(I)$ with dilation parameters $m > 0$ and translation parameters $n \geq 0$ is expressed as:

$$\psi_{n,m}^{a,b;d} = \mathcal{D}_m(\tau_n^{a,b;d} \psi)(t) = \mathcal{D}_m \psi^{a,b;d}(n, t) = \psi_m^{a,b;d}(n, t) \quad (1.12)$$

$$= m^{-\frac{5}{2}-2\mu} e^{i\left(a\left(\frac{1}{m^2}-1\right)(t^2+n^2)+d\left(\frac{1}{m}-1\right)(t+n)\right)} \psi^{a,b;d}\left(\frac{n}{m}, \frac{t}{m}\right)$$

$$= m^{-\frac{5}{2}-2\mu} e^{i\left(a\left(\frac{1}{m^2}-1\right)(t^2+n^2)+d\left(\frac{1}{m}-1\right)(t+n)\right)} \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^{\infty} \psi(z) D_{\mu}^{a,b;d}\left(\frac{n}{m}, \frac{t}{m}, z\right) e^{i(az^2+dz)} z^{1+2\mu} dz. \quad (1.13)$$

Here, \mathcal{D}_m represents the dilation operator.

The wavelet transform involving *Quadratic-Phase Hankel* (QPH) wavelet is define as:

$$W_{\psi}^{a,b;d} f(n, m) = \int_0^{\infty} f(t) \overline{\psi_{n,m}^{a,b;d}(t)} t^{1+2\mu} dt, \quad (1.14)$$

and accordingly, the admissibility condition for the QPH-wavelet is as follows:

$$C_{\psi}^{\mu,a,b;d} = \int_0^{\infty} \frac{|(H_{\mu}^{a,b,c;d,e} e^{i(a(\cdot)^2+d(\cdot))} \psi)(\omega)|^2}{\omega} d\omega < \infty. \quad (1.15)$$

The reconstruction formula (inversion formula) for (1.14) is as follows:

$$f(t) = \frac{1}{b^2 C_{\psi,\phi}^{\mu,a,b;d}} \int_0^{\infty} \int_0^{\infty} (W_{\psi}^{a,b;d} f)(n, m) \phi_{n,m}^{a,b;d}(t) n^{1+2\mu} dn dm. \quad (1.16)$$

There are four sections of the paper. A brief introduction to QPHT, the wavelet transform related to the specific QPHT instance, and Calderón's formula are provided in Section 1. In Section 2, some estimates related to quadratic-phase Hankel wavelet, convolution, and translation are presented. A few properties of the quadratic-phase Hankel wavelet transform are studied in Section 3. Calderón's reproducing formula involving quadratic-phase Hankel wavelet transform is obtained in the last section.

2. Preliminaries

In this section, preliminary results for the quadratic-phase Hankel translation, convolution, and dilation are listed:

Lemma 2.1. Let $f \in L_{\mu}^p(I)$ and $g \in L_{\mu}^q(I)$, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$(i) \ \|(\tau_t^{a,b;d} f)(\omega)\|_{L_{\mu}^p} \leq \frac{1}{(2b)^{\mu} \Gamma(\mu+1)} \|f\|_{L_{\mu}^p},$$

$$(ii) \ \|e^{-i(a(\cdot)^2+d(\cdot))} f \star g\|_{L_{\mu}^{\infty}} \leq \frac{1}{2^{\mu} b^{\mu+1} \Gamma(\mu+1)} \|f\|_{L_{\mu}^p} \|g\|_{L_{\mu}^q}.$$

Proof. (i): Let $f \in L^p_\mu(I)$. Then from (1.10), we have

$$|(\tau_t^{a,b;d} f)(\omega)| \leq \left| \frac{1}{b} \right| \int_0^\infty |f(z)| |D_\mu^{a,b;d}(t, \omega, z)| z^{1+2\mu} dz.$$

Using Hölder’s inequality, we get

$$\leq \left| \frac{1}{b} \right| \left(\int_0^\infty |f(z)|^p |D_\mu^{a,b;d}(t, \omega, z)| z^{1+2\mu} dz \right)^{\frac{1}{p}} \left(\int_0^\infty |D_\mu^{a,b;d}(t, \omega, z)| z^{1+2\mu} dz \right)^{\frac{1}{q}}.$$

Using (1.11),

$$|(\tau_t^{a,b;d} f)(\omega)| \leq \left(\left| \frac{1}{b} \right| \right)^{\frac{1}{p}} \left(\int_0^\infty |f(z)|^p |D_\mu^{a,b;d}(t, \omega, z)| z^{1+2\mu} dz \right)^{\frac{1}{p}} \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^{\frac{1}{q}}$$

and

$$|(\tau_t^{a,b;d} f)(\omega)|^p \leq \left| \frac{1}{b} \right|^p \int_0^\infty |f(z)|^p |D_\mu^{a,b;d}(t, \omega, z)| z^{1+2\mu} dz \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^{p-1}.$$

Now applying Fubini’s theorem, we have

$$\begin{aligned} & \int_0^\infty |(\tau_t^{a,b;d} f)(\omega)|^p \omega^{1+2\mu} d\omega \\ & \leq \left| \frac{1}{b} \right|^p \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^{p-1} \int_0^\infty \int_0^\infty |f(z)|^p |D_\mu^{a,b;d}(t, \omega, z)| \omega^{1+2\mu} d\omega z^{1+2\mu} dz \\ & = \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^{p-1} \int_0^\infty |f(z)|^p \left| \frac{1}{b} \right|^p \int_0^\infty |D_\mu^{a,b;d}(t, \omega, z)| \omega^{1+2\mu} d\omega z^{1+2\mu} dz \\ & = \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^{p-1} \int_0^\infty |f(z)|^p \frac{1}{(2b)^\mu \Gamma(\mu + 1)} z^{1+2\mu} dz \\ & = \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right)^p \int_0^\infty |f(z)|^p z^{1+2\mu} dz. \end{aligned}$$

Now

$$\left(\int_0^\infty |(\tau_t^{a,b;d} f)(\omega)|^p \omega^{1+2\mu} d\omega \right)^{\frac{1}{p}} \leq \left(\frac{1}{(2b)^\mu \Gamma(\mu + 1)} \right) \left(\int_0^\infty |f(z)|^p z^{1+2\mu} dz \right)^{\frac{1}{p}}.$$

Therefore,

$$\|(\tau_t^{a,b;d} f)(\omega)\|_{L^p_\mu} \leq \frac{1}{(2b)^\mu \Gamma(\mu + 1)} \|f\|_{L^p_\mu}.$$

(ii): Let $f \in L^p_\mu(I)$ and $g \in L^q_\mu(I)$. Then we have

$$\begin{aligned} \|e^{-i(a(\cdot)^2+d(\cdot))} f \star g\|_{L^\infty_\mu} &= \sup_{t \in I} |(e^{-i(a(\cdot)^2+d(\cdot))} f \star g)(t)| \\ &= \left| \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty e^{-i(a(\omega)^2+d(\omega))} f(\omega) (\tau_t^{a,b;d} g)(\omega) e^{i(a\omega^2+d\omega)} \omega^{1+2\mu} d\omega \right| \\ &\leq \frac{1}{b} \int_0^\infty |f(\omega)| |(\tau_t^{a,b;d} g)(\omega)| \omega^{1+2\mu} d\omega \\ &\leq \frac{1}{b} \left(\int_0^\infty |f(\omega)|^p \omega^{1+2\mu} d\omega \right)^{\frac{1}{p}} \left(\int_0^\infty |(\tau_t^{a,b;d} g)(\omega)|^q \omega^{1+2\mu} d\omega \right)^{\frac{1}{q}}. \end{aligned}$$

Using Lemma 2.1(i), we have

$$\|e^{-i(a(\cdot)^2+d(\cdot))} f \star g\|_{L_\mu^\infty} \leq \frac{1}{b} \|f\|_{L_\mu^p} \frac{1}{(2b)^\mu \Gamma(\mu+1)} \|g\|_{L_\mu^q}.$$

Hence,

$$\|e^{-i(a(\cdot)^2+d(\cdot))} f \star g\|_{L_\mu^\infty} \leq \frac{1}{2^\mu b^{\mu+1} \Gamma(\mu+1)} \|f\|_{L_\mu^p} \|g\|_{L_\mu^q}. \quad \square$$

Lemma 2.2. *If the function $\psi \in L_\mu^p(I)$ and $\psi_m^{a,b;d}$ is quadratic-phase dilation of ψ with $m > 0$ given by*

$$\psi_m^{a,b;d} = m^{-\frac{5}{2}-2\mu} e^{i\left(a\left(\frac{1}{m^2}-1\right)t^2+d\left(\frac{1}{m}-1\right)t\right)} \psi\left(\frac{t}{m}\right), \tag{2.1}$$

then

$$\|\psi_m^{a,b;d}\|_{L_\mu^p} \leq m^{-\frac{5}{2}-2\mu+\frac{2+2\mu}{p}} \|\psi\|_{L_\mu^p}.$$

Proof. Let $\psi \in L_\mu^p(I)$ and $\psi_m^{a,b;d}$ is defined as (2.1) for $m > 0$. Then

$$|\psi_m^{a,b;d}(t)| \leq m^{-\frac{5}{2}-2\mu} \left| \psi\left(\frac{t}{m}\right) \right|.$$

Then,

$$\int_0^\infty |\psi_m^{a,b;d}(t)|^p t^{1+2\mu} dt \leq (m^{-\frac{5}{2}-2\mu})^p \int_0^\infty \left| \psi\left(\frac{t}{m}\right) \right|^p t^{1+2\mu} dt.$$

Hence,

$$\|\psi_m^{a,b;d}\|_{L_\mu^p} \leq m^{-\frac{5}{2}-2\mu+\frac{2+2\mu}{p}} \|\psi\|_{L_\mu^p}. \quad \square$$

Lemma 2.3. *For $\psi, \phi \in L_\mu^2(I)$, we have*

$$\begin{aligned} \text{(i)} \quad & H_\mu^{a,b,c;d,e} \left(e^{-i\left(a\left(\frac{\cdot}{m}\right)^2+d\left(\frac{\cdot}{m}\right)\right)} \psi_m^{a,b;d} \right) (\omega) \\ &= \frac{1}{\sqrt{m}} e^{-i(c(m^2-1)\omega^2+e(m-1)\omega)} H_\mu^{a,b,c;d,e} \left(e^{-i(a(\cdot)^2+d(\cdot))} \psi \right) (m\omega), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \text{(ii)} \quad & H_\mu^{a,b,c;d,e} \left(e^{-i(a(\cdot)^2+d(\cdot))} \overline{\psi_m^{a,b;d}} \right) (\omega) \\ &= \frac{e^{-i\pi(1+\mu)}}{\sqrt{m}} e^{i(c(m^2+1)\omega^2+e(m+1)\omega)} \overline{H_\mu^{a,b,c;d,e} \left(e^{-i(am^2(\cdot)^2+dm(\cdot))} \psi \right) (m\omega)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \text{(iii)} \quad & H_\mu^{a,b,c;d,e} \left(\phi \star e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi \right) (\omega) \\ &= e^{-i(c\omega^2+e\omega)} H_\mu^{a,b,c;d,e} \left(e^{-i(a(\cdot)^2+d(\cdot))} \phi \right) (\omega) H_\mu^{a,b,c;d,e} \left(e^{-i\left(a\left(\frac{\cdot}{m}\right)^2+d\left(\frac{\cdot}{m}\right)\right)} \psi \right) (\omega), \end{aligned} \tag{2.4}$$

where $\psi_m^{a,b;d}$ is given by (2.1).

Proof. When we apply quadratic-phase Hankel transform $H_\mu^{a,b,c;d,e}$ as (1.6) on (2.1), we obtain the results (i) and (ii). The proof of (iii) is straightforward as (Prasad et al. [10, p. 13]). \square

3. Normalized Wavelet Transform Involving Quadratic-phase Hankel Wavelet

According to references Pinsky [8], Prasad and Kumar [9], and Upadhyay and Singh [12], the normalized continuous quadratic-phase Hankel wavelet is defined as follows:

Definition 3.1. The function ψ in the space $L^2_\mu(I)$ is considered a normalized quadratic-phase Hankel wavelet if its norm $\|\psi\|_{L^2_\mu} = 1$ and it satisfies the admissibility condition given as (1.15).

If the function ψ belongs to the space $L^2_\mu(I)$ and is a normalized continuum quadratic-phase Hankel wavelet then it must satisfy the condition $H^{a,b,c;d,e}_\mu \psi(0) = 0$ as the continuity of $H^{a,b,c;d,e}_\mu \psi$ and $H^{a,b,c;d,e}_\mu \psi(0) \neq 0$ would contradict the convergence of the integral in the equation (1.15). By adjusting the scale of the spatial coordinate, we can suppose that both $\|\psi\|_{L^2_\mu} = 1$ and $C^{\mu,a,b;d}_\psi = 1$.

The wavelet transform, given by the equation (1.14), can be represented in the form of quadratic-phase Hankel convolution as

$$(W^{a,b;d}_\psi f)(n, m) = b e^{i\frac{\pi}{2}(1+\mu)} \left(e^{-i(a(\cdot)^2+d(\cdot))} f \star \overline{\psi_m^{a,b;d}} \right)(n), \tag{3.1}$$

where $\psi_m^{a,b;d}$ is quadratic-phase dilation defined as (2.1).

As per Prasad and Kumar [9], we define

$$N^{a,b;d}_\mu(m) = \left(\int_0^\infty |(W^{a,b;d}_\psi f)(n, m)|^2 n^{1+2\mu} dn \right)^{\frac{1}{2}}. \tag{3.2}$$

Lemma 3.2. Consider $\psi \in L^2_\mu(I)$ as a normalized quadratic-phase Hankel wavelet and let $f \in L^2_\mu(I)$. Then

$$(i) \quad |W^{a,b;d}_\psi f(n, m)| \leq \frac{m^{-\frac{3-2\mu}{2}}}{(2b)^\mu \Gamma(\mu + 1)} \|f\|_{L^2_\mu}.$$

(ii) For $m > 0$, $n \rightarrow W^{a,b;d}_\psi f(n, m)$ belongs to $L^2_\mu(I)$ and the norm $N^{a,b;d}_\mu(m)$ satisfies

$$\int_0^\infty [N^{a,b;d}_\mu(m)]^2 dm = b^2 \|f\|_{L^2_\mu}^2. \tag{3.3}$$

Proof. (i) From (3.1), we have

$$(W^{a,b;d}_\psi f)(n, m) = b e^{i\frac{\pi}{2}(1+\mu)} \left(e^{-i(a(\cdot)^2+d(\cdot))} f \star \overline{\psi_m^{a,b;d}} \right)(n).$$

By applying Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned} |(W^{a,b;d}_\psi f)(n, m)| &\leq |b| \frac{1}{2^\mu b^{\mu+1} \Gamma(\mu + 1)} \|f\|_{L^2_\mu} \|\overline{\psi_m^{a,b;d}}\|_{L^2_\mu} \\ &\leq \|f\|_{L^2_\mu} \frac{m^{-\frac{3-2\mu}{2}}}{(2b)^\mu \Gamma(\mu + 1)} \|\overline{\psi}\|_{L^2_\mu} \\ &= \frac{m^{-\frac{3-2\mu}{2}}}{(2b)^\mu \Gamma(\mu + 1)} \|f\|_{L^2_\mu}, \end{aligned}$$

as $\|\psi\|_{L^2_\mu} = 1$.

(ii) Let $f \in L^1_\mu(I) \cap L^2_\mu(I)$. Then using (3.1) and Parseval’s relation (1.8), we have

$$\begin{aligned} [N_\mu^{a,b;d}(m)]^2 &= \int_0^\infty |(W_\psi^{a,b;d} f)(n, m)|^2 n^{1+2\mu} dn \\ &= b^2 \left| \left(e^{-i(a(\cdot)^2+d(\cdot))} f \star \overline{\psi_m^{a,b;d}} \right) (n) \right|^2 n^{1+2\mu} dn \\ &= b^2 \int_0^\infty \left| H_\mu^{a,b,c;d,e} \left[e^{-i(a(\cdot)^2+d(\cdot))} f \star \overline{\psi_m^{a,b;d}} \right] (\omega) \right|^2 \omega^{1+2\mu} d\omega \\ &= b^2 \int_0^\infty \left| H_\mu^{a,b,c;d,e} (e^{-2i(a(\cdot)^2+d(\cdot))} f)(\omega) \right|^2 \left| H_\mu^{a,b,c;d,e} (\overline{e^{i(a(\cdot)^2+d(\cdot))} \psi_m^{a,b;d}})(\omega) \right|^2 \omega^{1+2\mu} d\omega. \end{aligned}$$

In particular, $n \rightarrow W_\psi^{a,b;d} f(n, m)$ is in $L^2_\mu(I)$ for every $m > 0$. Then, using (2.3)

$$\begin{aligned} &\int_0^\infty [N_\mu^{a,b;d}(m)]^2 dm \\ &= b^2 \int_0^\infty \int_0^\infty |H_\mu^{a,b,c;d,e} (e^{-2i(a(\cdot)^2+d(\cdot))} f)(\omega)|^2 \\ &\quad \times \left| \frac{e^{-i\pi(1+\mu)}}{\sqrt{m}} e^{i(c(1+m^2)\omega^2+e(1+m)\omega)} \overline{H_\mu^{a,b,c;d,e} (e^{-i(a(\cdot)^2+d(\cdot))} \psi)(m\omega)} \right|^2 \omega^{1+2\mu} d\omega dm \\ &= b^2 \int_0^\infty |H_\mu^{a,b,c;d,e} (e^{-2i(a(\cdot)^2+d(\cdot))} f)(\omega)|^2 \int_0^\infty \frac{|H_\mu^{a,b,c;d,e} (e^{-i(a(\cdot)^2+d(\cdot))} \psi)(m\omega)|^2}{m} dm \omega^{1+2\mu} d\omega \\ &= b^2 \int_0^\infty |H_\mu^{a,b,c;d,e} (e^{-2i(a(\cdot)^2+d(\cdot))} f)(\omega)|^2 \omega^{1+2\mu} d\omega \\ &= b^2 \|f\|_{L^2_\mu}^2. \end{aligned}$$

This completes the proof. □

Definition 3.3. The partial inverse transform of a function $f \in L^2_\mu(I)$ is defined as

$$S_\epsilon^{a,b;d} f(x) = \int_{m>\epsilon} \left(\int_0^\infty W_\psi^{a,b;d} f(n, m) \psi_{n,m}^{a,b;d}(x) n^{1+2\mu} dn \right) dm, \quad \text{for } \epsilon > 0. \tag{3.4}$$

Theorem 3.4. The partial inverse transform of a function $f \in L^2_\mu(I)$ can be written as

$$S_\epsilon^{a,b;d} f(x) = b e^{i\frac{\pi}{2}(1+\mu)} \int_{m>\epsilon} \left(W_\psi^{a,b;d} f(n, m) \star e^{-i(a(\cdot)^2(\frac{1}{m^2}-1)+d(\cdot)(\frac{1}{m}-1))} \psi_m^{a,b;d} \right) (x) dm. \tag{3.5}$$

Proof. From (1.14), we have

$$\begin{aligned} &\int_0^\infty W_\psi^{a,b;d} f(n, m) \psi_{n,m}^{a,b;d}(x) n^{1+2\mu} dn \\ &= m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{i(a(\frac{1}{m^2}-1)(x^2+n^2)+d(\frac{1}{m}-1)(x+n))} \frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \psi(z) \\ &\quad \times D_\mu^{a,b;d} \left(\frac{n}{m}, \frac{x}{m}, z \right) e^{i(az^2+dz)} z^{1+2\mu} dz W_\psi^{a,b;d} f(n, m) n^{1+2\mu} dn \\ &= m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{i(a(\frac{1}{m^2}-1)(x^2+n^2)+d(\frac{1}{m}-1)(x+n))} W_\psi^{a,b;d} f(n, m) \\ &\quad \times \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \left(\int_0^\infty \left(\frac{n}{m\xi} \right)^{-\mu} J_\mu \left(\frac{n}{bm\xi} \right) \left(\frac{x}{m\xi} \right)^{-\mu} J_\mu \left(\frac{x}{bm\xi} \right) \xi^{1+2\mu} \right. \end{aligned}$$

$$\begin{aligned} & \times \left(\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty e^{i(az^2+c\xi^2+dz+d\xi)} (z\xi)^{-\mu} J_\mu \left(\frac{z\xi}{b} \right) z^{1+2\mu} e^{-i(az^2+dz)} \psi(z) dz \right) e^{-i(c\xi^2+e\xi)} d\xi \\ & \times e^{-i\left(a\left(\frac{n^2}{m^2}+\frac{x^2}{m^2}\right)+d\left(\frac{n}{m}+\frac{x}{m}\right)\right)} n^{1+2\mu} dn \\ & = m^{-\frac{5}{2}-2\mu} \int_0^\infty e^{-i(ax^2+dx)} H_\mu^{a,b,c;d,e} \left(e^{-i(a(\cdot)^2+d(\cdot))} \psi \right) (\xi) e^{-i(c\xi^2+e\xi)} \xi^{1+2\mu} \left(\frac{x}{m} \xi \right)^{-\mu} \\ & \times J_\mu \left(\frac{x}{mb} \xi \right) e^{i\pi(1+\mu)} e^{-i\left(c\frac{\xi^2}{m^2}+e\frac{\xi}{m}\right)} H_\mu^{a,b,c;d,e} \left(e^{-2i(a(\cdot)^2+d(\cdot))} W_\psi^{a,b;d} f(\cdot, m) \right) \left(\frac{\xi}{m} \right) d\xi. \end{aligned}$$

Now setting $\frac{\xi}{m} = \omega$ and using (2.2), (2.4), we get

$$\begin{aligned} & \int_0^\infty W_\psi^{a,b;d} f(n, m) \psi_{n,m}^{a,b;d}(x) n^{1+2\mu} dn \\ & = m^{-\frac{3}{2}-2\mu} e^{i\pi(1+\mu)} \int_0^\infty e^{-i(ax^2+dx)} H_\mu^{a,b,c;d,e} \left(e^{-i(a(\cdot)^2+d(\cdot))} \psi \right) (m\omega) e^{-i(c(m\omega)^2+e(m\omega))} \\ & \quad \times (m\omega)^{1+2\mu} (x\omega)^{-\mu} J_\mu \left(\frac{x\omega}{b} \right) e^{-i(c\omega^2+e\omega)} H_\mu^{a,b,c;d,e} \left(e^{-2i(a(\cdot)^2+d(\cdot))} W_\psi^{a,b;d} f(\cdot, m) \right) (\omega) d\omega \\ & = e^{i\pi(1+\mu)} \int_0^\infty e^{-i(ax^2+d\omega^2+dx+e\omega)} (x\omega)^{-\mu} J_\mu \left(\frac{x\omega}{b} \right) \omega^{1+2\mu} \\ & \quad \times H_\mu^{a,b,c;d,e} \left(e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \star W_\psi^{a,b;d} f(\cdot, m) \right) (\omega) d\omega. \end{aligned}$$

Applying inverse QPHT (1.7), we have

$$\begin{aligned} & \int_0^\infty W_\psi^{a,b;d} f(n, m) \psi_{n,m}^{a,b;d}(x) n^{1+2\mu} dn \\ & = b e^{i\frac{\pi}{2}(1+\mu)} \left(W_\psi^{a,b;d} f(n, m) \star e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \right) (x). \end{aligned}$$

Thus, we have

$$S_\epsilon^{a,b;d} f(x) = b e^{i\frac{\pi}{2}(1+\mu)} \int_{m>\epsilon} \left(W_\psi^{a,b;d} f(n, m) \star e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \right) (x) dm. \quad \square$$

Theorem 3.5. Let the function $f \in L^2_\mu(I)$ and $\psi \in L^2_\mu(I)$ be a normalized continuous wavelet. Then $\forall \epsilon > 0$ and $x \in I$, the partial inverse transform $S_\epsilon^{a,b;d} f(x)$ has a pointwise bound

$$|S_\epsilon^{a,b;d} f(x)| \leq \frac{b \|f\|_{L^2_\mu}}{(2b)^\mu \Gamma(\mu + 1)} C_\epsilon,$$

where $C_\epsilon = \left(\int_{m>\epsilon} \frac{1}{m^{3+2\mu}} dm \right)^{\frac{1}{2}}$.

Proof. From (3.4) and (3.5), we get

$$\begin{aligned} S_\epsilon^{a,b;d} f(x) &= \int_{m>\epsilon} \left(\int_0^\infty W_\psi^{a,b;d} f(n, m) \psi_{n,m}^{a,b;d}(x) n^{1+2\mu} dn \right) dm \\ &= b e^{i\frac{\pi}{2}(1+\mu)} \int_{m>\epsilon} \left(W_\psi^{a,b;d} f(n, m) \star e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \right) (x) dm. \end{aligned}$$

Applying Lemma 2.1, we obtain

$$\left| W_\psi^{a,b;d} f(n, m) \star e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \right|$$

$$\begin{aligned} &\leq \frac{1}{|b|} \|W_\psi^{a,b;d} f(n, m)\|_{L_\mu^2} \left\| \tau_t^{a,b;d} \left(e^{-i\left(a(\cdot)^2\left(\frac{1}{m^2}-1\right)+d(\cdot)\left(\frac{1}{m}-1\right)\right)} \psi_m^{a,b;d} \right) \right\|_{L_\mu^2} \\ &\leq \frac{1}{|b|} \|W_\psi^{a,b;d} f(n, m)\|_{L_\mu^2} \frac{m^{-\frac{3}{2}-\mu}}{(2b)^\mu \Gamma(\mu+1)} \|\psi\|_{L_\mu^2}. \end{aligned}$$

Now using (3.2) and (3.3),

$$\begin{aligned} |S_\epsilon^{a,b;d} f(x)| &\leq \int_{m>\epsilon} \|W_\psi^{a,b;d} f(n, m)\|_{L_\mu^2} \frac{m^{-\frac{3}{2}-\mu}}{(2b)^\mu \Gamma(\mu+1)} dm \\ &= \frac{1}{(2b)^\mu \Gamma(\mu+1)} \int_{m>\epsilon} [N_\mu^{a,b;d}(m)] m^{-\frac{3}{2}-\mu} dm \\ &\leq \frac{1}{(2b)^\mu \Gamma(\mu+1)} \left(\int_{m>\epsilon} [N_\mu^{a,b;d}(m)]^2 dm \right)^{\frac{1}{2}} \left(\int_{m>\epsilon} (m^{-\frac{3}{2}-\mu})^2 dm \right)^{\frac{1}{2}} \\ &\leq \frac{b \|f\|_{L_\mu^2}}{(2b)^\mu \Gamma(\mu+1)} \left(\int_{m>\epsilon} \frac{1}{m^{3+2\mu}} dm \right)^{\frac{1}{2}} \\ &= \frac{b \|f\|_{L_\mu^2}}{(2b)^\mu \Gamma(\mu+1)} C_\epsilon, \end{aligned}$$

where for each $\epsilon > 0$, $C_\epsilon = \left(\int_{m>\epsilon} \frac{1}{m^{3+2\mu}} dm \right)^{\frac{1}{2}}$ is convergent for $\mu > -1$. □

4. Calderón’s Reproducing Formula

Calderón’s reproducing identity is a powerful method used across various scientific and engineering fields for function reconstruction and analysis through wavelet transforms. In this section, we derive Calderón’s reproducing formula utilizing the properties of quadratic-phase Hankel transformation and Hankel convolution.

Theorem 4.1. *Let ψ and $\phi \in L_\mu^2(I)$ be basic quadratic-phase Hankel wavelets such that the following admissibility condition holds:*

$$C_{\psi,\phi}^{\mu,a,b,c;d,e} = \int_0^\infty \frac{|(H_\mu^{a,b,c;d,e} e^{-i(a(\cdot)^2+d(\cdot)})\psi)(\omega)| \overline{|(H_\mu^{a,b,c;d,e} e^{-i(a(\cdot)^2+d(\cdot)})\phi)(\omega)|}}{\omega} d\omega = 1.$$

Then, Calderón’s reproducing formula for f is given by

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \left((e^{-i(an^2+dn)} (e^{-i(a(\cdot)^2+d(\cdot)}) f(\cdot) \star \overline{\psi_m^{a,b;d}(\cdot)})(n)) \star \phi_m^{a,b;d}(t) \right) dm.$$

Proof. From equation (1.16), we have

$$f(t) = \frac{1}{b^2 C_{\psi,\phi}^{\mu,a,b,c;d,e}} \int_0^\infty \int_0^\infty (W_\psi^{a,b;d} f)(n, m) \phi_{n,m}^{a,b;d}(t) n^{1+2\mu} dn dm.$$

Using equations (3.1) and (1.12), we have

$$\begin{aligned} f(t) &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \int_0^\infty (e^{-i(a(\cdot)^2+d(\cdot)}) f(\cdot) \star \overline{\psi_m^{a,b;d}(\cdot)})(n) \phi_m^{a,b;d}(n, t) n^{1+2\mu} dn dm \\ &= \frac{e^{i\frac{\pi}{2}(1+\mu)}}{b} \int_0^\infty \int_0^\infty e^{-i(an^2+dn)} (e^{-i(a(\cdot)^2+d(\cdot)}) f(\cdot) \star \overline{\psi_m^{a,b;d}(\cdot)})(n) \phi_m^{a,b;d}(n, t) e^{i(an^2+dn)} n^{1+2\mu} dn dm. \end{aligned}$$

Applying equation (1.9), we obtain the desired result as:

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \left((e^{-i(an^2+dn)}(e^{-i(a(\cdot)^2+d(\cdot)})f(\cdot) \star \overline{\psi_m^{a,b;d}(\cdot)}(n)) \star \phi_m^{a,b;d})(t) dm. \quad \square$$

Remark 4.2. If $\psi = \phi$, then the following holds:

$$\int_0^\infty \frac{|(H_\mu^{a,b,c;d,e} e^{-i(a(\cdot)^2+d(\cdot)})\psi)(\omega)|^2}{\omega} d\omega = 1.$$

and the Calderón's reproducing formula for f is given by

$$f(t) = e^{i\pi(1+\mu)} \int_0^\infty \left((e^{-i(an^2+dn)}(e^{-i(a(\cdot)^2+d(\cdot)})f(\cdot) \star \overline{\psi_m^{a,b;d}(\cdot)}(n)) \star \psi_m^{a,b;d})(t) dm.$$

5. Conclusion

In this work, we have successfully established the Calderón's reproducing formula in the framework of quadratic phase Hankel transformation. The Calderón's reproducing formula is a combination of two convolutions. This work may have the future scope in the signal and image processing.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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