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Research Article

# Applications of Linear Differential Operator on Varying Arguments

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**Abstract.** In the present work, using Al-Oboudi operator and Carlson-Shaffer operator, we introduce a new Linear operator $\mathcal{AS}^{\delta}_{\lambda,q}$. The objective is to define the new subclasses of analytic functions $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$, $\mathcal{VS}^{\alpha}_{\lambda,\delta}(a,c,n;q)$ using the above linear operator and for functions belonging to these classes we obtain coefficient estimates and many more related properties like extreme points, integral means, unified radii results etc.

**Keywords.** Al-Oboudi $q$-differential operator, Univalent functions, Analytic functions and Carlson-Shaffer operator, Linear operator

**Mathematics Subject Classification (2020).** Primary 30C45; Secondary 30C50



## 1. Introduction

Linear differential operators play a crucial role in geometric function theory, which is a branch of mathematics that studies the properties of functions and their mappings in geometric settings. In particular, linear differential operators are used to study the properties of conformal mappings, quasi-conformal mappings, and other types of mappings between Riemann surfaces and other geometric objects.

Linear operators are used to study various properties of functions and mappings, such as their regularity, smoothness, and geometric properties like curvature and conformality.

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U}$ and normalized by $f(0) = f'(0) - 1 = 0$.

The class $\mathcal{A}$ is closed under convolution or Hadamard product

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j, \quad z \in \mathcal{U}, \tag{1.2}$$

where $f$ is given by (1.1) and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$.

Using the idea of convolution, now we introduce a linear operator $\mathcal{AS}_{\lambda,q}^{\delta} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{AS}_{\lambda,q}^{\delta} f(z) = [(1-\lambda)[1+(j-1)\delta]^n + \lambda\phi(a,c)] * f(z).$$

For functions $f \in \mathcal{A}$ of the form (1.1), we have

$$\mathcal{AS}_{\lambda,q}^{\delta} f(z) = z + \sum_{j=2}^{\infty} B_{\lambda}^{\delta}(a,c,j,n;q) a_j z^j, \tag{1.3}$$

where

$$B_{\lambda}^{\delta}(a,c,j,n;q) = \left[[1+(j-1)\delta]^n(1-\lambda) + \lambda\frac{(a)_{j-1}}{(c)_{j-1}}\right]_q,$$

$n \in \mathbb{N}_0$, $\lambda \geq 0$, $\delta \geq 0$ and $a, c \in \mathbb{R} \setminus \mathbb{Z}$.

Here $(a)_j$ is the Pochhammer symbol defined in terms of the Gamma function by,

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = \begin{cases} 1, & \text{for } j = 0, \\ a(a+1)(a+2)\cdots(a+j-1), & \text{for } j \in \mathbb{N}. \end{cases}$$

For a different parametric values of $q \to 1^-$, $\lambda = 0$, we get the Al-Oboudi differential operator [2].

For a different parametric values of $q \to 1^-$, $\lambda = 1$, we get the Carlson-Shaffer operator [3].

For a different parametric values of $\lambda = 0$, we get the differential operator studied by Dileep and Rajeev [6].

For $q \to 1^-$ and $\delta = 1$ we get operator studied by Dileep and Latha [4].

Now using the linear operator $\mathcal{AS}_{\lambda,q}^{\delta}$ we define the class $\mathcal{S}_{\lambda,\delta}^{\alpha}(a,c,n;q)$ consisting functions of the form (1.1) satisfying the condition:

$$\Re\left\{\frac{z(\mathcal{AS}_{\lambda,q}^{\delta} f(z))'}{\mathcal{AS}_{\lambda,q}^{\delta} f(z)} - \alpha\right\} \geq \left|\frac{z(\mathcal{AS}_{\lambda,q}^{\delta} f(z))'}{\mathcal{AS}_{\lambda,q}^{\delta} f(z)} - 1\right|, \quad 0 \leq \alpha < 1. \tag{1.4}$$

Silverman [15] defined the class $\mathcal{V}(\theta_j)$ as the class of all functions in $\mathcal{A}$ such that $\arg a_j = \theta_j$, for all $j$. If further there exists a real number $t$ such that $\theta_j + (j-1)t \equiv \pi \pmod{2\pi}$, then $f$ is said to be in the class $\mathcal{V}(\theta_j, t)$. The union of $\mathcal{V}(\theta_j, t)$ taken over all possible sequences $\{\theta_j\}$ and all possible real numbers $t$ is denoted by $\mathcal{V}$.

Further, we define $\mathcal{VS}_{\lambda,\delta}^{\alpha}(a,c,n;q) = \mathcal{S}_{\lambda,\delta}^{\alpha}(a,c,n;q) \cap \mathcal{V}$.

**Definition 1.1.** A function $f \in \mathcal{V}$ of the form (1.1) is in $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$ if $f$ satisfies the analytic condition

$$\Re\left\{\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)}\right\} \geq \beta\left|\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - 1\right| + \alpha, \tag{1.5}$$

where $\alpha,\beta \geq 0$ and $z \in \mathcal{U}$.

Theses classes stem essentially from the classes studied earlier by Vijaya and Murugusundaramoorthy [17].

In the next section, we shall make a systematic study of the class $\mathcal{VS}^{\alpha}_{\lambda,\delta}(a,c,n;q)$.

## 2. Main Results

**Theorem 2.1.** *A function $f$ of the form* (1.1) *is in $\mathcal{VS}^{\alpha}_{\lambda,\delta}(a,c,n;q)$ if and only if*

$$\sum_{j=2}^{\infty}(2j-1-\alpha)B^{\delta}_{\lambda}(a,c,j,n;q)|a_j| \leq 1-\alpha. \tag{2.1}$$

*Proof.* From (1.4), it suffices to show that

$$\left|\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - 1\right| \leq \Re\left\{\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - \alpha\right\},$$

i.e.,

$$\left|\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - 1\right| - \Re\left\{\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - 1\right\} \leq 2\left|\frac{z(\mathcal{AS}^{\delta}_{\lambda,q}f(z))'}{\mathcal{AS}^{\delta}_{\lambda,q}f(z)} - 1\right|$$

$$\leq 2\frac{\sum_{j=2}^{\infty}(j-1)B^{\delta}_{\lambda}(a,c,j,n;q)|a_j|\,|z|^{j-1}}{1-\sum_{j=2}^{\infty}B^{\delta}_{\lambda}(a,c,j,n;q)|a_j|\,|z|^{j-1}}.$$

Now the last expression is bounded by $(1-\alpha)$ if

$$\sum_{j=2}^{\infty}(2j-1-\alpha)B^{\delta}_{\lambda}(a,c,j,n;q)|a_j| \leq 1-\alpha.$$

Conversely, if $f \in \mathcal{VS}^{\alpha}_{\lambda,\delta}(a,c,n;q)$ then by definition,

$$\left|\frac{z+\sum_{j=2}^{\infty}jB^{\delta}_{\lambda}(a,c,j,n;q)a_jz^j}{z+\sum_{j=2}^{\infty}B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^j} - 1\right| \leq \Re\left\{\frac{z+\sum_{j=2}^{\infty}jB^{\delta}_{\lambda}(a,c,j,n;q)a_jz^j}{z+\sum_{j=2}^{\infty}B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^j} - \alpha\right\},$$

i.e.,

$$\left|\frac{\sum_{j=2}^{\infty}(j-1)B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^{j-1}}\right| \leq \Re\left\{\frac{(1-\alpha)+\sum_{j=2}^{\infty}(j-\alpha)B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^{j-1}}{1+\sum_{j=2}^{\infty}B^{\delta}_{\lambda}(a,c,j,n;q)a_jz^{j-1}}\right\}.$$

Since $f \in \mathcal{V}$ and $f$ lies in $\mathcal{V}(\theta_j, t)$ for some sequence $\theta_j$ and a real number $t$ such that $\theta_j + (j-1)t \equiv \pi (\operatorname{mod} 2\pi)$ set $z = re^{it}$ in the above inequality:

$$\left| \frac{\sum_{j=2}^{\infty} (j-1) B_{\lambda}^{\delta}(a,c,j,n;q) a_j r^{j-1}}{1 - \sum_{j=2}^{\infty} B_{\lambda}^{\delta}(a,c,j,n;q) a_j r^{j-1}} \right| \leq \Re \left\{ \frac{(1-\alpha) - \sum_{j=2}^{\infty} (j-\alpha) B_{\lambda}^{\delta}(a,c,j,n;q) a_j r^{j-1}}{1 - \sum_{j=2}^{\infty} B_{\lambda}^{\delta}(a,c,j,n;q) a_j r^{j-1}} \right\}.$$

Letting $r \to 1$, leads the desired inequality

$$\sum_{j=2}^{\infty} (2j-1-\alpha) B_{\lambda}^{\delta}(a,c,j,n;q) |a_j| \leq 1-\alpha. \qquad \square$$

**Corollary 2.2.** *If* $f \in \mathcal{VS}_{\lambda,\delta}^{\alpha}(a,c,n;q)$ *then*

$$|a_j| \leq \frac{1-\alpha}{(2j-1-\alpha) B_{\lambda}^{\delta}(a,c,j,n;q)}, \quad \text{for } j \geq 2.$$

*The sharpness follows for the function*

$$f(z) = z + \sum_{j=2}^{\infty} \frac{(1-\alpha)}{(2j-1-\alpha) B_{\lambda}^{\delta}(a,c,j,n;q)}, \quad \text{for } j \geq 2,\ z \in \mathcal{U}.$$

Similar to the proof of Theorem 2.1 we get the following result:

**Theorem 2.3.** *A function $f$ of the form* (1.1), *belongs to* $\mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$ *if and only if*

$$\sum_{j=2}^{\infty} E_j B_{\lambda}^{\delta}(a,c,j,n;q) |a_j| \leq 1-\alpha, \tag{2.2}$$

*where* $E_j = \beta(j-1) + j - \alpha$.

The result obtained in our next theorem unifies the radii results concerning close-to-convexity, starlikeness etc.

**Theorem 2.4.** *Let* $f \in \mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$. *Then* $\left| \frac{f*\Phi}{f*\Psi} - 1 \right| < 1-\eta$, *in* $|z| < r$ *with* $\Phi(z) = z + \sum_{j=2}^{\infty} \gamma_j z^j$ *and* $\Psi(z) = z + \sum_{j=2}^{\infty} \mu_j z^j$, *are analytic in* $\mathcal{U}$ *with the conditions* $\gamma_j, \mu_j \geq 0$, $\gamma_j \geq \mu_j$, *for* $j \geq 2$ *and* $f(z) * \Psi(z) \neq 0$, *where*

$$r = \inf_j \left[ \frac{E_j B_{\lambda}^{\delta}(a,c,j,n;q)(1-\delta)}{(1-\alpha)[(\gamma_j - \mu_j) + \mu_j(1-\eta)]} \right]^{\frac{1}{j-1}}, \quad j \geq 2. \tag{2.3}$$

*Proof.* Consider,

$$\left| \frac{f*\Phi}{f*\Psi} - 1 \right| = \left| \frac{z - \sum_{j=2}^{\infty} \gamma_j a_j z^j}{z - \sum_{j=2}^{\infty} \mu_j a_j z^j} - 1 \right| \leq \left| \frac{z - \sum_{j=2}^{\infty} \gamma_j a_j z^j - z + \sum_{j=2}^{\infty} \mu_j a_j z^j}{z - \sum_{j=2}^{\infty} \mu_j a_j z^j} \right| \leq \frac{\sum_{j=2}^{\infty} a_j [\gamma_j - \mu_j] |z|^{j-1}}{1 - \sum_{j=2}^{\infty} \mu_j a_j |z|^{j-1}} < 1-\eta, \tag{2.4}$$

$$\sum_{j=2}^{\infty} a_j [(\gamma_j - \mu_j) + (1-\eta)\mu_j] \leq 1-\eta, \quad (|z| < r; 0 \leq \eta < 1), \tag{2.5}$$

where $r$ is given by (2.3). From Theorem 2.3, (2.5) will be true if,

$$\frac{[(\gamma_j-\mu_j)+(1-\eta)\mu_j]}{1-\eta}|z|^{j-1} \le \frac{E_jB_\lambda^\delta(a,c,j,n;q)(1-\eta)}{(1-\alpha)[(\gamma_j-\mu_j)+\mu_j(1-\eta)]},$$

that is, if

$$|z| = \left[\frac{E_jB_\lambda^\delta(a,c,j,n;q)(1-\eta)}{(1-\alpha)[(\gamma_j-\mu_j)+\mu_j(1-\eta)]}\right]^{\frac{1}{j-1}}. \tag{2.6}$$

□

As corollaries to the above theorem we get the following results:

By choosing $\Phi(z)=\frac{z}{(1-z)^2}$ and $\Psi(z)=z$, we have

**Corollary 2.5.** *Let the function $f$ defined by* (1.1) *is in* $\mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$. *Then $f$ is close-to-convex of order $\eta$ $(0\le\eta<1)$, hence univalent in the disc $|z|<r_1$, where*

$$r_1 = \inf_j \left[\frac{E_jB_\lambda^\delta(a,c,j,n;q)(1-\eta)}{(1-\alpha)j}\right]^{\frac{1}{j-1}}, \quad j\ge 2. \tag{2.7}$$

*The result is sharp.*

For $\Phi(z)=\frac{z}{(1-z)^2}$ and $\Psi(z)=\frac{z}{1-z}$, we have

**Corollary 2.6.** *Let the function $f$ defined by* (1.1) *belongs to* $\mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$. *Then $f$ is starlike of order $\eta$ $(0\le\eta<1)$, hence univalent in the disc $|z|<r_2$, where*

$$r_2 = \inf_j \left[\frac{E_jB_\lambda^\delta(a,c,j,n;q)(1-\eta)}{(1-\alpha)(j-\eta)}\right]^{\frac{1}{j-1}}, \quad j\ge 2. \tag{2.8}$$

*The result is sharp.*

If $\Phi(z)=\frac{z+z^2}{(1-z)^3}$ and $\Psi(z)=\frac{z}{(1-z)^2}$, then we have

**Corollary 2.7.** *Let the function $f$ be defined by* (1.1) *belongs to* $\mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$. *Then $f$ is convex of order $\eta$ $(0\le\eta<1)$, hence univalent in the disc $|z|<r_3$, where*

$$r_3 \le \inf_j \left[\frac{E_jB_\lambda^\delta(a,c,j,n)(1-\eta)}{j(1-\alpha)(j-\eta)}\right]^{\frac{1}{j-1}}, \quad j\ge 2. \tag{2.9}$$

*The result is sharp.*

Using the coefficient inequality proved above we can easily prove the following growth and distortion theorems.

**Theorem 2.8.** *Let $f$ of the form* (1.1) *to be in* $\mathcal{VS}_{\lambda,\delta}^{\alpha,\beta}(a,c,n;q)$. *Then*

$$r-\frac{1-\alpha}{E_2B_\lambda^\delta(a,c,2,n;q)}r^2 \le |f(z)| \le r+\frac{1-\alpha}{E_2B_\lambda^\delta(a,c,2,n;q)}r^2, \text{ and}$$

$$1-\frac{2(1-\alpha)}{E_jB_\lambda^\delta(a,c,2,n;q)}r \le |f'(z)| \le 1+\frac{2(1-\alpha)}{E_2B_\lambda^\delta(a,c,2,n;q)}r.$$

*The result is sharp.*

*Proof.* Let $f$ of the form (1.1) belongs to $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$,

$$|f(z)| = \left|z + \sum_{j=2}^{\infty} a_j z^j\right| \le |z| + |z|^2 \sum_{j=2}^{\infty} |a_j|,$$

since $f \in \mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$ and by Theorem 2.3, we have

$$E_2 B^{\delta}_{\lambda}(a,c,2,n;q) \sum_{j=2}^{\infty} |a_j| \le \sum_{j=2}^{\infty} E_j B^{\delta}_{\lambda}(a,c,j,n;q)|a_j| \le 1-\alpha.$$

Thus

$$|f(z)| \le |z| + \frac{1-\alpha}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)}|z|^2,$$

i.e.,

$$|f(z)| \le r + \frac{1-\alpha}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)} r^2 .$$

Similarly, we get

$$|f(z)| \ge r - \frac{1-\alpha}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)} r^2.$$

On the other hand

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1},$$

and

$$|f'(z)| = 1 + \sum_{j=2}^{\infty} j|a_j||z|^{j-1} \le 1 + |z| \sum_{j=2}^{\infty} j|a_j|.$$

Since $f \in \mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$.

Then, by Theorem 2.3 we have

$$\sum_{j=2}^{\infty} j|a_j| \le \frac{2(1-\alpha)}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)}.$$

Thus

$$|f'(z)| \le 1 + \frac{2(1-\alpha)}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)} r.$$

Similarly, we get

$$|f'(z)| \ge 1 - \frac{2(1-\alpha)}{E_2 B^{\delta}_{\lambda}(2,n;q)} r.$$

This completes the result. □

**Theorem 2.9.** *A function $f$ of the form* (1.1) *belongs to $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$, with* $\arg a_j = \theta_j$, *where* $[\theta_j + (j-1)t] = \pi \,(\mathrm{mod}\ 2\pi)$. *Define $f_1(z) = z$ and*

$$f_2(z) = z + \frac{1-\alpha}{E_j B^{\delta}_{\lambda}(a,c,j,n;q)} e^{i\theta_j} z^j, \quad j \ge 2,\ z \in \mathcal{U}.$$

*Then* $f \in \mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$ *if and only if* $f$ *expressed in the form*

$$f(z) = \sum_{j=2}^{\infty} \mu_j f_j(z),$$

*where* $\mu_j \geq 0$ *and* $\sum_{j=2}^{\infty} \mu_j = 1$.

*Proof.* If $f(z) = \sum_{j=2}^{\infty} \mu_j f_j(z)$ with $\sum_{j=2}^{\infty} \mu_j = 1$ and $\mu_j \geq 0$, then

$$\sum_{j=2}^{\infty} E_m B^{\delta}_{\lambda}(a,c,j,n;q) \frac{(1-\alpha)}{E_j B^{\delta}_{\lambda}(a,c,j,n;q)} \mu_j = \sum_{j=2}^{\infty} \mu_j (1-\alpha) = (1-\mu_1)(1-\alpha) \geq 1-\alpha.$$

Hence $f \in \mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$.

Conversely, let the function $f$ defined by (1.1) be in the class $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$, since

$$|a_j| \leq \frac{1-\alpha}{E_j B^{\delta}_{\lambda}(a,c,j,n;q)}, \quad j = 2,3,\cdots.$$

We may set $\mu_j = \frac{E_j B^{\delta}_{\lambda}(a,c,j,n;q)|a|_j}{1-\alpha}$, $j \geq 2$ and $\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j$.

Then $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z)$, this completes the proof. □

**Lemma 2.10** ([8])**.** *If for the functions* $f$ *and* $g$ *are analytic in* $\mathcal{U}$ *with* $g \prec f$, *then for* $k > 0$ *and* $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^k d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^k d\theta,$$

In [14] Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve the integral means inequality, conjectured in [12] and settled in [13], such that

$$\int_0^{2\pi} |f(re^{i\theta})|^k d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^k d\theta,$$

for all $f \in \mathcal{V}$, $k > 0$ and $0 < r < 1$.

In [13], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$ of $T$.

**Theorem 2.11.** *Let* $f$ *of the form* (1.1) *belongs to* $\mathcal{VS}^{\alpha,\beta}_{\lambda,\delta}(a,c,n;q)$ *and* $f_2$ *is defined by* $f_2(z) = z - \frac{(1-\alpha)}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)} z^2$ *then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^k d\theta \leq \int_0^{2\pi} |f_2(z)|^k d\theta \tag{2.10}$$

*Proof.* For $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$, eq. (2.10) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{j=2}^{\infty} |a_j| z^{j-1} \right|^k d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\alpha)}{E_2 B^{\delta}_{\lambda}(a,c,2,n;q)} z \right|^k d\theta.$$

By Lemma 2.10 it suffices to show that

$$1-\sum_{j=2}^{\infty}|a_j|z^{j-1}\prec 1-\frac{1-\alpha}{E_2B_\lambda^\delta(a,c,2,n;q)}z.$$

Setting

$$1-\sum_{j=2}^{\infty}|a_j|z^{j-1}=1-\frac{1-\alpha}{E_2B_\lambda^\delta(a,c,2,n;q)}\omega(z)$$

and using (2.2) we obtain

$$\begin{aligned}\omega(z)&=\left|\sum_{j=2}^{\infty}\frac{E_2B_\lambda^\delta(a,c,j,n;q)}{1-\alpha}|a_j|z^{j-1}\right|\\&\le|z|\sum_{j=2}^{\infty}\frac{E_2B_\lambda^\delta(a,c,j,n;q)}{1-\alpha}|a_j|\\&\le|z|.\end{aligned}$$

This completes the proof. □

In Theorems 2.4, 2.8, 2.9, and 2.11 if we substitute $\beta=1$, then we get the results for the class $\mathcal{VS}_{\lambda,\delta}^{\alpha}(a,c,n;q)$.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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