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Research Article

Two Versions of Quadratic-Phase Hankel Transformations of Random Order

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Abstract. In this study, we have eliminated the restriction on the remaining parameters for the two versions of *quadratic phase Hankel transformations* (QPHT) with the aid of a new parameter. Furthermore, the QPHT of random order has been used to solve a few differential equations.

Keywords. Hankel transformation, Quadratic-phase Hankel transformation, Zemanian type spaces, Distribution

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1. Introduction

The Hankel transformations H_μ of a conventional function f is defined by

$$(H_\mu f)(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy) f(x) dx, \quad \mu \ge -\frac{1}{2}, \tag{1.1}$$

where J_μ is the Bessel function of first kind having order μ. Earlier various type of Hankel transformations has been considered by Linares and Pérez [2], Malgonde and Debnath [3], Malgonde and Bandewar [4], Mendez [5], Pathak [6], Pérez and Robayna [7], Prasad and Kumar [8], Torre [10], and Zemanian [11]. Quadratic-phase Hankel transformations are the generalization of most of the Hankel transformations defined so far.

Prasad *et al*. [9] defined the first and second quadratic-phase Hankel transformations $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}$ and $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}$ depending on five real parameters a,b,c,d,e and four additional real-valued parameters μ,ν,α and β where $b\neq 0$ with $\nu\mu+2\nu-\alpha\geq 1$ of any functions f and g on $I=(0,\infty)$ as follows:

$$(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f)(y)=\int_0^\infty K^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}(y,x)f(x)dx \tag{1.2}$$

and

$$(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}g)(y)=\int_0^\infty K^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}(y,x)g(x)dx, \tag{1.3}$$

where

$$K^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}(y,x)=\nu\beta\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}y^{-1-2\alpha+2\nu}e^{i\beta(ax^{2\nu}+cy^{2\nu}+dx^{\nu}+ey^{\nu})}(xy)^{\alpha}J_\mu\left(\frac{\beta}{b}(xy)^{\nu}\right) \tag{1.4}$$

and

$$K^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}(y,x)=\nu\beta\frac{e^{-i\frac{\pi}{2}(1+\mu)}}{b}x^{-1-2\alpha+2\nu}e^{i\beta(ax^{2\nu}+cy^{2\nu}+dx^{\nu}+ey^{\nu})}(xy)^{\alpha}J_\mu\left(\frac{\beta}{b}(xy)^{\nu}\right). \tag{1.5}$$

The inversion formula of (1.2) and (1.3) are, respectively, given as:

$$f(x)=\left(H^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta}(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f)(y)\right)(x)=\int_0^\infty K^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta}(x,y)(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f)(y)dy \tag{1.6}$$

and

$$g(x)=\left(H^{-c,-b,-a;-e,-d}_{2,\mu,\nu,\alpha,\beta}(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}g)(y)\right)(x)=\int_0^\infty K^{-c,-b,-a;-e,-d}_{2,\mu,\nu,\alpha,\beta}(x,y)(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}g)(y)dy. \tag{1.7}$$

Prasad *et al*. [9] defined the space containing all smooth functions $\phi(x)$ defined on $I=(0,\infty)$ which satisfies

$$\gamma^{1,\mu,\nu,\alpha,\beta}_{q,k,a;d}(\phi)=\sup_{x\in I}|x^q(x^{1-2\nu}D_x)^k e^{\pm i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu+\alpha-2\nu+1}\phi(x)|<\infty, \tag{1.8}$$

exist for all non-negative integers q,k is known as Zemanian type space $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}$.

Further, on the other hand $\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}$ is the space containing all smooth functions $\psi(x)$ on I such that

$$\gamma^{2,\mu,\nu,\alpha,\beta}_{q,k,a;d}(\psi)=\sup_{x\in I}|x^q(x^{1-2\nu}D_x)^k e^{\pm i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu-\alpha}\psi(x)|<\infty, \tag{1.9}$$

exist for all non-negative integers q,k.

The dual of the space $\mathbb{H}^{a;d}_{p,\mu,\nu,\alpha,\beta}$ is represented by $\mathbb{H}'^{,a;d}_{p,\mu,\nu,\alpha,\beta}$, $p=1,2$ and their members are generalized function of slow growth.

According to Prasad *et al*. [9] we have the following differential operators:

$$M^{a;d}_{\mu,\nu,\alpha,\beta}=e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu-\alpha}D_x x^{\nu\mu+\alpha-\nu+1}e^{i\beta(ax^{2\nu}+dx^{\nu})}, \tag{1.10}$$

$$N^{a;d}_{\mu,\nu,\alpha,\beta}=e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{\nu\mu-\alpha+\nu}D_x x^{-\nu\mu+\alpha-2\nu+1}e^{i\beta(ax^{2\nu}+dx^{\nu})}, \tag{1.11}$$

$$\begin{aligned}\Delta^{a;d}_{1,\mu,\nu,\alpha,\beta}&=M^{a;d}_{\mu,\nu,\alpha,\beta}N^{a;d}_{\mu,\nu,\alpha,\beta}\\&=e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu-\alpha}D_x x^{2\nu\mu+1}D_x x^{-\nu\mu+\alpha-2\nu+1}e^{i\beta(ax^{2\nu}+dx^{\nu})}\\&=x^{-2\nu+2}D_x^2+[(3-4\nu+2\alpha)x^{-2\nu+1}+2id\beta\nu x^{-\nu+1}+4ia\beta\nu x]D_x-(4a^2\beta^2\nu^2x^{2\nu})\\&\quad+4ia\beta\nu(\alpha+1-\nu)+[id\beta\nu(2-3\nu+2\alpha)x^{-\nu}-(d^2\beta^2\nu^2)-4ad\nu^2\beta^2x^{\nu}]\\&\quad+((\alpha+1-2\nu)^2-\nu^2\mu^2)x^{-2\nu},\end{aligned} \tag{1.12}$$

$$M^{*,a;d}_{\mu,\nu,\alpha,\beta} = -e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{\nu\mu+\alpha-\nu+1}D_x x^{-\nu\mu-\alpha}e^{i\beta(ax^{2\nu}+dx^{\nu})}, \tag{1.13}$$

$$N^{*,a;d}_{\mu,\nu,\alpha,\beta} = -e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu+\alpha-2\nu+1}D_x x^{\nu\mu-\alpha+\nu}e^{i\beta(ax^{2\nu}+dx^{\nu})}, \tag{1.14}$$

$$\begin{aligned}\Delta^{a;d}_{2,\mu,\nu,\alpha,\beta} &= N^{*,a;d}_{\mu,\nu,\alpha,\beta}M^{*,a;d}_{\mu,\nu,\alpha,\beta}\\ &= e^{-i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu+\alpha-2\nu+1}D_x x^{2\nu\mu+1}D_x x^{-\nu\mu-\alpha}e^{i\beta(ax^{2\nu}+dx^{\nu})}\\ &= x^{-2\nu+2}D_x^2+[(1-2\alpha)x^{-2\nu+1}+2id\beta\nu x^{-\nu+1}+4ia\beta\nu x]D_x-(4a^2\beta^2\nu^2x^{2\nu})\\ &\quad +d\beta\nu(i\nu x^{-\nu}-d\beta\nu-2i\alpha x^{-\nu}-4a\beta\nu x^{\nu})+4ia\beta\nu(\nu-\alpha)+(\alpha^2-\nu^2\mu^2)x^{-2\nu}.\end{aligned} \tag{1.15}$$

The differential operators $N^{a;d}_{\mu,\nu,\alpha,\beta}$ and $M^{*,a;d}_{\mu,\nu,\alpha,\beta}$ have an inverse that is given by

$$(N^{a;d}_{\mu,\nu,\alpha,\beta})^{-1}\phi(x) = x^{\nu\mu-\alpha+2\nu-1}e^{-i\beta(ax^{2\nu}+dx^{\nu})}\int_{\infty}^{x}t^{-\nu\mu+\alpha-\nu}e^{i\beta(at^{2\nu}+dt^{\nu})}\phi(t)\mathrm{d}t, \tag{1.16}$$

$$(M^{*,a;d}_{\mu,\nu,\alpha,\beta})^{-1}\phi(x) = -x^{\nu\mu+\alpha}e^{-i\beta(ax^{2\nu}+dx^{\nu})}\int_{\infty}^{x}t^{-\nu\mu-\alpha+\nu-1}e^{i\beta(at^{2\nu}+dt^{\nu})}\phi(t)\mathrm{d}t. \tag{1.17}$$

The following results are obtained by repeating steps (1.11), (1.13), (1.16) and (1.17) in order:

$$\begin{aligned}&N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\dots N^{a;d}_{\mu+1,\nu,\alpha,\beta}N^{a;d}_{\mu,\nu,\alpha,\beta}\phi(y)\\ &\quad = e^{-i\beta(ay^{2\nu}+dy^{\nu})}y^{\nu\mu-\alpha+\nu(m+2)-1}(y^{1-2\nu}D_y)^m e^{i\beta(ay^{2\nu}+dy^{\nu})}y^{-\nu\mu+\alpha-2\nu+1}\phi(y),\end{aligned} \tag{1.18}$$

$$\begin{aligned}&(N^{a;d}_{\mu,\nu,\alpha,\beta})^{-1}(N^{a;d}_{\mu+1,\nu,\alpha,\beta})^{-1}\cdots(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta})^{-1}\phi(x)\\ &\quad = x^{\nu\mu-\alpha+2\nu-1}e^{-i\beta(ax^{2\nu}+dx^{\nu})}\int_{\infty}^{x}t_0^{2\nu-1}\int_{\infty}^{t_0}t_1^{2\nu-1}\cdots\int_{\infty}^{t_{m-2}}t_{m-1}^{-\nu\mu-\nu m+\alpha}\\ &\qquad \times e^{i\beta(at_{m-1}^{2\nu}+dt_{m-1}^{\nu})}\phi(t_{m-1})\mathrm{d}t_{m-1}\dots\mathrm{d}t_1\,\mathrm{d}t_0,\end{aligned} \tag{1.19}$$

$$\begin{aligned}&M^{*,a;d}_{\mu+m-1,\nu,\alpha,\beta}\cdots M^{*a;d}_{\mu+1,\nu,\alpha,\beta}M^{*,a;d}_{\mu,\nu,\alpha,\beta}\phi(y)\\ &\quad = (-1)^m e^{-i\beta(ay^{2\nu}+dy^{\nu})}y^{\nu(\mu+m)+\alpha}(y^{1-2\nu}D_y)^m y^{-\nu\mu-\alpha}e^{i\beta(ay^{2\nu}+dy^{\nu})}\phi(y),\end{aligned} \tag{1.20}$$

$$\begin{aligned}&(M^{*,a;d}_{\mu,\nu,\alpha,\beta})^{-1}(M^{*,a;d}_{\mu+1,\nu,\alpha,\beta})^{-1}\cdots(M^{*,a;d}_{\mu+m-1,\nu,\alpha,\beta})^{-1}\phi(x)\\ &\quad = (-1)^m x^{\nu\mu+\alpha}e^{-i\beta(ax^{2\nu}+dx^{\nu})}\int_{\infty}^{x}t_0^{2\nu-1}\int_{\infty}^{t_0}t_1^{2\nu-1}\cdots\int_{\infty}^{t_{m-2}}t_{m-1}^{-\nu(\mu+m-2)-\alpha-1}\\ &\qquad \times e^{i\beta(at_{m-1}^{2\nu}+dt_{m-1}^{\nu})}\phi(t_{m-1})\mathrm{d}t_{m-1}\dots\mathrm{d}t_1\,\mathrm{d}t_0.\end{aligned} \tag{1.21}$$

We have the following theorems as per [9].

Theorem 1.1. *For $\nu\mu+2\nu-\alpha\ge 1$, the transformation $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}$ is a continuous linear map from $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$, and its inverse is also continuous linear map from $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$.*

Theorem 1.2. *For $\nu\mu+2\nu-\alpha\ge 1$, the transformation $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta}$ is a continuous linear map from $\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{2,\mu,\nu,\alpha,\beta}(I)$, and its inverse is a also continuous linear map from $\mathbb{H}^{-c;-e}_{2,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$.*

For further work, the following recurrence relations will be utilized:

(i) $$(x^{1-2\nu}D_x)^n[x^{\nu\mu}J_\mu(\beta x^\nu)] = (\nu\beta)^n x^{\nu(\mu-n)}J_{\mu-n}(\beta x^\nu), \tag{1.22}$$

(ii) $$(x^{1-2\nu}D_x)^n[x^{-\nu\mu}J_\mu(\beta x^\nu)] = (-\nu\beta)^n x^{-\nu(\mu+n)}J_{\mu+n}(\beta x^\nu). \tag{1.23}$$

This work is split into four sections. Section 1 provides an overview of quadratic-phase Hankel transformations and Zemanian-type spaces. Also, some differential and integral operators are listed. Section 2 is devoted to the study of quadratic-phase Hankel transformations of random order and its inverse. Also, the continuity of these transformations across Zemanian-type spaces are analyzed. A few fundamental characteristics of QPHT of random order using differential operators are provided. The generalized QPHT of random order are introduced in Section 3. The last section discusses the use of the QPHT of random order in the solution of generalized partial differential equations.

2. Quadratic-Phase Hankel Transformations (QPHT) of Random Order

Motivated by the work of Linares and Pérez [2], Pathak [6] and Zemanian [11] for the conventional Hankel transforms, we have introduced the quadratic-phase Hankel transformation of random order. The first and second quadratic-phase Hankel transformations are defined in (1.2) and (1.3) for the condition $\nu\mu+2\nu-\alpha\geq 1$. In this part, we have eliminated this restriction by using an additional positive integer parameter m together with μ,ν,α,β and a,b,c,d,e as previously mentioned, such that $\nu\mu+2\nu-\alpha+m\geq 1$ and defining the extended first and second quadratic-phase Hankel transformations $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ and $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}$ of any function $f\in\mathbb{H}^{a;d,}_{1,\mu,\nu,\alpha,\beta}(I)$ and $g\in\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$, respectively, as:

$$\begin{aligned}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y) &= F(y)\\ &= (-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}\left(H^{a,b,c;d,e}_{1,\mu+m,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\cdots N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g\right)(y) &= G(y)\\ &= \left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}\left(H^{a,b,c;d,e}_{2,\mu+m,\nu,\alpha,\beta}\left(M^{*,a;d}_{\mu+m-1,\nu,\alpha,\beta}\cdots M^{*,a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y).\end{aligned} \tag{2.2}$$

The inverse of (2.1) and (2.2) respectively are:

$$\begin{aligned}&\left(\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}F(y)\right)(x) = f(x)\\ &= (-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^m\left(\left(N^{a;d}_{\mu,\nu,\alpha,\beta}\right)^{-1}\cdots\left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\right)^{-1}\left(H^{-c,-b,-a;-e,-d}_{1,\mu+m,\nu,\alpha,\beta}y^{\nu m}F(y)\right)\right)(x)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\left(\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}\right)^{-1}G(y)\right)(x) = g(x)\\ &= \left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^m\left(\left(M^{*,a;d}_{\mu,\nu,\alpha,\beta}\right)^{-1}\cdots\left(M^{*,a;d}_{\mu+m-1,\nu,\alpha,\beta}\right)^{-1}\left(H^{-c,-b,-a;-e,-d}_{2,\mu+m,\nu,\alpha,\beta}y^{\nu m}G(y)\right)\right)(x).\end{aligned} \tag{2.4}$$

Theorem 2.1. *Let μ,ν,α,β be any real number and m a positive integer such that $\nu\mu+2\nu-\alpha+m\geq 1$. Then, the first QPHT of random order $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ is a continuous linear map from $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$. Furthermore, its inverse $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}$ is also continuous linear map from $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$.*

Proof. For all $q,k\in\mathbb{N}_0$ and $\psi\in\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$, evidently it is clear that

$$\gamma^{1,\mu+1,\nu,\alpha,\beta}_{q,k,a;d}\left(N^{a;d}_{\mu,\nu,\alpha,\beta}\psi\right)=\gamma^{1,\mu,\nu,\alpha,\beta}_{q,k+1,a;d}(\psi).$$

Thus, we get that $\psi\to N^{a;d}_{\mu,\nu,\alpha,\beta,}\psi$ is a continuous linear mapping from $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{1,\mu+1,\nu,\alpha,\beta}(I)$.

(i) Continuing in a similar manner, we have $\psi\to N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\dots N^{a;d}_{\mu,\nu,\alpha,\beta}\psi$ is a continuous linear mapping from $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{1,\mu+m,\nu,\alpha,\beta}(I)$.

(ii) Also, we note that $\psi\to H^{a,b,c;d,e}_{1,\mu+m,\nu,\alpha,\beta}\psi$ is a continuous linear mapping from $\mathbb{H}^{a;d}_{1,\mu+m,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{1,\mu+m,\nu,\alpha,\beta}(I)$.

(iii) As, $\gamma^{1,\mu,\nu,\alpha,\beta}_{q,k,-c;-e}\left(\left(\frac{\nu\beta}{b}x^{\nu}\right)^{-m}\psi\right)=\frac{\nu\beta}{b}\gamma^{1,\mu+m,\alpha,\beta}_{q,k,-c;-e}(\psi)$. Consequently, it follows that $\psi\to\left(\frac{\nu\beta}{b}x^{\nu}\right)^{-m}\psi$ is a continuous linear mapping from $\mathbb{H}^{-c;-e}_{1,\mu+m,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$.

When we combine (i), (ii), and (iii), we get that $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ is a continuous linear mapping from $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$. In the same way, the inverse transformation $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}$ is also continuous linear mapping from $\mathbb{H}^{-c;-e}_{1,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$. □

By using the same argument as in the preceding Theorem, we may prove the following remark for $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}$.

Remark 2.2. Let μ,ν,α,β be any real number and m a positive integer such that $\nu\mu+2\nu-\alpha+m\geq 1$. Then, the second QPHT of random order $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}$ is a continuous linear map from $\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{-c;-e}_{2,\mu,\nu,\alpha,\beta}(I)$. Furthermore, its inverse $\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}\right)^{-1}$ is also continuous linear map from $\mathbb{H}^{-c;-e}_{2,\mu,\nu,\alpha,\beta}(I)$ into $\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$.

Theorem 2.3. *For every positive integer m and whenever $\nu\mu+2\nu-\alpha\geq 1$, the first QPHT of random order $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ coincides with $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}$. Additionally, the inverse of first QPHT of random order $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}$ coincides with $H^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta}$.*

Proof. Suppose $\nu\mu+2\nu-\alpha\geq 1$ and $f\in\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$; then for $m=1$, we have

$$\begin{aligned}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,1}f\right)(y)&=-\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^{\nu}\right)^{-1}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta}\left(N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)\\&=-e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+\nu}e^{i\beta(cy^{2\nu}+ey^{\nu})}\int_0^{\infty}x^{\nu(\mu+1)}J_{\mu+1}\left(\frac{\beta}{b}(xy)^{\nu}\right)\\&\quad\times D_x\left[x^{-\nu\mu+\alpha-2\nu+1}e^{i\beta(ax^{2\nu}+dx^{\nu})}f(x)\right]\mathrm{d}x.\end{aligned}$$

Using integration by parts and (1.22), we have

$$= e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+\nu}e^{i\beta(cy^{2\nu}+ey^{\nu})}\int_0^\infty D_x\left[x^{\nu(\mu+1)}J_{\mu+1}\left(\frac{\beta}{b}(xy)^\nu\right)\right]$$
$$\times x^{-\nu\mu+\alpha-2\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)\mathrm{d}x$$
$$= e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+\nu}e^{i\beta(cy^{2\nu}+ey^{\nu})}\int_0^\infty x^{2\nu-1}(\frac{\nu\beta}{b}y^\nu)x^{\nu\mu}J_\mu\left(\frac{\beta}{b}(xy)^\nu\right)$$
$$\times x^{-\nu\mu+\alpha-2\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)\mathrm{d}x$$
$$= \frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-2\alpha+2\nu}\int_0^\infty e^{i\beta(ax^{2\nu}+cy^{2\nu}+dx^\nu+ey^\nu)}(xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu\right)f(x)\mathrm{d}x$$
$$= \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f\right)(y).$$

For $m=2$

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,2}f\right)(y) = \left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^{-2}\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu+2)}y^{-1-\alpha}e^{i\beta(cy^{2\nu}+ey^\nu)}\int_0^\infty x^{\nu\mu+2\nu}$$
$$\times J_{\mu+2}\left(\frac{\beta}{b}(xy)^\nu\right)D_x x^{1-2\nu}D_x e^{i\beta(ax^{2\nu}+dx^\nu)}x^{-\nu\mu+\alpha-2\nu+1}f(x)\mathrm{d}x$$
$$= -\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^{-2}\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu+2)}y^{-1-\alpha}e^{i\beta(cy^{2\nu}+ey^\nu)}\int_0^\infty\left(\frac{\nu\beta}{b}y^\nu\right)$$
$$\times x^{\nu(\mu+1)}J_{\mu+1}\left(\frac{\beta}{b}(xy)^\nu\right)D_x\left[e^{i\beta(ax^{2\nu}+dx^\nu)}x^{-\nu\mu+\alpha-2\nu+1}f(x)\right]\mathrm{d}x$$
$$= \frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-2\alpha+2\nu}\int_0^\infty e^{i\beta(ax^{2\nu}+cy^{2\nu}+dx^\nu+ey^\nu)}(xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu\right)f(x)\mathrm{d}x$$
$$= \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f\right)(y).$$

In the same way, induction may now be used to prove the statement.

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y) = \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta}f\right)(y).$$

Now, for the inverse $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}$.

$$f(x) = \left(\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}F(y)\right)(x)$$
$$= (-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^m\left(\left(N^{a;d}_{\mu,\nu,\alpha,\beta}\right)^{-1}\cdots\left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\right)^{-1}\left(H^{-c,-b,-a;-e,-d}_{1,\mu+m,\nu,\alpha,\beta}y^{\nu m}F(y)\right)\right)(x).$$

Considering $m=1$ and using the recurrence relation (1.22), we obtain

$$\left(\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,1}\right)^{-1}F(y)\right)(x) = -\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}}\left(\left(N^{a;d}_{\mu,\nu,\alpha,\beta}\right)^{-1}\left(H^{-c,-b,-a;-e,-d}_{1,\mu+1,\nu,\alpha,\beta}y^\nu F(y)\right)\right)(x)$$
$$= -\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}}x^{\nu\mu-\alpha+2\nu-1}e^{-i\beta(ax^{2\nu}+dx^\nu)}\int_\infty^x t^{-\nu\mu+\alpha-\nu}e^{i\beta(at^{2\nu}+dt^\nu)}$$
$$\times\left(e^{i\frac{\pi}{2}(1+\mu+1)}\frac{\nu\beta}{b}t^{-1-2\alpha+2\nu}\int_0^\infty e^{-i\beta(at^{2\nu}+cy^{2\nu}+dt^\nu+ey^\nu)}(ty)^\alpha\right.$$
$$\left.\times J_{\mu+1}\left(\frac{\beta}{b}(ty)^\nu\right)y^\nu F(y)\mathrm{d}y\right)\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{\nu\beta}{b} e^{i\frac{\pi}{2}(1+\mu)} x^{\nu\mu-\alpha+2\nu-1} e^{-i\beta(ax^{2\nu}+dx^{\nu})} \int_0^\infty y^\alpha e^{-i\beta(cy^{2\nu}+ey^{\nu})} F(y) \\
&\quad \times \left(\int_\infty^x \left(-\frac{\nu\beta}{b} y^\nu \right) t^{-\nu\mu-1+\nu} J_{\mu+1}\left(\frac{\beta}{b}(ty)^\nu \right) \mathrm{d}t \right) \mathrm{d}y \\
&= \frac{\nu\beta}{b} e^{i\frac{\pi}{2}(1+\mu)} x^{-1-2\alpha+2\nu} \int_0^\infty e^{-i\beta\left(ax^{2\nu}+cy^{2\nu}+dx^\nu+ey^\nu\right)} \\
&\quad \times (xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu \right) F(y)\mathrm{d}y \\
&= \left(H^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta} F(y) \right)(x).
\end{aligned}
$$

For $m = 2$,

$$
\begin{aligned}
\left(\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,2} \right)^{-1} F(y) \right)(x) &= (-1)^2 \left(\frac{\nu\beta}{b} e^{-i\frac{\pi}{2}} \right)^2 \left(\left(N^{a;d}_{\mu,\nu,\alpha,\beta} \right)^{-1} \left(N^{a;d}_{\mu+1,\nu,\alpha,\beta} \right)^{-1} \right. \\
&\quad \times \left. \left(H^{-c,-b,-a;-e,-d}_{1,\mu+2,\nu,\alpha,\beta} y^{2\nu} F(y) \right) \right)(x) \\
&= \left(\frac{\nu\beta}{b} e^{-i\frac{\pi}{2}} \right)^2 x^{\nu\mu-\alpha+2\nu-1} e^{-i\beta(ax^{2\nu}+dx^\nu)} \int_\infty^x t_0^{2\nu-1} \int_\infty^{t_0} t_1^{-\nu\mu-1} \\
&\quad \times e^{i\beta(at_1^{2\nu}+dt_1^\nu)} \frac{\nu\beta}{b} e^{i\frac{\pi}{2}(\mu+3)} \int_0^\infty e^{-i\beta\left(at_1^{2\nu}+cy^{2\nu}+dt_1^\nu+ey^\nu\right)} y^\alpha \\
&\quad \times J_{\mu+2}\left(\frac{\beta}{b}(t_1 y)^\nu \right) F(y) y^{2\nu} \mathrm{d}y\ \mathrm{d}t_1\ \mathrm{d}t_0 \\
&= -\left(\frac{\nu\beta}{b} \right)^2 x^{\nu\mu-\alpha+2\nu-1} e^{-i\beta(ax^{2\nu}+dx^\nu)} \int_\infty^x t_0^{2\nu-1} \int_0^\infty e^{i\frac{\pi}{2}(1+\mu)} e^{-i\beta(cy^{2\nu}+ey^\nu)} \\
&\quad \times F(y)\ y^{\nu+\alpha}\ t_0^{-\nu(\mu+1)} J_{\mu+1}\left(\frac{\beta}{b}(t_0 y)^\nu \right) \mathrm{d}y\ \mathrm{d}t_0 \\
&= \frac{\nu\beta}{b}\ e^{i\frac{\pi}{2}(1+\mu)} x^{-1-2\alpha+2\nu} \int_0^\infty e^{-i\beta\left(ax^{2\nu}+cy^{2\nu}+dx^\nu+ey^\nu\right)} (xy)^\alpha J_\mu\left(\frac{\beta}{b}(xy)^\nu \right) \\
&\quad \times F(y)\mathrm{d}y \\
&= \left(H^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta} F(y) \right)(x).
\end{aligned}
$$

Hence, proceeding similarly, we obtain

$$
\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} \right)^{-1} = \left(H^{-c,-b,-a;-e,-d}_{1,\mu,\nu,\alpha,\beta} \right). \qquad \square
$$

Lemma 2.4. *Let $m,k \in \mathbb{Z}_+$ be any positive integers such that $m,k \geq -(\nu\mu+2\nu-\alpha)$. Then the extended quadratic-phase Hankel transform $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ coincides with $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,k}$ on $\mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$.*

Proof. From definition of QPHT of random order defined as (2.1), it is clear that $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ remains independent of the choice of m whenever $\nu\mu+2\nu-\alpha \geq 1$. Indeed for $m > k > 0$ using the Theorem 2.3, we observe that $H^{a,b,c;d,e}_{1,\mu+k,\nu,\alpha,\beta,m-k} = H^{a,b,c;d,e}_{1,\mu+k,\nu,\alpha,\beta}$, hence

$$
\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f \right)(y) = (-1)^m \left(e^{-i\frac{\pi}{2}} \frac{\nu\beta}{b} y^\nu \right)^{-m} \left(H^{a,b,c;d,e}_{1,\mu+m,\nu,\alpha,\beta} \left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta} \dots N^{a;d}_{\mu,\nu,\alpha,\beta} f \right) \right)(y)
$$

$$= (-1)^k \left(e^{-i\frac{\pi}{2}} \frac{\nu\beta}{b} y^\nu\right)^{-k} (-1)^{m-k} \left(e^{-i\frac{\pi}{2}} \frac{\nu\beta}{b} y^\nu\right)^{-(m-k)}$$
$$\times \left(H^{a,b,c;d,e}_{1,\mu+k+m-k,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+k+m-k-1,\nu,\alpha,\beta} \dots N^{a;d}_{\mu,\nu,\alpha,\beta} f\right)\right)(y).$$

Applying Theorem 2.3, the previous expression becomes

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y) = (-1)^k \left(e^{-i\frac{\pi}{2}} \frac{\nu\beta}{b} y^\nu\right)^{-k} \left(H^{a,b,c;d,e}_{1,\mu+k,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+k-1,\nu,\alpha,\beta} \dots N^{a;d}_{\mu,\nu,\alpha,\beta} f\right)\right)(y)$$
$$= \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,k} f\right)(y).$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let μ,ν,α,β be any real numbers and m be a positive integer such that $\nu\mu + 2\nu - \alpha + m \geq 1$. Then for $f \in \mathbb{H}^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)$, we have*

(i) $N^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y) = -i\dfrac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}(x^\nu f)\right)(y),$

(ii) $\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}\left(N^{a;d}_{\mu,\nu,\alpha,\beta} f\right)\right)(y) = i\dfrac{\nu\beta}{b} y^\nu \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y),$

(iii) $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(\Delta^{a;d}_{1,\mu,\nu,\alpha,\beta} f\right)\right)(y) = -\left(\dfrac{\nu\beta}{b} y^\nu\right)^2 \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y),$

(iv) $\Delta^{-c;-e}_{1,\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y) = -\left(\dfrac{\nu\beta}{b}\right)^2 \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} x^{2\nu} f\right)(y).$

If the function $f \in \mathbb{H}^{a;d}_{1,\mu+1,\nu,\alpha,\beta}(I)$, then

(v) $\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(M^{a;d}_{\mu,\nu,\alpha,\beta} f\right)\right)(y) = i\dfrac{\nu\beta}{b} y^\nu \left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y),$

(vi) $M^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y) = -i\dfrac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}(x^\nu f)\right)(y).$

Proof.

(i) $$N^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y)$$
$$= (-1)^m e^{im\frac{\pi}{2}} \left(\frac{\nu\beta}{b}\right)^{-m} e^{i\beta(cy^{2\nu}+ey^\nu)} y^{\nu\mu-\alpha+\nu} \int_0^\infty \left(\frac{\nu\beta}{b}\right) e^{-i\frac{\pi}{2}(1+\mu+m)}$$
$$\times D_y\left(y^{-\nu(\mu+m)} J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)\right) x^\alpha e^{i\beta(ax^{2\nu}+dx^\nu)} \left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta} \dots N^{a;d}_{\mu,\nu,\alpha,\beta} f\right)(x)\mathrm{d}x. \tag{2.5}$$

Now,

$$x^\nu N^{a;d}_{\mu+m-1,\nu,\alpha,\beta} \dots N^{a;d}_{\mu,\nu,\alpha,\beta} f(x)$$
$$= e^{-i\beta(ax^{2\nu}+dx^\nu)} x^{\nu(\mu+1)-\alpha+\nu(m+2)-1} \left(x^{1-2\nu} D_x\right)^m e^{i\beta(ax^{2\nu}+dx^\nu)} x^{-\nu(\mu+1)+\alpha-2\nu+1} x^\nu f(x)$$
$$= N^{a;d}_{\mu+m,\nu,\alpha,\beta} \dots N^{a;d}_{\mu+1,\nu,\alpha,\beta} x^\nu f(x). \tag{2.6}$$

Using (2.6) in (2.5), we have

$$N^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m} f\right)(y)$$
$$= (-1)^{m+1} \left(\frac{\nu\beta}{b}\right) e^{i\frac{\pi}{2}} \left(e^{-i\frac{\pi}{2}} \frac{\nu\beta}{b} y^\nu\right)^{-m} \frac{\nu\beta}{b} e^{-i\frac{\pi}{2}(2+\mu+m)} y^{-1-2\alpha+2\nu}$$

$$\times \int_0^\infty (xy)^\alpha J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right) e^{i\beta(ax^{2\nu}+cy^{2\nu}+dx^\nu+ey^\nu)}\left(N^{a;d}_{\mu+m,\nu,\alpha,\beta}\dots N^{a;d}_{\mu+1,\nu,\alpha,\beta}x^\nu f(x)\right)(x)\mathrm{d}x$$
$$= -i\frac{\nu\beta}{b}(-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}\left(H^{a,b,c;d,e}_{1,\mu+m+1,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+m,\nu,\alpha,\beta}\dots N^{a;d}_{\mu+1,\nu,\alpha,\beta}x^\nu f\right)\right)(y)$$
$$= -i\frac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}x^\nu f\right)(y).$$

Thus the result (i) is obtained.

(ii) $\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)(y)$

$$= (-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}\left(H^{a,b,c;d,e}_{1,\mu+1+m,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+1+m-1,\nu,\alpha,\beta}\dots N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)$$
$$= (-1)\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)(-1)^{m+1}\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-(m+1)}\left(H^{a,b,c;d,e}_{1,\mu+m+1,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+m,\nu,\alpha,\beta}\dots N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)$$
$$= i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m+1}f\right)(y).$$

Using Lemma 2.4,

$$\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}N^{a;d}_{\mu,\nu,\alpha,\beta}f\right)(y) = i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y).$$

Thus, we get the required result (ii).

In order to prove the property (v), let $f \in \mathbb{H}^{a;d}_{1,\mu+1,\nu,\alpha,\beta}$. Then, applying (1.18), we obtain

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(M^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)$$
$$= (-1)^m\left(\frac{\nu\beta}{b}y^\nu\right)^{-m}\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu}\int_0^\infty e^{i\beta(cy^{2\nu}+ey^\nu)}J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)$$
$$\times x^{\nu\mu+\nu(m+2)-1}\left(x^{1-2\nu}D_x\right)^m x^{-2\nu\mu-2\nu+1}D_x x^{\nu\mu+\alpha-\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)\mathrm{d}x$$
$$= (-1)^m\left(\frac{\nu\beta}{b}y^\nu\right)^{-m}\frac{\nu\beta}{b}e^{-\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu}e^{i\beta(cy^{2\nu}+ey^\nu)}\int_0^\infty J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)$$
$$\times x^{\nu\mu+\nu(m+2)-1}\Big[2\nu\mu\left(x^{1-2\nu}D_x\right)^m x^{-\nu\mu+\alpha-3\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)$$
$$+\left(x^{1-2\nu}D_x\right)^{m+1}x^{-\nu\mu+\alpha-\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)\Big]\mathrm{d}x.$$

Suppose that $f_1(x) = x^{-\nu\mu+\alpha-3\nu+1}e^{i\beta(ax^{2\nu}+dx^\nu)}f(x)$ and reducing the second term on the right side, we obtain

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(M^{a;d}_{\mu,\nu,\alpha,\beta}f\right)\right)(y)$$
$$= (-1)^m\left(\frac{\nu\beta}{b}y^\nu\right)^{-m}\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu}e^{i\beta(cy^{2\nu}+ey^\nu)}\int_0^\infty J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)$$
$$\times x^{\nu\mu+\nu(m+2)-1}\left[2\nu(\mu+m+1)\left(x^{1-2\nu}D_x\right)^m f_1(x) + x^{2\nu}\left(x^{1-2\nu}D_x\right)^{m+1}f_1(x)\right]\mathrm{d}x. \tag{2.7}$$

We now have to show $i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}f\right)(y)$ is equivalent to (2.7). Therefore, using (1.23), we obtain

$$i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}f\right)(y)$$

$$= i\left(\frac{\nu\beta}{b}\right)^2 y^\nu \left(-\frac{\nu\beta}{b}y^\nu\right)^{-m} e^{-i\frac{\pi}{2}(2+\mu)} y^{-1-\alpha+2\nu} \int_0^\infty e^{i\beta(cy^{2\nu}+ey^\nu)} J_{\mu+m+1}$$

$$\left(\frac{\beta}{b}(xy)^\nu\right) x^{\nu(\mu+1)+\nu(m+2)-1} \left(x^{1-2\nu}D_x\right)^m e^{i\beta(ax^{2\nu}+dx^\nu)} x^{-\nu(\mu+1)+\alpha-2\nu+1} f(x)\mathrm{d}x, \tag{2.8}$$

we know that

$$J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right) = \left(-\frac{\nu\beta}{b}y^\nu\right)^{-1} x^{1-2\nu} x^{\nu(\mu+m+1)} D_x\left[x^{-\nu(\mu+m)} J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)\right].$$

Therefore, (2.8) becomes

$$i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y)$$

$$= (-1)^{m+1}\left(\frac{\nu\beta}{b}y^\nu\right)^{-m} \frac{\nu\beta}{b} e^{-i\frac{\pi}{2}(1+\mu)} y^{-1-\alpha+2\nu}$$

$$\times e^{i\beta(cy^{2\nu}+ey^\nu)} \int_0^\infty x^{2\nu\mu+2\nu m+2\nu} D_x\left[x^{-\nu(\mu+m)} J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)\right]$$

$$\times \left(x^{1-2\nu}D_x\right)^m e^{i\beta(ax^{2\nu}+dx^\nu)} x^{-\nu(\mu+1)+\alpha-2\nu+1} f(x)\mathrm{d}x$$

$$= (-1)^m \left(\frac{\nu\beta}{b}y^\nu\right)^{-m} \frac{\nu\beta}{b} e^{-i\frac{\pi}{2}(1+\mu)} y^{-1-\alpha+2\nu} e^{i\beta(cy^{2\nu}+ey^\nu)} \int_0^\infty J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)$$

$$\times x^{\nu(\mu+m+2)-1}\left[2\nu(\mu+m+1)\left(x^{1-2\nu}D_x\right)^m f_1(x) + x^{2\nu}\left(x^{1-2\nu}D_x\right)^{m+1} f_1(x)\right]\mathrm{d}x. \tag{2.9}$$

Using (2.7) and (2.9), we conclude that

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(M^{a;d}_{\mu,\nu,\alpha,\beta} f\right)\right)(y) = i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y).$$

For (vi), let $f \in \mathbb{H}^{a;d}_{1,\mu+1,\nu,\alpha,\beta}(I)$ be any function. Next, we have

$$M^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y)$$

$$= e^{i\beta(cy^{2\nu}+ey^\nu)} y^{-\nu\mu-\alpha} D_y y^{\nu\mu+\alpha-\nu+1} e^{-i\beta(cy^{2\nu}+ey^\nu)} (-1)^m \left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}$$

$$\times \int_0^\infty J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right) x^{\nu(\mu+m+3)-1}\left(x^{1-2\nu}D_x\right)^m e^{i\beta(ax^{2\nu}+dx^\nu)} x^{-\nu\mu+\alpha-3\nu+1} f(x)\mathrm{d}x.$$

Let $e^{i\beta(ax^{2\nu}+dx^\nu)} x^{-\nu\mu+\alpha-3\nu+1} f(x) = f_1(x)$ and using (1.22). Then

$$M^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m} f\right)(y)$$

$$= (-1)^m \left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^{-m} e^{i\beta(cy^{2\nu}+ey^\nu)} y^{-\nu\mu-\alpha} \frac{\nu\beta}{b} e^{-i\frac{\pi}{2}(2+\mu+m)}$$

$$\times \int_0^\infty \left[-2\nu m y^{\nu(\mu-m+1)-1} J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right)\right.$$

$$\left. + y^{-2\nu m} D_y\left(y^{\nu(\mu+m+1)} J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right)\right)\right] x^{\nu(\mu+m+3)-1}\left(x^{1-2\nu}D_x\right)^m f_1(x)\mathrm{d}x$$

$$= 2\nu m(-1)^{m+1}\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^{-m} e^{i\beta(cy^{2\nu}+ey^\nu)} \frac{\nu\beta}{b} e^{-i\frac{\pi}{2}(2+\mu+m)} y^{-\nu m+\nu-\alpha-1}$$

$$\times \int_0^\infty x^{\nu(\mu+m+3)} x^{-1} J_{\mu+m+1}\left(\frac{\beta}{b}(xy)^\nu\right)\left(x^{1-2\nu}D_x\right)^m f_1(x)\mathrm{d}x$$

$$+(-1)^m\left(e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\right)^{-m}\left(\frac{\nu\beta}{b}\right)^2 e^{-i\frac{\pi}{2}(2+\mu+m)}y^{2\nu-\nu m-\alpha-1}$$
$$\times\int_0^\infty x^{\nu(\mu+m+4)-1}J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)\left(x^{1-2\nu}D_x\right)^m f_1(x)\mathrm{d}x.$$

When the first integral is integrated by parts, the above expression becomes

$$e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}\left(-e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^\nu\right)^{-m}\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu+m)}y^{-1-\alpha+2\nu}e^{i\beta(cy^{2\nu}+ey^\nu)}$$
$$\times\int_0^\infty J_{\mu+m}\left(\frac{\beta}{b}(xy)^\nu\right)x^{\nu\mu+\nu(m+2)-1}\left(x^{1-2\nu}D_x\right)^m e^{i\beta(ax^{2\nu}+dx^\nu)}x^{-\nu\mu+\alpha-\nu+1}f(x)\mathrm{d}x$$
$$=-i\frac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\left(x^\nu f\right)\right)(y).$$

In a similar manner, we can easily verify (iii) by using (ii) and (v) and obtain (iv) using (i) and (vi). □

Similar calculations lead to the following results for $H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}$.

Lemma 2.6. *Let μ,ν,α,β be any real numbers and m be a positive integer such that $\nu\mu+2\nu-\alpha+m\geq 1$. Then, for any $g\in\mathbb{H}^{a;d}_{2,\mu,\nu,\alpha,\beta}(I)$, we have*

(i) $M^{*,-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g\right)(y)=i\frac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{2,\mu+1,\nu,\alpha,\beta,m}\left(x^\nu g\right)\right)(y)$,

(ii) $\left(H^{a,b,c;d,e}_{2,\mu+1,\nu,\alpha,\beta,m}\left(M^{*,a;d}_{\mu,\nu,\alpha,\beta}g\right)\right)(y)=-i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g\right)(y)$,

(iii) $\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}\left(\Delta^{a;d}_{2,\mu,\nu,\alpha,\beta}g\right)\right)(y)=-\left(\frac{\nu\beta}{b}y^\nu\right)^2\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g\right)(y)$,

(iv) $\Delta^{-c;-e}_{2,\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g\right)(y)=-\left(\frac{\nu\beta}{b}\right)^2\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}x^{2\nu}g\right)(y)$.

If $g\in\mathbb{H}^{a;d}_{2,\mu+1,\nu,\alpha,\beta}(I)$, then

(v) $\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}\left(N^{*,a;d}_{\mu,\nu,\alpha,\beta}g\right)\right)(y)=-i\frac{\nu\beta}{b}y^\nu\left(H^{a,b,c;d,e}_{2,\mu+1,\nu,\alpha,\beta,m}g\right)(y)$,

(vi) $N^{*,-c;-e}_{\mu,\nu,\alpha,\beta}\left(H^{a,b,c;d,e}_{2,\mu+1,\nu,\alpha,\beta,m}g\right)(y)=i\frac{\nu\beta}{b}\left(H^{a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}\left(x^\nu g\right)\right)(y)$.

3. Generalized Quadratic-Phase Hankel Transformations of Random Order

For any positive integer m, assume that μ,ν,α,β are all real parameters such that $\nu\mu+2\nu-\alpha+m\geq 1$. Then the first and second generalized quadratic-phase Hankel transformation of random order $H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ and $H'^{,a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}$ respectively defined on $\mathbb{H}'^{,a;d}_{2,\mu,\nu,\alpha,\beta}(I)$ and $\mathbb{H}'^{,a;d}_{1,\mu,\nu,\alpha,\beta}(I)$ as the adjoint operator of $H^{c,b,a;e,d}_{2,\mu,\nu,\alpha,\beta,m}$ acting on $\mathbb{H}^{c;e}_{2,\mu,\nu,\alpha,\beta}(I)$ and adjoint operator of $H^{c,b,a;e,d}_{1,\mu,\nu,\alpha,\beta,m}$ acting on $\mathbb{H}^{c;e}_{1,\mu,\nu,\alpha,\beta}(I)$ respectively defined as

$$\left\langle H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f,\phi\right\rangle=\left\langle f,H^{c,b,a;e,d}_{2,\mu,\nu,\alpha,\beta,m}\phi\right\rangle,\quad\forall\ f\in\mathbb{H}'^{,a;d}_{2,\mu,\nu,\alpha,\beta}(I)\ \text{ and }\ \phi\in\mathbb{H}^{c;e}_{2,\mu,\nu,\alpha,\beta}(I)$$

and

$$\left\langle H'^{,a,b,c;d,e}_{2,\mu,\nu,\alpha,\beta,m}g,\psi\right\rangle=\left\langle g,H^{c,b,a;e,d}_{1,\mu,\nu,\alpha,\beta,m}\psi\right\rangle,\quad\forall\ g\in\mathbb{H}'^{a;d}_{1,\mu,\nu,\alpha,\beta}(I)\ \text{ and }\ \psi\in\mathbb{H}^{c;e}_{1,\mu,\nu,\alpha,\beta}(I).$$

Theorem 3.1. *With q being a positive integer and μ,ν,α,β being any real numbers, let $\nu\mu+2\nu-\alpha+m\geq 1$. Then, we get the following properties for $f\in\mathbb{H}'^{,\alpha;d}_{2,\mu,\nu,\alpha,\beta}(I)$ as*

(i) $N^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y)=-i\dfrac{\nu\beta}{b}\left(H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}x^{\nu}f\right)(y)$,

(ii) $\left(H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}(N^{a;d}_{\mu,\nu,\alpha,\beta}f)\right)(y)=i\dfrac{\nu\beta}{b}y^{\nu}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y)$,

(iii) $\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}(\Delta^{a;d}_{1,\mu,\nu,\alpha,\beta}f)\right)(y)=-\left(\dfrac{\nu\beta}{b}y^{\nu}\right)^{2}\left(H'^{,a,b,c,;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y)$,

(iv) $\Delta^{-c;-e}_{1,\mu,\nu,\alpha,\beta}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y)=-\left(\dfrac{\nu\beta}{b}\right)^{2}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}(x^{2\nu}f)\right)(y)$.

If $f\in\mathbb{H}'^{,\alpha;d}_{2,\mu+1,\nu,\alpha,\beta}(I)$, then

(v) $\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}(M^{a;d}_{\mu,\nu,\alpha,\beta}f)\right)(y)=i\dfrac{\nu\beta}{b}y^{\nu}\left(H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}f\right)(y)$,

(vi) $\left(M^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}f\right)(y)\right)=-i\dfrac{\nu\beta}{b}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}(x^{\nu}f)\right)(y)$.

Proof. For $f\in\mathbb{H}'^{,a;d}_{2,\mu,\nu,\alpha,\beta}(I)$ and using Lemma 2.6(v), we have

(i)
$$\begin{aligned}\left\langle N^{-c;-e}_{\mu,\nu,\alpha,\beta}H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f,\ \phi\right\rangle &=\left\langle H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f, N^{*,c;e}_{\mu,\nu,\alpha,\beta}\phi\right\rangle\\ &=\left\langle f, H^{c,b,a;e,d}_{2,\mu,\nu,\alpha,\beta,m}N^{*,c;e}_{\mu,\nu,\alpha,\beta}\phi\right\rangle\\ &=\left\langle f,-i\frac{\nu\beta}{b}y^{\nu}H^{c,b,a;e,d}_{2,\mu+1,\nu,\alpha,\beta,m}\phi\right\rangle\\ &=\left\langle -i\frac{\nu\beta}{b}H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}x^{\nu}f,\ \phi\right\rangle.\end{aligned}$$

Comparing the above result, we have

$$N^{-c;-e}_{\mu,\nu,\alpha,\beta}\left(H'^{,a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}f\right)(y)=-i\frac{\nu\beta}{b}\left(H'^{,a,b,c;d,e}_{1,\mu+1,\nu,\alpha,\beta,m}x^{\nu}f\right)(y).$$

Similarly, we can prove the other part. □

In the same manner as Theorem 3.1, we will get the results for the second generalized quadratic-phase Hankel transformation.

4. Applications

Here the theory of QPHT of random order has been applied to solve generalized partial differential equations. Utilizing the subsequent findings, the following generalized partial differential equations will be solved:

Problem 4.1. *If $\mu,\nu,\alpha,\beta\in\mathbb{R}$ and $m\in\mathbb{N}$ such that $\nu\mu+2\nu-\alpha+m\geq 1$, then for any $l>0$, we have*

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\delta(x-l)\right)(y)=\frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}l^{\alpha}y^{-1-\alpha+2\nu}e^{i\beta(al^{2\nu}+cy^{2\nu}+dl^{\nu}+ey^{\nu})}J_{\mu}\left(\frac{\beta}{b}(ly)^{\nu}\right).$$

Proof. Using (2.1), we have

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\delta(x-l)\right)(y)$$

$$= \left(-e^{-i\frac{\pi}{2}}\frac{\nu\beta}{b}y^{\nu}\right)^{-m}\left(H^{a,b,c;d,e}_{1,\mu+m,\nu,\alpha,\beta}\left(N^{a;d}_{\mu+m-1,\nu,\alpha,\beta}\dots N^{a;d}_{\mu,\nu,\alpha,\beta}\delta(x-l)\right)\right)(y)$$

$$= (-1)^m\left(\frac{\nu\beta}{b}\right)^{-m+1}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu-\nu m}e^{i\beta(cy^{2\nu}+ey^{\nu})}\int_0^\infty x^{\nu(\mu+m)}$$

$$\times\ J_{\mu+m}\left(\frac{\beta}{b}(xy)^{\nu}\right)D_x\left(x^{1-2\nu}D_x\right)^{m-1}e^{i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu+\alpha-2\nu+1}\delta(x-l)\mathrm{d}x$$

$$= (-1)^{m+1}\left(\frac{\nu\beta}{b}\right)^{-m+2}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu-\nu(m-1)}e^{i\beta(cy^{2\nu}+ey^{\nu})}\int_0^\infty x^{\nu(\mu+m+1)-1}$$

$$\times J_{\mu+m-1}\left(\frac{\nu\beta}{b}(xy)^{\nu}\right)\left(x^{1-2\nu}D_x\right)^{m-1}e^{i\beta(ax^{2\nu}+dx^{\nu})}x^{-\nu\mu+\alpha-2\nu+1}\delta(x-l)\mathrm{d}x.$$

Continuing in this manner $m-1$ times, we get

$$\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\delta(x-l)\right)(y) = \frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}l^{\alpha}y^{-1-\alpha+2\nu}e^{i\beta\left(al^{2\nu}+cy^{2\nu}+dl^{\nu}+ey^{\nu}\right)}J_{\mu}\left(\frac{\beta}{b}(ly)^{\nu}\right).$$ □

Problem 4.2. *Consider the following problem:*

$$\phi(x) - \Delta^{a,d}_{1,\mu,\nu,\alpha,\beta}\phi(x) = \delta(x-l), \tag{4.1}$$

where $0 < l < \infty$ *and* $\Delta^{a;d}_{1,\mu,\nu,\alpha,\beta}$ *is as* (1.12).

Taking first quadratic-phase Hankel transformation of random order $H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}$ on both sides of (4.1) and using the Lemma 2.5 (iii), we have

$$\left(1+\left(\frac{\nu\beta}{b}y^{\nu}\right)^2\right)\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\phi\right)(y) = \left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\delta(x-l)\right)(y).$$

Now using the result of problem 4.1, we get

$$\left(1+\left(\frac{\nu\beta}{b}y^{\nu}\right)^2\right)\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\phi\right)(y) = \frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}y^{-1-\alpha+2\nu}l^{\alpha}e^{i\beta\left(al^{2\nu}+cy^{2\nu}+dl^{\nu}+ey^{\nu}\right)}J_{\mu}\left(\frac{\beta}{b}(ly)^{\nu}\right).$$

Therefore,

$$\phi(x) = \frac{\nu\beta}{b}e^{-i\frac{\pi}{2}(1+\mu)}l^{\alpha}e^{i\beta(al^{2\nu}+dl^{\nu})}\left(H^{a,b,c;d,e}_{1,\mu,\nu,\alpha,\beta,m}\right)^{-1}\left(\frac{y^{-1-\alpha+2\nu}e^{i\beta(cy^{2\nu}+ey^{\nu})}J_{\mu}\left(\frac{\beta}{b}(ly)^{\nu}\right)}{1+\left(\frac{\nu\beta}{b}y^{\nu}\right)^2}\right)$$

$$= (-1)^m\left(\frac{\nu\beta}{b}\right)^{m+2}l^{\alpha}e^{i\beta\left(a\left(l^{2\nu}-x^{2\nu}\right)+d(l^{\nu}-x^{\nu})\right)}x^{\nu\mu-\alpha+2\nu-1}\int_\infty^x t_0^{2\nu-1}\int_\infty^{t_0} t_1^{2\nu-1}$$

$$\cdots\int_\infty^{t_{m-2}} t_{m-1}^{-\nu\mu-\nu m+2\nu-1}\int_0^\infty J_{\mu+m}\left(\frac{\beta}{b}(yt_{m-1})^{\nu}\right)$$

$$\times\left(\frac{y^{-1+2\nu+\nu m}J_{\mu}\left(\frac{\beta}{b}(ly)^{\nu}\right)}{1+\left(\frac{\nu\beta}{b}y^{\nu}\right)^2}\right)\mathrm{d}y\ \mathrm{d}t_{m-1}\dots\mathrm{d}t_1\ \mathrm{d}t_0.$$

Changing the order of integration between t_{m-1} and y and using the relation (1.23), we have

$$\phi(x) = (-1)^{m-1}\left(\frac{\nu\beta}{b}\right)^{m+1}l^{\alpha}e^{i\beta(al^{2\nu}+dl^{\nu})}x^{\nu\mu-\alpha+2\nu-1}e^{-i\beta(ax^{2\nu}+dx^{\nu})}\int_\infty^x t_0^{2\nu-1}$$

$$\int_{\infty}^{t_0} t_1^{2\nu-1} \dots \int_{\infty}^{t_{m-3}} t_{m-2}^{2\nu-1} \int_0^{\infty} \frac{y^{2\nu-1+\nu m-\nu} J_\mu\left(\frac{\beta}{b}(ly)^\nu\right)}{1+\left(\frac{\nu\beta}{b} y^\nu\right)^2} \int_{\infty}^{t_{m-2}} D_{t_{m-1}}$$
$$\times \left[t_{m-1}^{-\nu(\mu+m-1)} J_{\mu+m-1}\left(\frac{\beta}{b}(t_{m-1}y)^\nu\right)\right] \mathrm{d}t_{m-1}\,\mathrm{d}y\,\mathrm{d}t_{m-2}\dots\mathrm{d}t_1\,\mathrm{d}t_0.$$

Similarly, if we repeat the previous step, we get

$$\phi(x) = \left(\frac{\nu\beta}{b}\right)^2 l^\alpha x^{-1-\alpha+2\nu} e^{i\beta(a(l^{2\nu}-x^{2\nu})+d(l^\nu-x^\nu))} \int_0^\infty \frac{y^{2\nu-1} J_\mu\left(\frac{\beta}{b}(ly)^\nu\right) J_\mu\left(\frac{\beta}{b}(xy)^\nu\right)}{1+\left(\frac{\nu\beta}{b}y^\nu\right)^2}\,\mathrm{d}y.$$

Using [1], we get the following solution for $\nu \neq 0$:

$$\phi(x) = l^\alpha x^{-1-\alpha+2\nu} e^{i\beta(a(l^{2\nu}-x^{2\nu})+d(l^\nu-x^\nu))} \frac{1}{\nu} \begin{cases} I_\mu\left(\frac{x^\nu}{\nu}\right) K_\mu\left(\frac{l^\nu}{\nu}\right), & \text{if } 0 < x \le l, \\ I_\mu\left(\frac{l^\nu}{\nu}\right) K_\mu\left(\frac{x^\nu}{\nu}\right), & \text{if } l < x < \infty, \end{cases}$$

where I_μ and K_μ are modified Bessel functions of the first and third kind, respectively, with order μ.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. Bateman and A. Erdélyi, *Tables of Integral Transformation*, Volume II, McGraw-Hill, New York (1954).

[2] M. L. Linares and J. M. R. M. Pérez, Hankel complementary integral transformations of arbitrary order, *International Journal of Mathematics and Mathematical Sciences* **15**(2) (1992), 323 – 332, DOI: 10.1155/S0161171292000401.

[3] S. P. Malgonde and L. Debnath, On Hankel type integral transformations of generalized functions, *Integral Transforms and Special Functions* **15**(5) (2004), 421 – 430, DOI: 10.1080/10652460410001686055.

[4] S. P. Malgonde and S. R. Bandewar, On the generalized Hankel-Clifford transformation of arbitrary order, *Proceedings Mathematical Sciences* **110** (2000), 293 – 304, DOI: 10.1007/BF02878684.

[5] J. M. Mendez, A mixed Parseval equation and the generalized Hankel transformations, *Proceedings of the American Mathematical Society* **102**(3) (1988), 619 – 624, DOI: 10.2307/2047234.

[6] R. S. Pathak, *Integral Transforms of Generalized Functions and Their Applications*, Gordon Breach Science Publishers, Amsterdam, 432 pages (1997).

[7] J. M. R. M. Pérez and M. M. S. Robayna, A pair of generalized Hankel-Clifford transformations and their applications, *Journal of Mathematical Analysis and Applications* **154**(2) (1991), 543 – 557, DOI: 10.1016/0022-247X(91)90057-7.

[8] A. Prasad and T. Kumar, A pair of linear canonical Hankel transformation of random order, *Mediterranean Journal of Mathematics* **16**(6) (2019), Article number 150, DOI: 10.1007/s00009-019-1421-z.

[9] A. Prasad, T. Kumar and A. Kumar, Convolution for a pair of quadratic-phase Hankel transforms, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **114** (2020), Article number 150, DOI: 10.1007/s13398-020-00873-9.

[10] A. Torre, Hankel-type integral transforms and their fractionalization: a note, *Integral Transforms and Special Functions* **19**(4) (2008), 277 – 292, DOI: 10.1080/10652460701827848.

[11] A. H. Zemanian, *Generalized Integral Transforms*, Interscience Publishers, New York (1968).