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Research Article

Fixed Point Technique: Hyers-Ulam Stability Results Deriving From Cubic Mapping in Fuzzy Normed Spaces

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Abstract. In this work, we introduce a novel finite-dimensional cubic functional equation

$$\phi\left(\sum_{a=1}^{l} a n_a\right)=\sum_{1\le a<b<c\le l}\phi(an_a+bn_b+cn_c)+(3-l)\sum_{1\le a<b\le l}\phi(an_a+bn_b)+\left(\frac{(l^2-5l+6)}{2}\right)\sum_{a=0}^{l-1}(a+1)^3\phi(n_{a+1}),$$

where $l\ge 4$ is an integer, and derive its general solution. The main purpose of this work is to examine the Hyers-Ulam stability of this functional equation in fuzzy normed spaces by means of direct approach and fixed point approach.

Keywords. Fuzzy normed spaces, Ulam stability, Cubic mapping

Mathematics Subject Classification (2020). 39B52, 26E50, 39B82

1. Introduction

A cubic mapping $f : U \to V$ between real vector spaces is defined as:

$$f(2u+v)+f(2u-v)=2f(u+v)+2f(u-v)+12f(u), \tag{1.1}$$

where u and v are in U. The equation (1.1) is known as a cubic functional equation. The problem of fuzzy stability in functional equations has received significant attention recently. Several fuzzy stability findings for various functional equations have been studied by Mirmostafaee and Moslehian [10–12], and Mirmostafaee *et al*. [13].

In addressing applied problems, it's common to encounter situations where only partial information is accessible, or where the parameters of a model are uncertain, or measurements are imprecise. These characteristics often motivate researchers to explore functional equations within the framework of fuzzy theory.

Over the past four decades, fuzzy theory has emerged as a vibrant field of study, witnessing significant advancements in the adaptation of classical set theory to fuzzy sets. This branch of mathematics has found extensive applications across various domains in science and engineering.

In 1984, Katsaras [7] introduced a fuzzy norm on a linear space. Following this, in 1991, Biswas [3] expanded on this concept and explored fuzzy inner product spaces within linear spaces. In 1992, Felbin [6] proposed an alternative notion of a fuzzy norm for linear topological structures within fuzzy normed linear spaces.

Subsequently, in 1994, Cheng and Mordeson [4] defined another type of fuzzy norm on a linear space, leading to the development of the induced fuzzy metric by Kramosil and Michalek [8]. In 2003, Bag and Samanta [1] modified the definition provided by Cheng and Mordeson [4] by removing a regular condition. Recent research has seen numerous authors delve into various aspects of these topics, as documented by Bag and Samanta [2], Mirmostafaee and Moslehian [10], and Shieh [16].

In this work, we introduce a novel finite-dimensional cubic functional equation

$$\phi\left(\sum_{a=1}^{l} an_a\right)=\sum_{1\le a<b<c\le l}\phi(an_a+bn_b+cn_c)+(3-l)\sum_{1\le a<b\le l}\phi(an_a+bn_b) + \left(\frac{(l^2-5l+6)}{2}\right)\sum_{a=0}^{l-1}(a+1)^3\phi(n_{a+1}), \tag{1.2}$$

where $l \ge 4$ is an integer, and derive its solution. The main aim of this study is to investigate the Hyers-Ulam stability of the above mentioned functional equation in fuzzy normed spaces, employing both direct and fixed point techniques.

2. Preliminaries

We review some fundamental facts about fuzzy normed spaces, as well as some preliminary findings. We follow the concept of fuzzy normed spaces in [1].

Definition 2.1 ([1]). *Let X be a real vector space. A function $N : X \times R \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in R$,*

(N_1) $N(x,t)=0$, *for* $t \le 0$;

(N_2) *$x = 0$ if and only if $N(x,t) = 1$, for all $t > 0$;*

(N_3) *$N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;*

(N_4) *$N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;*

(N_5) *$N(x,\cdot)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$;*

(N_6) *for $x \neq 0$, $N(x,\cdot)$ is continuous on R.*

The pair (X,N) is called a fuzzy normed vector space.

Definition 2.2. *Let (X,N) denote a fuzzy normed space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. x is the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, denoted by N-$\lim x_n = x$.*

The limit of the convergent sequence $\{x_n\}_{n=1}^{\infty}$ in a fuzzy normed space (X,N) is unique, as seen in [14].

Definition 2.3. *In a fuzzy normed space (X,N), a sequence $\{x_n\}_{n=1}^{\infty}$ is defined as a Cauchy sequence if, for every $\varepsilon > 0$ and each $t > 0$, there exists an $M \in N$ such that for all $n \geq M$ and every $p > 0$, the condition $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ is satisfied.*

The condition (N_4) states that all convergent sequences in a fuzzy normed space are Cauchy sequences. A fuzzy normed space (X,N) is referred to as a *fuzzy Banach space* if all Cauchy sequences in X converge.

A mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if any sequence $\{x_n\}$ that converges to x_0 in X also converges to $f(x_0)$. If $f : X \to Y$ is continuous at all $x \in X$, it is considered *continuous* on X.

In fixed point theory, we shall apply the fundamental result presented below.

Theorem 2.4 ([5])**.** *Let (X,d) denote a generalized complete metric space, and let $\Lambda : X \to X$ represent a strictly contractive function with a Lipschitz constant $L < 1$. Assume there exists an element $a \in X$ such that a nonnegative integer k satisfies $d(\Lambda^{k+1}a, \Lambda^k a) < \infty$. Then,*

(i) *the sequence $\{\Lambda^n a\}_{n=1}^{\infty}$ converges to a fixed point $b \in X$ of Λ;*

(ii) *b is only one fixed point of Λ in the set $Y = \{y \in X : d(\Lambda^k a, y) < \infty\}$;*

(iii) *$d(y,b) \leq \frac{1}{1-L} d(y, \Lambda y)$, for every $y \in Y$.*

3. Solution of Equation (1.2)

Theorem 3.1. *If a mapping $\phi : A \to B$ fulfills the functional equation* (1.2)*, then the function $\phi : A \to B$ fulfills* (1.1)*.*

Proof. Assume that $\phi : A \to B$ fulfills (1.2), for every $n_1, n_2, \cdots, n_l \in X$. Substituting $(n_1, n_2, \cdots, n_l)$ by $(0, 0, \cdots, 0)$ in (1.2), we receive

$$\phi(0) = 0,$$

for all $n \in A$. Replacing $(n_1, n_2, \cdots, n_l)$ by $(n, 0, \cdots, 0)$ in (1.2), we arrive

$$\phi(-n) = -\phi(n),$$

for every $n \in A$. Hence ϕ is odd function. Again replacing $(n_1, n_2, \cdots, n_l)$ by $\left(n, \frac{n}{2}, 0, \cdots, 0\right)$ in (1.2), we have

$$\phi(2n) = 2^3\phi(n), \tag{3.1}$$

for all $n \in A$. Now, letting n by $2n$ in (3.1), we get

$$\phi(4n) = 4^3\phi(n), \tag{3.2}$$

for all $n \in A$. In general, for any positive integer a, we get

$$\phi(an) = a^3\phi(n). \tag{3.3}$$

Setting $(n_1, n_2, \cdots, n_l)$ by $\left(u, \frac{-u}{2}, \frac{u}{3}, \frac{v}{4}, 0, \cdots, 0\right)$ in (1.2) and utilizing (3.1), we receive

$$3\phi(u+v) = -6\phi(u) + 3\phi(v) + \phi(2u+v) + \phi(u-v), \tag{3.4}$$

for every $u, v \in A$. Substituting v by $-v$ in (3.4), we reach

$$3\phi(u-v) = -6\phi(u) - 3\phi(v) + \phi(2u-v) + \phi(u+v), \tag{3.5}$$

for every u, v in A. Adding (3.4) and (3.5), we archive our result (1.1). □

In the subsequent sections of this paper, we designate A as a linear space, (B, P) as a fuzzy Banach space, and (Z, Q) as a fuzzy normed space. To simplify notation, we introduce the abbreviation for a mapping $\phi : A \to B$ as follows:

$$D\phi(n_1, n_2, \cdots, n_l) = \phi\left(\sum_{a=1}^{l} a n_a\right) - \sum_{1 \le a < b < c \le l} \phi(a n_a + b n_b + c n_c) - (3-l) \sum_{1 \le a < b \le l} \phi(a n_a + b n_b)$$
$$- \left(\frac{(l^2 - 5l + 6)}{2}\right) \sum_{a=0}^{l-1} (a+1)^3 \phi(n_{a+1}),$$

for every $n_1, n_2, \cdots, n_l \in A$.

4. Ulam Stability of Equation (1.2): Direct Technique

Theorem 4.1. *Let $u \in \{-1, 1\}$ be fixed and let a mapping $\chi : A^l \to Z$ such that $\varsigma > 0$ and $\left(\frac{\varsigma}{2^3}\right)^u < 1$,*

$$Q(\chi(2^u n, 2^u n, , 0, \cdots, 0), \epsilon) \ge Q(\varsigma^u \chi(n, n, 0, \cdots, 0), \epsilon) \tag{4.1}$$

and

$$\lim_{m \to \infty} Q(\chi(2^{um} n_1, 2^{um} n_2, \cdots, 2^{um} n_l), 2^{3um}\epsilon) = 1,$$

for every $n, n_1, n_2, \cdots, n_l \in A$ and $\epsilon > 0$. If an odd function $\phi : A \to B$ fulfills $\phi(0) = 0$ and

$$P(D\phi(n_1, n_2, \cdots, n_l), \epsilon) \ge Q(\chi(n_1, n_2, \cdots, n_l), \epsilon), \tag{4.2}$$

for all $n_1, n_2, \cdots, n_l \in A$ and $\epsilon > 0$. Then, the limit

$$C(n) = P - \lim_{m \to \infty} \frac{\phi(2^{um} n)}{2^{3um}}$$

exists for every $n \in A$ and a unique cubic mapping $C : A \to B$ fulfilling

$$P(\phi(n) - C(n), \epsilon) \ge Q(\chi(n, n, 0, \cdots, 0), (l^2 - 5l + 6)\epsilon |2^3 - \varsigma|), \tag{4.3}$$

for all $n \in A$ and $\epsilon > 0$.

Proof. Let $u = 1$. Switching $(n_1, n_2, \cdots, n_l)$ by $(n, n, 0, \cdots, 0)$ in (4.2), we reach

$$P((l^2 - 5l + 6)\phi(2n) - 8(l^2 - 5l + 6)\phi(n), \epsilon) \ge Q(\chi(n, n, 0, \cdots, 0), \epsilon), \quad n \in A, \epsilon > 0.$$

Then, we have

$$P\left(\frac{\phi(2n)}{2^3}-\phi(n),\frac{\epsilon}{8(l^2-5l+6)}\right)\geq Q(\chi(n,n,0,\cdots,0),\epsilon),\quad n\in A,\epsilon>0. \tag{4.4}$$

Switching n by $2^m n$ in (4.4), we acquire

$$P\left(\frac{\phi(2^{m+1}n)}{2^3}-\phi(2^m n),\frac{\epsilon}{8(l^2-5l+6)}\right)\geq Q(\chi(2^m n,2^m n,0,\cdots,0),\epsilon),\quad n\in A,\epsilon>0. \tag{4.5}$$

Using (4.1) and the condition (N_3) in (4.5), we obtain

$$P\left(\frac{\phi(2^{m+1}n)}{2^{3(m+1)}}-\frac{\phi(2^m n)}{2^{3m}},\frac{\epsilon}{2^{3(m+1)}(l^2-5l+6)}\right)\geq Q\left(\chi(n,n,0,\cdots,0),\frac{\epsilon}{\varsigma^m}\right),\quad n\in A,\epsilon>0. \tag{4.6}$$

Switching ϵ by $\varsigma^m\epsilon$ in (4.6), we reach

$$P\left(\frac{\phi(2^{3(m+1)}m)}{2^{3(m+1)}}-\frac{\phi(2^m n)}{2^{3m}},\frac{\varsigma^m\epsilon}{2^{3(m+1)}(l^2-5l+6)}\right)\geq Q(\chi(n,n,0,\cdots,0),\epsilon),\quad n\in A,\epsilon>0. \tag{4.7}$$

From (4.7), we obtain

$$\begin{aligned}
&P\left(\frac{\phi(2^m n)}{2^{3m}}-\phi(n),\sum_{a=0}^{m-1}\frac{\epsilon\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}\right)\\
&\quad=P\left(\sum_{a=0}^{m-1}\left[\frac{\phi(2^{a+1}n)}{2^{3(a+1)}}-\frac{\phi(2^a n)}{2^{3a}}\right],\sum_{a=0}^{m-1}\frac{\epsilon\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}\right)\\
&\quad\geq\min_{0\leq a\leq m-1}P\left(\frac{\phi(2^{a+1}n)}{2^{3(a+1)}}-\frac{\phi(2^a n)}{2^{3a}},\frac{\epsilon\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}\right)\\
&\quad\geq Q(\chi(n,n,0,\cdots,0),\epsilon),
\end{aligned} \tag{4.8}$$

for all $n\in A$, $\epsilon>0$ and every $m\in N$. Switching n by $2^s n$ in (4.8) and using (4.1) with (N_3), we attain

$$\begin{aligned}
P\left(\frac{\phi(2^{m+s}n)}{2^{3(m+s)}}-\frac{\phi(2^s n)}{2^{3s}},\sum_{a=0}^{m-1}\frac{\epsilon\varsigma^a}{2^{3(a+s+1)}(l^2-5l+6)}\right)&\geq Q(\chi(2^s n,2^s n,0,\cdots,0),\epsilon)\\
&\geq Q\left(\chi(n,n,0,\cdots,0),\frac{\epsilon}{\varsigma^s}\right),
\end{aligned}$$

and so

$$P\left(\frac{\phi(2^{m+s}n)}{2^{3(m+s)}}-\frac{\phi(2^s n)}{2^{3s}},\sum_{a=s}^{m+s-1}\frac{\epsilon\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}\right)\geq Q(\chi(n,n,0,\cdots,0),\epsilon),$$

for every $n\in A$, $\epsilon>0$ and all integers $s,m\geq 0$. Replacing ϵ by $\frac{\epsilon}{\sum_{a=s}^{m+s-1}\frac{\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}}$ in the above inequality, we obtain

$$P\left(\frac{\phi(2^{m+s}n)}{2^{3(m+s)}}-\frac{\phi(2^s n)}{2^{3s}},\epsilon\right)\geq Q\left(\chi(n,n,0,\cdots,0),\frac{\epsilon}{\sum_{a=s}^{m+s-1}\frac{\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}}\right), \tag{4.9}$$

for all $n\in A$, $\epsilon>0$ and all integers $s,m\geq 0$. Since $\sum_{a=0}^{\infty}\left(\frac{\varsigma}{8(l^2-5l+6)}\right)^a<\infty$, it follows from (4.9) and (N_5) that $\{\frac{\phi(2^m n)}{2^{3m}}\}_{m=1}^{\infty}$ is a Cauchy sequence in (B,P) for each $n\in A$. As (B,P) is a fuzzy Banach space, $\{\frac{\phi(2^m n)}{2^{3m}}\}_{m=1}^{\infty}$ converges to a point $C(n)\in B$ for every $n\in A$. Consequently, we may define the mapping $C : A \to B$ as

$$C(n) := P - \lim_{m\to\infty} \frac{\phi(2^m n)}{2^{3m}}, \quad n \in A.$$

As ϕ is an odd function, C inherits the same property of being odd. Substituting $s = 0$ into (4.9), we obtain:

$$P\left(\frac{\phi(2^m n)}{2^{3m}} - \phi(n), \epsilon\right) \geq Q\left(\chi(n,n,0,\cdots,0), \frac{\epsilon}{\sum_{a=0}^{m-1} \frac{\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}}\right), \tag{4.10}$$

for all $n \in A$, $\epsilon > 0$ and every $m \geq 1$. Then

$$\begin{aligned} P(\phi(n) - C(n), \epsilon + \alpha) &\geq \min\left\{P\left(\frac{\phi(2^m n)}{2^{3m}} - \phi(n), \epsilon\right), P\left(\frac{\phi(2^m n)}{2^{3m}} - C(n), \alpha\right)\right\} \\ &\geq \min\left\{Q\left(\chi(n,n,0,\cdots,0), \frac{\epsilon}{\sum_{a=0}^{m-1} \frac{\varsigma^a}{2^{3(a+1)}(l^2-5l+6)}}\right), P\left(\frac{\phi(2^m n)}{2^{3m}} - C(n), \alpha\right)\right\}, \end{aligned}$$

for all $n \in A$, $\epsilon, \alpha > 0$ and every $m \geq 1$. Thus, by passing the limit as $m \to \infty$ in the last inequality and utilizing property (N_6), we obtain:

$$P(\phi(n) - C(n), \epsilon + \alpha) \geq Q(\chi(n,n,0,\cdots,0), (l^2 - 5l + 6)(2^3 - \varsigma)\epsilon), \quad n \in A, \epsilon, \alpha > 0.$$

By taking the limit as α approaches 0, we arrive at equation (4.3).

Now, we claim that C is cubic. It is evident that

$$\begin{aligned} P(DC(n_1,n_2,\cdots,n_l), 2\epsilon) &\geq \min\left\{P\left(DC(n_1,n_2,\cdots,n_l) - \frac{1}{2^{3m}} D\phi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), \epsilon\right),\right. \\ &\qquad \left. P\left(\frac{1}{2^{3m}} D\phi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), \epsilon\right)\right\}, \\ \text{By (4.2)} &\geq \min\left\{P\left(DC(n_1,n_2,\cdots,n_l) - \frac{1}{2^{3m}} D\phi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), \epsilon\right),\right. \\ &\qquad \left. Q(\chi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), 2^{3m}\epsilon)\right\}, \quad n \in A, \epsilon > 0. \end{aligned}$$

Since

$$\lim_{m\to\infty} P\left(DC(n_1,n_2,\cdots,n_l) - \frac{1}{2^{3m}} D\phi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), \epsilon\right) = 1,$$

$$\lim_{m\to\infty} C(\chi(2^m n_1, 2^m n_2, \cdots, 2^m n_l), 2^{3m}\epsilon) = 1,$$

we infer $P(DC(n_1,n_2,\cdots,n_l), 2\epsilon) = 1$, for all $n_1,n_2,\cdots,n_l \in A$ and all $\epsilon > 0$. Then (N_2) implies $DC(n_1,n_2,\cdots,n_l) = 0$, for all $n_1,n_2,\cdots,n_l \in A$. Therefore, Theorem 3.1 implies that $C : A \to B$ is cubic function. To demonstrate the uniqueness of C, let us consider one more cubic mapping $D : A \to B$ which fulfilling (4.3). Because $C(2^m n) = 2^{3m} C(n)$ and $D(2^m n) = 2^{3m} D(n)$, for every $n \in A$ and every $m \in N$, then

$$\begin{aligned} P(C(n) - D(n), \epsilon) &= P\left(\frac{C(2^m n)}{2^{3m}} - \frac{D(2^m n)}{2^{3m}}, \epsilon\right) \\ &\geq \min\left\{P\left(\frac{C(2^m n)}{2^{3m}} - \frac{\phi(2^m n)}{2^{3m}}, \frac{\epsilon}{2}\right), P\left(\frac{\phi(2^m n)}{2^{3m}} - \frac{D(2^m n)}{2^{3m}}, \frac{\epsilon}{2}\right)\right\} \end{aligned}$$

$$\geq Q\left(\chi(2^m n, 2^m n, 0, \cdots, 0), \frac{(l^2-5l+6)(2^3-\varsigma)\epsilon}{2}\right)$$
$$\geq Q\left(\chi(n, n, 0, \cdots, 0), \frac{(l^2-5l+6)(2^3-\varsigma)\epsilon}{2\varsigma^m}\right),$$

for every $n \in A$, $\epsilon > 0$ and every $m \in N$. Since $\lim_{m\to\infty} \frac{((l^2-5l+6))(2^3-\varsigma)\epsilon}{2\varsigma^m} = \infty$, we obtain

$$\lim_{m\to\infty} Q\left(\chi(n, n, 0, \cdots, 0), \frac{(l^2-5l+6)(2^3-\varsigma)\epsilon}{2\varsigma^m}\right) = 1.$$

Consequently, $P(C(n)-D(n),\epsilon) = 1$, for every $n \in A$ and every $\epsilon > 0$. So $C(n) = D(n)$, for every $n \in A$. For $u = -1$, we may illustrate the result utilizing a similar technique. The proof of the theorem is now accomplished. □

5. Ulam Stability of Equation (1.2): Fixed Point Technique

Radu [15] presented a new approach for investigating the stability associated with functional equations employing the fixed point alternative method.

In this section, we explore the Ulam-Hyers stability of (1.2) in fuzzy normed spaces utilizing the fixed point approach.

First, let us define ψ_a as a constant such that:

$$\psi_a = \begin{cases} 2, & \text{if } a = 0, \\ \frac{1}{2}, & \text{if } a = 1 \end{cases}$$

and we consider $\Upsilon = \{v : A \to B : v(0) = 0\}$.

Theorem 5.1. *Let $\phi : A \to B$ be an odd function, where $\phi(0) = 0$ and there is a mapping $\chi : A^l \to Z$ subject to*

$$\lim_{m\to\infty} Q(\chi(\psi_a^m n_1, \psi_a^m n_2, \cdots, \psi_a^m n_l), \psi_a^{3m}\epsilon) = 1, \quad n_1, n_2, \cdots, n_l \in A, \epsilon > 0, \tag{5.1}$$

and satisfying the inequality

$$P(D\phi(n_1, n_2, \cdots, n_l), \epsilon) \geq Q(\chi(n_1, n_2, \cdots, n_l), \epsilon), \quad n_1, n_2, \cdots, n_l \in A, \epsilon > 0. \tag{5.2}$$

Let $\sigma(n) = \frac{1}{(l^2-5l+6)}\chi\left(\frac{n}{2}, \frac{n}{2}, 0, \cdots, 0\right)$ for every $n \in A$. If there is $L = L_a \in (0,1)$ such that

$$Q\left(\frac{1}{\psi_a^3}\sigma(\psi_a n), \epsilon\right) \geq Q(L\sigma(n), \epsilon), \quad n \in A, \epsilon > 0, \tag{5.3}$$

then there is only one cubic mapping $C : A \to B$ fulfilling

$$P(\phi(n) - C(n), \epsilon) \geq Q\left(\frac{L^{1-a}}{1-L}\sigma(n), \epsilon\right), \quad n \in A, \epsilon > 0. \tag{5.4}$$

Proof. Suppose ς is the generalised metric on Υ:

$$\varsigma(v, w) = \inf\{r \in (0,\infty) : P(v(n) - w(n), \epsilon) \geq Q(r\sigma(n), \epsilon), n \in A, \epsilon > 0\},$$

and as usual, we use $\inf \emptyset = +\infty$. Miheţ and Radu [9, Lemma 2.1] demonstrates that (Υ, ς) is the complete generalised metric space. We may define $\Phi_a : \Upsilon \longrightarrow \Upsilon$ by $\Phi_a v(n) = \frac{1}{\psi_a^3} v(\psi_a n)$ for all $n \in A$. Let v, w in Υ be given such that $\varsigma(v, w) \leq \alpha$. Then

$$P(v(n) - w(n), \epsilon) \geq Q(\alpha\sigma(n), \epsilon), \quad n \in A, \epsilon > 0,$$

whence

$$P(\Phi_a v(n)-\Phi_a w(n),\epsilon)\ge Q\left(\frac{\alpha}{\psi_a^3}\sigma(\psi_a n),\epsilon\right),\quad n\in A,\epsilon>0.$$

According to (5.3),

$$P(\Phi_a v(n)-\Phi_a w(n),\epsilon)\ge Q(\alpha L\sigma(n),\epsilon),\quad n\in A,\epsilon>0.$$

Hence, we have $\varsigma(\Phi_a v,\Phi_a w)\le \alpha L$. This shows $\varsigma(\Phi_a v,\Phi_a w)\le L\varsigma(v,w)$, i.e., Φ_a is strictly contractive function on Υ with L. Switching $(n_1,n_2,\cdots,n_l)$ by $(n,n,0,\cdots,0)$ in (5.2) and using (N_3), we obtain

$$P\left(\frac{\phi(2n)}{2^3}-\phi(n),\epsilon\right)\ge Q\left(\frac{\chi(n,n,0,\cdots,0)}{2^3(l^2-5l+6)},\epsilon\right),\quad n\in A,\ \epsilon>0. \tag{5.5}$$

If $a=0$, we deduce from (5.5) that

$$P\left(\frac{\phi(2n)}{2^3}-\phi(n),\epsilon\right)\ge Q(L\sigma(n),\epsilon),\quad n\in A,\ \epsilon>0.$$

Therefore,

$$\varsigma(\Phi_0\phi,\phi)\le L=L^{1-a}. \tag{5.6}$$

Replacing n by $\frac{n}{2}$ in (5.5), we obtain

$$\begin{aligned}P\left(\phi(n)-2^3\phi\left(\frac{n}{2}\right),2^3\epsilon\right)&\ge Q\left(\chi\left(\frac{n}{2},\frac{n}{2},0,\cdots,0\right),2^3(l^2-5l+6)\epsilon\right)\\&=Q(\sigma(n),2^3(l^2-5l+6)\epsilon),\quad n\in A,\ \epsilon>0.\end{aligned}$$

Therefore,

$$\varsigma(\Phi_1\phi,\phi)\le 1=L^{1-a}. \tag{5.7}$$

Based on (5.6) and (5.7), we may deduce that $\varsigma(\Phi_a\phi,\phi)\le L^{1-a}<\infty$. The Fixed Point Alternative Theorem 2.4 asserts that there exists a fixed point C of Φ_a in Υ such that

(i) $\Phi_a C=C$ and $\lim_{m\to\infty}\varsigma(\Phi_a^m\phi,C)=0$;

(ii) C is the only one fixed point of Φ in $E=\{v\in\Upsilon : d(\phi,v)<\infty\}$;

(iii) $\varsigma(\phi,C)\le\frac{1}{1-L}\varsigma(\phi,\Phi_a\phi)$.

Setting $\varsigma(\Phi_a^m\phi,C)=\alpha_m$, we get $P(\Phi_a^m\phi(n)-C(n),\epsilon)\ge Q(\alpha_m\sigma(n),\epsilon)$, for all $n\in A$ and all $\epsilon>0$. Since $\lim_{m\to\infty}\alpha_m=0$, we infer

$$C(n)=P-\lim_{m\to\infty}\frac{\phi(\psi_a^m n)}{\psi_a^{3m}},\quad n\in A.$$

Switching $(n_1,n_2,\cdots,n_l)$ by $(\psi_a^m n_1,\psi_a^m n_2,\cdots,\psi_a^m n_l)$ in (5.2), we obtain

$$P\left(\frac{1}{\psi_a^{3m}}D\phi(\psi_a^m n_1,\psi_a^m n_2,\cdots,\psi_a^m n_l),\epsilon\right)\ge Q(\chi(\psi_a^m n_1,\psi_a^m n_2,\cdots,\psi_a^m n_l),\psi_a^{3m}\epsilon),$$

for all $\epsilon>0$ and all $n_1,n_2,\cdots,n_l\in A$. Applying a similar approach as the proof of Theorem 4.1, we can argue that the function $C:A\to B$ is cubic. As $\varsigma(\Phi_a\phi,\phi)\le L^{1-a}$, from (iii) that $\varsigma(\phi,C)\le\frac{L^{1-a}}{1-L}$, implying (5.4). To demonstrate the function C is unique, let $D:A\to B$ be an one more cubic function satisfying (5.4). As $C(2^m n)=2^{3m}C(n)$ and $D(2^m n)=2^{3m}D(n)$, we obtain

$$\begin{aligned}P(C(n)-D(n),\epsilon)&=P\left(\frac{C(2^m n)}{2^{3m}}-\frac{D(2^m n)}{2^{3m}},\epsilon\right)\\&\ge\min\left\{P\left(\frac{C(2^m n)}{2^{3m}}-\frac{\phi(2^m n)}{2^{3m}},\frac{\epsilon}{2}\right),P\left(\frac{\phi(2^m n)}{2^{3m}}-\frac{D(2^m n)}{2^{3m}},\frac{\epsilon}{2}\right)\right\}\end{aligned}$$

$$\geq Q\left(\frac{L^{1-a}}{1-L}\sigma(2^m n), \frac{2^{3m}\epsilon}{2}\right).$$

By (5.1), we have

$$\lim_{m\to\infty} Q\left(\frac{L^{1-a}}{1-L}\sigma(2^m n), \frac{2^{3m}\epsilon}{2}\right) = 1.$$

Consequently, $P(C(n)-D(n),\epsilon)=1$ for every $n\in A$ and every $\epsilon>0$. So $C(n)=D(n)$ for all $n\in A$, this concludes the proof. □

Corollary 5.2. *If an odd mapping $\phi: A\to B$ satisfies $\phi(0)=0$ and inequality*

$$P(D\phi(n_1,n_2,\cdots,n_l),\epsilon)\geq Q\left(\tau+\theta\prod_{a=1}^{l}\|n_a\|^q,\epsilon\right),$$

for every $n_1,n_2,\cdots,n_l\in A$ and every $\epsilon>0$. Then, there is only one cubic function $C:A\to B$ such that

$$P(\phi(n)-C(n),\epsilon)\geq Q(\tau,7\epsilon), \quad n\in A, \epsilon>0,$$

where q,θ,τ are in R^+ with $lq\in(0,3)$.

Corollary 5.3. *If an odd mapping $\phi: A\to B$ such that $\phi(0)=0$ and*

$$P(D\phi(n_1,n_2,\cdots,n_l),\epsilon)\geq Q\left(\alpha\sum_{a=1}^{l}\|n_a\|^p+\theta\prod_{a=1}^{l}\|n_a\|^q,\epsilon\right),$$

for every $n_1,n_2,\cdots,n_l\in A$ and every $\epsilon>0$. Then, there is only one cubic function $C:A\to B$ such that

$$P(\phi(n)-C(n),\epsilon)\geq Q(2\alpha\|n\|^p,(l^2-5l+6)|2^3-2^p|\epsilon), \quad n\in A,\ \epsilon>0,$$

where q,p,α and θ are in R^+ with $p,lq\in(0,3)\cup(3,+\infty)$.

Corollary 5.4. *If an odd mapping $\phi: A\to B$ such that $\phi(0)=0$ and*

$$P(D\phi(n_1,n_2,\cdots,n_l),\epsilon)\geq Q\left(\theta\prod_{a=1}^{l}\|n_a\|^q,\epsilon\right),$$

for every $n_1,n_2,\cdots,n_l\in A$ and every $\epsilon>0$, where q and θ are in R^+ with $0<lq\neq 3$. Then, the function ϕ is cubic.

6. Conclusion

We introduced a novel finite-dimensional cubic functional equation and derive its general solution. The main purpose of this work is to examined the Hyers-Ulam stability of this functional equation in fuzzy normed spaces by means of direct approach and fixed point approach.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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