



On the Bihyperbolic k -Fibonacci Numbers

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Received: February 14, 2024

Accepted: April 9, 2024

Abstract. In this paper, we extend the concept of bihyperbolic numbers to the k -Fibonacci numbers. We will find the main formulas that affect these numbers, such as the recurrence formulas and the sum of the first n terms, as well as the identities of Binet, Catalan and d’Ocagne. We finish the study by finding the generating function of the bihyperbolic k -Fibonacci numbers.

Keywords. k -Fibonacci numbers, Hiperbolic and bihyperbolic numbers, Convolution, Binet, Catalan and d’Ocagne identities, Generating function

Mathematics Subject Classification (2020). 11B39, 11C20

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1. Introduction

In this section, we will review the concepts of hyperbolic and bihyperbolic numbers, as well as the k -Fibonacci numbers and some of their properties.

1.1 Hyperbolic Numbers

Hyperbolic numbers [6, 7],¹ can be defined as follows: if x and y are two real numbers, a bihyperbolic number is $z = x + jy$ with $j^2 = 1$, $j \neq 1$.

The set of hyperbolic numbers is denoted as C^+ or H_1 . The hyperbolic numbers are also called double numbers, split complex numbers and perplex numbers. For two hyperbolic numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ it is possible to define the following operations:

¹See URL: https://en.wikipedia.org/wiki/Split-complex_number.

- *Addition*: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$.
- *Multiplication*: $z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)$.

The conjugate of $z = x + jy$ is $\bar{z} = x - jy$ and satisfies similar properties to usual complex conjugate. Namely, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$, $\overline{(\bar{z})} = z$.

The modulus of a bihyperbolic number $z = x + jy$ is $\|z\|^2 = z\bar{z} = x^2 - y^2$. Then, $\|z\| = \sqrt{x^2 - y^2}$ if $|x| > |y|$, or $\|z\| = \sqrt{y^2 - x^2}$ if $|y| > |x|$.

A bihyperbolic number is invertible if and only if its modulus is nonzero, $\|z\| \neq 0$. The multiplicative inverse of an invertible element is given by $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{\|z\|^2}$. Then, the inverse number of z , if exists ($\|z\| \neq 0$), is $z^{-1} = \frac{x}{x^2 - y^2} - j\frac{y}{x^2 - y^2}$. So, the division of two hyperbolic numbers, if exists, is $\frac{z_1}{z_2} = \frac{x_1 x_2 - y_1 y_2}{x_2^2 - y_2^2} - j\frac{x_1 y_2 - x_2 y_1}{x_2^2 - y_2^2}$.

1.2 Bihyperbolic Numbers

Now, we will extend the concept of hiperbolic numbers to the bihyperbolic numbers.

Definition 1. Let H_2 be the set of bihyperbolic numbers defined by

$$z = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1, j_2, j_3 \notin \mathbb{R}$ are operators such that $j_1^2 = j_2^2 = j_3^2 = 1$ and $j_1 j_2 = j_2 j_1 = j_3$, $j_1 j_3 = j_3 j_1 = j_2$, $j_2 j_3 = j_3 j_2 = j_1$.

The addition and multiplication on H_2 are commutative and associative. Also, the multiplication is distributive over addition. Hence $(H_2, +, \cdot)$ is a commutative ring. In this paper we will study the bihyperbolic k -Fibonacci numbers.

1.3 k -Fibonacci Numbers

One of the more studied integer sequences is the Fibonacci sequence [5, 8], and it has been generalized in many ways, as [4]. Here, we use the following one-parameter generalization of the Fibonacci sequence [2, 3].

Definition 2. For any integer number $k \geq 1$, the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined recurrently by:

$$F_{k,0} = 0, F_{k,1} = 1 \text{ and } F_{k,n+1} = k F_{k,n} + F_{k,n-1}, \text{ for } n \geq 1.$$

The first k -Fibonacci numbers are

$$\{F_{k,0}, F_{k,1}, F_{k,2}, F_{k,3}, F_{k,4}, F_{k,5}, \dots\} = \{0, 1, k, k^2 + 1, k^3 + 2k, k^4 + 3k^2 + 1, \dots\}.$$

Note that for $k = 1$ the classical Fibonacci sequence is obtained and for $k = 2$ it is the Pell sequence.

The characteristic equation of the definition is $r^2 = kr + 1$ whose solutions are $\sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $\sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2}$, that verify $\sigma_1 \cdot \sigma_2 = -1$, $\sigma_1 + \sigma_2 = k$, $\sigma_1 - \sigma_2 = \sqrt{k^2 + 4}$, $\sigma^2 = k\sigma + 1$, $\sigma_1 > 0$, $\sigma_2 < 0$.

Generating function of the k -Fibonacci numbers is $f(k, x) = \frac{x}{1 - kx - x^2}$.

For the properties of the k -Fibonacci numbers, see [2, 3].

$$\text{Binet Identity : } F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}, \tag{1.1}$$

$$\text{Sum : } \sum_{i=0}^n F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1), \tag{1.2}$$

$$\text{Convolution : } F_{k,a+b} = F_{k,a}F_{k,b-1} + F_{k,a+1}F_{k,b}. \tag{1.3}$$

2. On the Bihyperbolic k -Fibonacci Numbers

In [1], the bihyperbolic Fibonacci, Jacobsthal and Pell numbers are studied. We will extend this concept to the k -Fibonacci numbers.

According to Definition 1, the n th hyperbolic k -Fibonacci number $BhF_{k,n}$ is defined as

$$BhF_{k,n} = F_{k,n} + j_1F_{k,n+1} + j_2F_{k,n+2} + j_3F_{k,n+3}. \tag{2.1}$$

Theorem 1 (Recurrence Relation). *Let $n \geq 2$ be an integer. Then $BhF_{k,n} = k BhF_{k,n-1} + BhF_{k,n-2}$, with initial conditions*

$$BhF_{k,0} = F_{k,0} + j_1F_{k,1} + j_2F_{k,2} + j_3F_{k,3} = j_1 + j_2k + j_3(k^2 + 1),$$

$$BhF_{k,1} = F_{k,1} + j_1F_{k,2} + j_2F_{k,3} + j_3F_{k,4} = 1 + j_1k + j_2(k^2 + 1) + j_3(k^3 + 2k).$$

Proof. Taking into account formula (2.1) and Definition of k -Fibonacci numbers:

$$\begin{aligned} k \cdot BhF_{k,n-1} + BhF_{k,n-2} &= k(F_{k,n-1} + j_1F_{k,n} + j_2F_{k,n+1} + j_3F_{k,n+2}) \\ &\quad + (F_{k,n-2} + j_1F_{k,n-1} + j_2F_{k,n} + j_3F_{k,n+1}) \\ &= (kF_{k,n-1} + F_{k,n-2}) + j_1(kF_{k,n} + F_{k,n-1}) \\ &\quad + j_2(kF_{k,n+1} + F_{k,n}) + j_3(kF_{k,n+2} + F_{k,n+1}) \\ &= F_{k,n} + j_1F_{k,n+1} + j_2F_{k,n+2} + j_3F_{k,n+3} \\ &= BhF_{k,n}. \end{aligned}$$

So, the sequence of bihyperbolic k -Fibonacci numbers $\{BhF_{k,n}\}_{n \in \mathbb{N}}$ is also a k -Fibonacci sequence, with initial conditions $BhF_{k,0}, BhF_{k,1}$.

In the next subsection, we give a recurrence relation between the bihyperbolic k -Fibonacci numbers.

2.1 Recurrence Relation

Let $n \geq 0$ be an integer. The bihyperbolic k -Fibonacci numbers verify the recurrence relation

$$BhF_{k,n} - j_1BhF_{k,n+1} - j_2BhF_{k,n+2} + j_3BhF_{k,n+3} = k^2(F_{k,n+4} + F_{k,n+2}).$$

Proof.

$$\begin{aligned} &BhF_{k,n} - j_1BhF_{k,n+1} - j_2BhF_{k,n+2} + j_3BhF_{k,n+3} \\ &= F_{k,n} + j_1F_{k,n+1} + j_2F_{k,n+2} + j_3F_{k,n+3} - j_1(F_{k,n+1} + j_1F_{k,n+2} + j_2F_{k,n+3} + j_3F_{k,n+4}) \\ &\quad - j_2(F_{k,n+2} + j_1F_{k,n+3} + j_2F_{k,n+4} + j_3F_{k,n+5}) + j_3(F_{k,n+3} + j_1F_{k,n+4} + j_2F_{k,n+5} + j_3F_{k,n+6}) \end{aligned}$$

$$\begin{aligned}
&= F_{k,n} + j_1 F_{k,n+1} + j_2 F_{k,n+2} + j_3 F_{k,n+3} - j_1 F_{k,n+1} - F_{k,n+2} - j_3 F_{k,n+3} - j_2 F_{k,n+4} \\
&\quad - j_2 F_{k,n+2} - j_3 F_{k,n+3} - F_{k,n+4} - j_1 F_{k,n+5} + j_3 F_{k,n+3} + j_2 F_{k,n+4} + j_1 F_{k,n+5} + F_{k,n+6} \\
&= F_{k,n} - F_{k,n+2} - F_{k,n+4} + F_{k,n+6} \\
&= k F_{k,n+5} - k F_{k,n+1} \\
&= k(F_{k,n+5} - F_{k,n+1}) \\
&= k(k F_{k,n+4} + F_{k,n+3} - F_{k,n+3} + k F_{k,n+2}) \\
&= k^2(F_{k,n+4} + F_{k,n+2}). \quad \square
\end{aligned}$$

In the next theorem, we study the Binet formula for these numbers.

Theorem 2 (Binet Identity). For $n \in \mathbb{N}$,

$$BhF_{k,n} = \frac{A\sigma_1^n - B\sigma_2^n}{\sigma_1 - \sigma_2}, \quad (2.2)$$

where $A = 1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3$ and $B = 1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3$.

Proof. In [3], it is proven that $\sigma^n = F_{k,n}\sigma + F_{k,n-1}$, for $\sigma = \sigma_{1,2}$. Using this formula and the Binet Identity (Formula (1.1)) for the k -Fibonacci numbers, $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$,

$$\begin{aligned}
BhF_{k,n} &= F_{k,n} + j_1 F_{k,n+1} + j_2 F_{k,n+2} + j_3 F_{k,n+3} \\
&= \frac{1}{\sigma_1 - \sigma_2} (\sigma_1^n - \sigma_2^n + j_1(\sigma_1^{n+1} - \sigma_2^{n+1}) + j_2(\sigma_1^{n+2} - \sigma_2^{n+2}) + j_3(\sigma_1^{n+3} - \sigma_2^{n+3})) \\
&= \frac{1}{\sigma_1 - \sigma_2} (\sigma_1^n(1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3) - \sigma_2^n(1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3)) \\
&= \frac{A\sigma_1^n - B\sigma_2^n}{\sigma_1 - \sigma_2},
\end{aligned}$$

where $A = 1 + j_1\sigma_1 + j_2\sigma_1^2 + j_3\sigma_1^3$ and $B = 1 + j_1\sigma_2 + j_2\sigma_2^2 + j_3\sigma_2^3$.

Other form of expressing A and B is the following:

$$\begin{aligned}
1 + j_1\sigma + j_2\sigma^2 + j_3\sigma^3 &= 1 + j_1\sigma + j_2(k\sigma + 1) + j_3((k^2 + 1)\sigma + k) \\
&= (1 + j_2 + k j_3) + (j_1 + k j_2 + (k^2 + 1)j_3)\sigma.
\end{aligned}$$

If $F_{k,-n} = (-1)^{n+1}F_{k,n}$, last formula is $BhF_{k,-1} + BhF_{k,0}\sigma$, and A and B take the form $A = BhF_{k,-1} + BhF_{k,0}\sigma_1$ and $B = BhF_{k,-1} + BhF_{k,0}\sigma_2$. \square

2.2 Some Properties of A and B

Taking into account the preceding concepts, $A = (1 + j_2 + j_3k) + (j_1 + j_2k + j_3(k^2 + 1))\sigma_1$ and $B = (1 + j_2 + j_3k) + (j_1 + j_2k + j_3(k^2 + 1))\sigma_2$, and $\sigma_1\sigma_2 = -1$ it is

$$\begin{aligned}
A - B &= (j_1 + j_2k + j_3(k^2 + 1))(\sigma_1 - \sigma_2) \\
&= BhF_{k,0}\sqrt{k^2 + 4}, \\
A + B &= 2(1 + j_2 + j_3k) + (j_1 + j_2k + j_3(k^2 + 1))k \\
&= 2 + j_1k + j_2(k^2 + 2) + j_3(k^3 + 3k)
\end{aligned}$$

$$\begin{aligned}
 &= 1 + (F_{k,1} + j_1 F_{k,2} + j_2 F_{k,3} + j_3 F_{k,4}) + j_2 + k j_3 \\
 &= (1 + j_2 + k j_3) + BhF_{k,1}, \\
 A \cdot B &= (1 + j_2 + j_3 k)^2 + (1 + j_2 + j_3 k)(j_1 + j_2 k + j_3(k^2 + 1))k - (j_1 + j_2 k + j_3(k^2 + 1))^2 \\
 &= 1 + 1 + k^2 + 2j_2 + 2j_3 k + 2j_1 k + j_1 k + j_2 k^2 + j_3(k^3 + k) + j_3 k + k^2 + j_1(k^3 + k) \\
 &\quad + j_2 k^2 + j_1 k^3 + (k^4 + k^2) - (1 + k^2 + (k^4 + 2k^2 + 1) + 2j_3 k + 2j_2(k^2 + 1) + 2j_1(k^3 + k)) \\
 &= 2j_1 k + j_3(k^3 + 2k) \\
 &= 2j_1 F_{k,2} + j_3 F_{k,4}.
 \end{aligned}$$

Catalan Identity for the k -Fibonacci numbers is [2] $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+r-1}F_{k,r}^2$. We will apply this formula to find the Catalan Identity for the bihyperbolic k -Fibonacci numbers.

Theorem 3 (Catalan Identity). *Let $n \geq r \geq 0$ be integers. Then*

$$BhF_{k,n-r}BhF_{k,n+r} - BhF_{k,n}^2 = (-1)^{n+r-1}(AB)F_{k,r}^2,$$

where $(AB) = 2k j_1 + (k^3 + 2k)j_3$.

Proof. Because the Binet Identity for the bihyperbolic k -Fibonacci numbers and $\sigma_1 \sigma_2 = -1$,

$$\begin{aligned}
 BhF_{k,n-r} &= \frac{A\sigma_1^{n-r} - B\sigma_2^{n-r}}{\sigma_1 - \sigma_2}; \quad BhF_{k,n+r} = \frac{A\sigma_1^{n+r} - B\sigma_2^{n+r}}{\sigma_1 - \sigma_2}, \\
 BhF_{k,n-r} \cdot BhF_{k,n+r} &= \frac{1}{(\sigma_1 - \sigma_2)^2} \left(A^2 \sigma_1^{2n} + B^2 \sigma_2^{2n} - (AB)(-1)^n \left(\left(\frac{\sigma_1}{\sigma_2} \right)^r + \left(\frac{\sigma_2}{\sigma_1} \right)^r \right) \right), \\
 BhF_{k,n}^2 &= \frac{1}{(\sigma_1 - \sigma_2)^2} (A^2 \sigma_1^{2n} + B^2 \sigma_2^{2n} - 2(AB)(-1)^n), \\
 BhF_{k,n-r} \cdot BhF_{k,n+r} - BhF_{k,n}^2 &= \frac{1}{(\sigma_1 - \sigma_2)^2} (AB)(-1)^{n-1}((-1)^r(\sigma_1^{2r} + \sigma_2^{2r}) - 2) \\
 &= (AB)(-1)^{n-1}(-1)^r \frac{(\sigma_1^r - \sigma_2^r)^2}{(\sigma_1 - \sigma_2)^2} \\
 &= (AB)(-1)^{n+r-1}F_{k,r}^2.
 \end{aligned}$$

Corollary 1. *If $r = 1$,*

$$BhF_{k,n-1} \cdot BhF_{k,n+1} - BhF_{k,n}^2 = (AB)(-1)^n = (2k j_1 + (k^3 + 2k)j_3)(-1)^n.$$

2.3 A New Relation

Between the bihyperbolic k -Fibonacci numbers and the k -Fibonacci numbers the following relation exists.

For $n \geq 0$,

$$BhF_{k,n} = (1 + j_2 + k j_3)F_{k,n} + BhF_{k,0}F_{k,n+1}. \tag{2.3}$$

Proof. Taking into account $BhF_{k,0} = j_1 + j_2 k + j_3(k^2 + 1)$ and $\sigma^r = F_{k,r} \sigma + F_{k,r-1}$, from the Binet Identity for the k -Fibonacci numbers (Formula (1.1)),

$$1 + j_1 \sigma + j_2 \sigma^2 + j_3 \sigma^3 = 1 + j_1 \sigma + j_2(k \sigma + 1) + j_3((k^2 + 1)\sigma + k)$$

$$\begin{aligned}
 &= (1 + j_2 + j_3k) + (j_1 + j_2k + j_3(k^2 + 1))\sigma \\
 &= (1 + j_2 + j_3k) + BhF_{k,0}\sigma, \\
 BhF_{k,n} &= (1 + j_2 + j_3k) \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} + BhF_{k,0} \frac{\sigma_1^{n+1} - \sigma_2^{n+1}}{\sigma_1 - \sigma_2} \\
 &= (1 + j_2 + j_3k)F_{k,n} + BhF_{k,0}F_{k,n+1}.
 \end{aligned}$$

On the other hand, $F_{k,-r} = (-1)^{r+1}F_{k,r} \rightarrow 1 + j_2j_3k = F_{-1} + j_1F_{k,0} + j_2F_{k,1} + j_3F_{k,2} = BhF_{k,-1}$. So, $BhF_{k,n} = BhF_{k,-1}F_{k,n} + BhF_{k,0}F_{k,n+1}$. □

2.4 Sum of the Bihyperbolic k -Fibonacci Numbers

For $n \geq 0$,

$$\sum_{i=0}^n BhF_{k,i} = \frac{1}{k} \left(BhF_{k,n+1} + BhF_{k,n} - 1 + \sum_{i=1}^3 (F_{k,i} - F_{k,i-1})j_i \right).$$

Proof. From equation (2.3) and taking into account formula (1.2):

$$\begin{aligned}
 \sum_{i=0}^n BhF_{k,i} &= (1 + j_2 + k j_3) \sum_{i=0}^n F_{k,i} + BhF_{k,0} \sum_{i=0}^n F_{k,i+1} \\
 &= (1 + j_2 + k j_3) \frac{1}{k} (F_{k,n+1} + F_{k,n} - 1) + BhF_{k,0} \left(\frac{1}{k} (F_{k,n+1} + F_{k,n} - 1) + F_{k,n+1} - F_{k,0} \right) \\
 &= (1 + j_2 + k j_3) \frac{1}{k} (F_{k,n+1} + F_{k,n} - 1) + BhF_{k,0} \frac{1}{k} (F_{k,n+1} + F_{k,n} + k F_{k,n+1} - 1) \\
 &= (1 + j_2 + k j_3) \frac{1}{k} (F_{k,n+1} + F_{k,n} - 1) + BhF_{k,0} \left(\frac{1}{k} (F_{k,n+2} + F_{k,n+1} - 1) \right) \\
 &= \frac{1}{k} ((1 + j_2 + k j_3)F_{k,n+1} + BhF_{k,0}F_{k,n+2} + (1 + j_2 + k j_3)F_{k,n} \\
 &\quad + BhF_{k,0}F_{k,n+1} - ((1 + j_2 + k j_3) + BhF_{k,0})) \\
 &= \frac{1}{k} (BhF_{k,n+1} + BhF_{k,n} - (1 + F_{k,0}j_1 + F_{k,1}j_2 + F_{k,2}j_3) + (F_{k,1}j_1 + F_{k,2}j_2 + F_{k,3}j_3)) \\
 &= \frac{1}{k} (BhF_{k,n+1} + BhF_{k,n} - BhF_{k,-1} + BhF_{k,0}).
 \end{aligned}$$
□

Theorem 4 (D’Ocagne Identity). *Let $m \geq n \geq 0$ be integer numbers. Then*

$$BhF_{k,m}BhF_{k,n+1} - BhF_{k,m+1}BhF_{k,n} = (-1)^n(AB)F_{k,m-n}.$$

Proof. Using the Binet Identity for the bihyperbolic k -Fibonacci numbers (formula (2.2)) and taking into account $\sigma_1 \cdot \sigma_2 = -1$,

$$\begin{aligned}
 &BhF_{k,m}BhF_{k,n+1} - BhF_{k,m+1}BhF_{k,n} \\
 &= \frac{1}{(\sigma_1 - \sigma_2)^2} ((A\sigma_1^m - B\sigma_2^m)(A\sigma_1^{n+1} - B\sigma_2^{n+1}) - (A\sigma_1^{m+1} - B\sigma_2^{m+1})(A\sigma_1^n - B\sigma_2^n)) \\
 &= \frac{1}{(\sigma_1 - \sigma_2)^2} (AB)(-\sigma_1^m \sigma_2^{n+1} - \sigma_1^{n+1} \sigma_2^m + \sigma_1^{m+1} \sigma_2^n + \sigma_1^n \sigma_2^{m+1}) \\
 &= \frac{1}{(\sigma_1 - \sigma_2)^2} (AB)(\sigma_1^m \sigma_2^n (\sigma_1 - \sigma_2) - \sigma_1^n \sigma_2^m (\sigma_1 - \sigma_2))
 \end{aligned}$$

$$\begin{aligned}
 &= (AB) \frac{\sigma_1^m \sigma_2^n - \sigma_1^n \sigma_2^m}{\sigma_1 - \sigma_2} \\
 &= (AB)(-1)^n \frac{\sigma_1^{m-n} - \sigma_2^{m-n}}{\sigma_1 - \sigma_2} \\
 &= (-1)^n (AB) F_{k,m-n}.
 \end{aligned}$$

Using convolution Formula (1.3) of the k -Fibonacci numbers, we will try a similar formula for bihyperbolic k -Fibonacci numbers.

2.5 Convolution Formula

If n, m are integer numbers $n, m \geq 0$ it is verified

$$BhF_{k,n+m} = F_{k,m} BhF_{k,n+1} + F_{k,m-1} BhF_{k,n}.$$

Proof. From the definition, and taking into account the convolution formula for the k -Fibonacci numbers (1.3):

$$\begin{aligned}
 BhF_{k,n+m} &= F_{k,n+m} + j_1 F_{k,n+m+1} + j_2 F_{k,n+m+2} + j_3 F_{k,n+m+3} \\
 &= F_{k,n+1} F_{k,m} + F_{k,n} F_{k,m-1} + j_1 (F_{k,n+2} F_{k,m} + F_{k,n+1} F_{k,m-1}) \\
 &\quad + j_2 (F_{k,n+3} F_{k,m} + F_{k,n+2} F_{k,m-1}) + j_3 (F_{k,n+4} F_{k,m} + F_{k,n+3} F_{k,m-1}) \\
 &= F_{k,m} (F_{k,n+1} + j_1 F_{k,n+2} + j_2 F_{k,n+3} + j_3 F_{k,n+4}) \\
 &\quad + F_{k,m-1} (F_{k,n} + j_1 F_{k,n+1} + j_2 F_{k,n+2} + j_3 F_{k,n+3}) \\
 &= F_{k,m} BhF_{k,n+1} + F_{k,m-1} BhF_{k,n}
 \end{aligned}$$

2.6 Generating Function

Following the same process as in [3], we will find the generating function of the bihyperbolic k -Fibonacci numbers.

Let $bhf(x)$ be the generating function of the sequence of bihyperbolic k -Fibonacci numbers, $\{BhF_{k,n}\}_{n \in \mathbb{N}}$. Then, taking into account that $BhF_{k,m} = k BhF_{k,m-1} + BhF_{k,m-2}$ for $m \geq 2$,

$$\begin{aligned}
 bhf(x) &= BhF_{k,0} + BhF_{k,1}x + BhF_{k,2}x^2 + BhF_{k,3}x^3 + \dots + BhF_{k,n}x^n + \dots, \\
 k bhf(x)x &= k BhF_{k,0}x + k BhF_{k,1}x^2 + k BhF_{k,2}x^3 + \dots + k BhF_{k,n-1}x^n + \dots, \\
 x^2 bhf(x) &= BhF_{k,0}x^2 + BhF_{k,1}x^3 + \dots + BhF_{k,n-2}x^n + \dots,
 \end{aligned}$$

$$bhf(x)(1 - kx - x^2) = BhF_{k,0} + (BhF_{k,1} - kBhF_{k,0})x^2.$$

On the other hand

$$\begin{aligned}
 BhF_{k,1} - kBhF_{k,0} &= F_{k,1} + (F_{k,2} - kF_{k,1})j_1 + (F_{k,3} - kF_{k,2})j_2 + (F_{k,4} - kF_{k,3})j_3 \\
 &= F_{k,1} + F_{k,0}j_1 + F_{k,1}j_2 + F_{k,2}j_3 \\
 &= 1 + j_2 + kj_3 \\
 &= BhF_{k,-1}.
 \end{aligned}$$

So, the generating function of the bihyperbolic k -Fibonacci numbers is

$$bhf(x) = \frac{j_1 + kj_2 + (k^2 + 1)j_3 + (1 + j_2 + kj_3)x}{1 - kx - x^2}.$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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