Communications in Mathematics and Applications
Vol. 15, No. 2, pp. 583–596, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i2.2591

http://www.rgnpublications.com

Research Article

Some New Results in Extended Cone *b*-Metric Space

Iqbal Kour, Pooja Saproo, Shallu Sharma[3] and Sahil Billawria*

Department of Mathematics, University of Jammu, Jammu & Kashmir, India
*__Corresponding author:__ sahilbillawria2@gmail.com

Received: February 13, 2024 **Accepted:** April 18, 2024

Abstract. The main aspect of this paper is to investigate some topological properties and Kannan-type contractions in extended cone b-metric spaces. Additionally, we have imposed some extra conditions such that a sequence in an extended cone b-metric space becomes a Cauchy sequence. Furthermore, in order to achieve new results the concept of asymptotic regularity has also been utilized.

Keywords. Cone metric space, Extended cone b-metric space, Fixed point, Completeness

Mathematics Subject Classification (2020). 47H10, 49J45, 54H25, 54E35,54A20

1. Introduction

The contraction mapping principle in many forms of generalised metric spaces serves as the foundation for advanced metric fixed point theory. In 1922, Banach [5] demonstrated the Banach contraction principle related to fixed point theory in metric space. Some examples of generalized metric space are fuzzy metric (Rano *et al.* [12]), b-metric (Bakhtin [4], and Czerwik [6]) space. b-metric space is one of such generalised metric spaces, which was first introduced by Bakhtin [4], and Czerwik [6]. The renowned Banach contraction principle in b-metric space was generalised by Bakhtin [4]. Aghajani *et al.* [1] obtained some fixed point results in partially ordered b-metric space. Extended b-metric was introduced by Kamran *et al.* [10] as a development of the b-metric. Aydi *et al.* [2] introduced the notion of a new extended b-metric space. Huang and Zhang [8] created cone metric space in 2007, replacing the set of real numbers with an ordered Banach space. Kannan [11] presented one of the most significant generalisations of the Banach contraction principle. Kannan's work [11] enhanced the Banach contraction mapping notion by presenting a new contraction, currently known as the Kannan contraction. Kannan fixed point results have been extended and generalised in the establishment of b-metric spaces (Czerwik [6]) and generalised metric spaces (Azam and Arshad [3]). Hussian and Shah [9] established the concept of cone b-metric space by combining the notions of b-metric and cone metric. Das and Beg [7] further introduced the notion of extended cone b-metric space. In this paper we investigate Kannan type contractions within the framework of new extended b-metric.

2. Preliminaries

Definition 2.1 ([6]). Let $\mathsf{X} \neq \phi$ be any set and $t \in [1,\infty)$. A b-metric is a function $\mathfrak{B} : \mathsf{X}\times\mathsf{X} \to [0,\infty)$ such that for every $x,y,z \in \mathsf{X}$ the following hold:

(i) $\mathfrak{B}(x,y) = 0 \iff x = y$,

(ii) $\mathfrak{B}(x,y) = \mathfrak{B}(y,x)$,

(iii) $\mathfrak{B}(x,z) \leq t[\mathfrak{B}(x,y) + \mathfrak{B}(y,z)]$.

Then $(\mathsf{X},\mathfrak{B})$ is said to be a b-metric space.

Definition 2.2 ([2]). Let $\mathsf{X} \neq \phi$ be any set and $\Theta : \mathsf{X}\times\mathsf{X}\times\mathsf{X} \to [1,\infty)$ be a function. A map $d : \mathsf{X}\times\mathsf{X} \to [0,\infty)$ is said to be an extended b-metric on X if for every $x,y,z \in \mathsf{X}$ the following hold:

(i) $d(x,y) = 0 \iff x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq \Theta(x,y,z)(d(x,y) + d(y,z))$.

Then (X,d) is said an extended b-metric space.

Definition 2.3 ([8]). Let $\mathsf{E} \subset \mathbb{R}$ be a Banach space. A set P contained in E is said to be a cone if it satisfies the following:

(i) P is non-empty, closed and $\mathsf{P} \neq \{0\}$.

(ii) $sx + ty \in \mathsf{P}$ whenever $x,y \in \mathsf{P}$ and $s,t \in \mathbb{R}_{\geq 0}$.

(iii) $x \in \mathsf{P}$ and $-x \in \mathsf{P}$ implies that $x = 0$.

Hereafter we assume that P is a cone contained in a real Banach space E.

Definition 2.4 ([8]). A partial ordering $\leq$ with respect to P is defined as:

(i) $x \leq y \iff y - x \in \mathsf{P}$ and $x < y$ denotes that $x \leq y$, $x \neq y$.

(ii) $x \ll y$ indicates that $y - x$ is an element of int(P) (interior P).

Definition 2.5 ([8]). P is said to be normal if there is a positive number N such that $x \leq y$ implies that $\|x\| \leq \mathsf{N}\|y\|$, for all $x,y \in \mathsf{E}$. The smallest N satisfying the above condition of normality is said to be the normal constant of P.

Hereafter we assume that P has non-empty interior and $\leq$ denotes the partial ordering with respect to P.

Definition 2.6 ([8]). Let $X \neq \phi$ be any set. A cone metric on X is a map $d : X \times X \to E$ such that for each $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x,y) > 0, d(x,y) = 0 \iff x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

Then (X,d) is a cone metric space.

Definition 2.7 ([8]). Let $\{x_n\}$ be a sequence in a cone metric space (X,d) and P be a normal cone with normal constant N. Then

(i) $\{x_n\}$ is convergent to x if for each $u \in E$, $0 \ll u$, there is a natural number M so that for all $m \geq M$, we have $d(x_n,x) \ll u$.

(ii) $\{x_n\}$ is a Cauchy sequence if for each $u \in E$, $0 \ll u$, there is a natural number M so that for all $m,n \geq M$, we have $d(x_n,x_m) \ll u$.

(iii) (X,d) is complete if every Cauchy sequence converges in X.

Lemma 2.1 ([13]). Let (X,d) be a cone metric space.

(i) For every $u_1 \gg 0$ and $u_2 \in P$ there exists $u_3 \gg 0$ such that $u \gg u_1$ and $u \gg u_2$.

(ii) For every $u_1 \gg 0$, $u_2 \gg 0$ there exists $u \gg 0$ such that $u \ll u_1$ and $u \ll u_2$.

Definition 2.8 ([7]). Let $X \neq \phi$ be any set and $\Theta : X \times X \times X \to [1,\infty)$ be a function. A map $d_\Theta : X \times X \to E$ is said to be an extended cone b-metric on X if for every $x, y, z \in X$ the following hold:

(i) $d_\Theta(x,y) > 0, d_\Theta(x,y) = 0 \iff x = y$,

(ii) $d_\Theta(x,y) = d_\Theta(y,x)$,

(iii) $d_\Theta(x,z) \leq \Theta(x,y,z)(d_\Theta(x,y) + d_\Theta(y,z))$.

Then (X, d_Θ) is said an extended cone b-metric space.

Definition 2.9 ([7]). Let $\{x_n\}$ be a sequence in an extended cone b-metric space (X,d_Θ) and P be a normal cone with normal constant N. Then

(i) $\{x_n\}$ is convergent to x if for each $u \in E$, $0 \ll u$, there is a natural number M so that for all $m \geq M$, we have $d_\Theta(x_n,x) \ll u$.

(ii) $\{x_n\}$ is a Cauchy sequence if for each $u \in E$, $0 \ll u$, there is a natural number M so that for all $m,n \geq M$, we have $d_\Theta(x_n,x_m) \ll u$.

(iii) (X,d_Θ) is complete if every Cauchy sequence in X converges in X.

Definition 2.10 ([7]). Let (X,d_Θ) be an extended cone b-metric space and $x \in X$, $0 \ll u$. The open and closed balls in X are defined as $B(a,u) = \{y \in X : d_\Theta(a,y) \ll u\}$ and $B[a,y] = \{y \in X : d_\Theta(a,y) \leq u\}$, respectively.

Definition 2.11 ([7]). Let (X,d_Θ) be an extended cone b-metric and $\{(x_n,y_n)\}$ be a sequence in $X \times X$. Then d_Θ is continuous if x_n is convergent to x and y_n is convergent to y implies that $d_\Theta(x_n,y_n)$ is convergent to $d_\Theta(x,y)$ in E.

3. Topological Properties of Extended Cone b-metric Space

Proposition 3.1. The family $\mathcal{B} = \{\mathsf{B}(x,u) : u \gg 0\}$ is a basis for the topology σ_{d_Θ} on X.

Proof. (i) Let $x \in \mathsf{X}$. Then there exists $u \gg 0$ such that $x \in \mathsf{B}(x,u)$. Hence $x \in \mathsf{B}(x,u) \subseteq \bigcup_{x\in\mathsf{X},u\gg 0} \mathsf{B}(x,u)$,

(ii) Let $x \in \mathsf{X}$, $u_1 \gg 0$, $u_2 \gg 0$ such that $x \in \mathsf{B}(x,u_1) \cap \mathsf{B}(x,u_2)$. Then by using Lemma 2.1 there exists $u \gg 0$ such that $u \ll u_1$ and $u \ll u_2$. Clearly, $x \in \mathsf{B}(x,u) \subseteq \mathsf{B}(x,u_1) \cap \mathsf{B}(x,u_2)$. □

Definition 3.2. Let (X, d_Θ) be an extended cone b-metric space. A set $\mathfrak{U} \subset (\mathsf{X}, d_\Theta)$ is said to be sequentially open if for $x \in \mathfrak{U}$ such that $x_n \to x$ then there exists a natural number N such that $x_n \in \mathfrak{U}$, for all $n > \mathsf{N}$.

Proposition 3.3. Let (X, d_Θ) be an extended cone b-metric space. Then, the sequential topology σ and the topology σ_{d_Θ} induced by d_Θ coincide.

Proof. Suppose $\mathfrak{U} \in \sigma$. Suppose $\mathfrak{U} \notin \sigma_{d_\Theta}$. Then, there is some $x \in \mathfrak{U}$ and $u_1 \gg 0$ such that $\mathsf{B}(x,u_1)$ is not contained in $\mathfrak{U}$. Let $x_n \in \mathsf{B}(x,u_1)$ such that $x_n \notin \mathfrak{U}$ for every natural number n. Then $d_\Theta(x_n,x) \ll u_1$. This implies that $x_n \to x$ in $(\mathfrak{X}, d_\Theta)$. Since $\mathfrak{U} \in \sigma$. Then there exists a natural number $\mathfrak{N}$ such that $x_n \in \mathfrak{U}$ for all $n > \mathfrak{N}$, a contradiction. Next suppose that $\mathfrak{U} \in \sigma_{d_\Theta}$. For all $x \in \mathfrak{U}$ such that $x_n \to x$ in $(\mathfrak{X}, \sigma_\Theta)$, we have $\mathsf{B}(x,u_1) \subset \mathfrak{U}$, for some $u_1 \gg 0$. This implies there exists a natural number $\mathfrak{N}_0$ such that $d_\theta(x_n,x) \ll u_1$, for all $n \geq \mathfrak{N}_0$. Hence $x_n \in \mathfrak{U}$, for all $n \geq \mathfrak{N}_0$. Thus, $\mathfrak{U} \in \sigma$. □

4. Kannan-type Contractions in Extended Cone b-metric Space

Proposition 4.1. Let (X, d_Θ) be an extended cone b-metric space and $\Theta : \mathsf{X} \times \mathsf{X} \times \mathsf{X} \to [1,\infty)$ be a map. If there exists $s \in [0,1)$ such that the sequence $\{x_{n_1}\}$ satisfies $\lim_{n_1,n_2\to\infty} \Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) < 1/s$ and

$$0 < d_\Theta(x_{n_1}, x_{n_1+1}) \leq s d_\Theta(x_{n_1-1}, x_{n_1}), \tag{4.1}$$

for $n_1 \in \mathbb{N}$, then the sequence $\{x_{n_1}\}$ is a Cauchy sequence.

Proof. Let $\{x_{n_1}\}$ be a sequence in X. Now,

$$\begin{aligned} d_\Theta(x_{n_1}, x_{n_1+1}) &\leq s d_\Theta(x_{n_1-1}, x_{n_1}) \\ &\leq s^2 d_\Theta(x_{n_1-2}, x_{n_1-1}) \\ &\vdots \\ &\leq s^{n_1} d_\Theta(x_0, x_1). \end{aligned} \tag{4.2}$$

Since $s \in [0,1)$, we see that

$$\lim_{n_1\to\infty} d_\Theta(x_{n_1}, x_{n_1+1}) = 0. \tag{4.3}$$

Using inequality, for $n_2 \geq n_1$, we get

$$d_\Theta(x_{n_1}, x_{n_2}) \leq \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})(d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(x_{n_1+1}, x_{n_2}))$$

$$
\begin{aligned}
&= \Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1},x_{n_1+1}) + \Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1+1},x_{n_2})\\
&\le \Theta(x_{n_1},x_{n_2},x_{n_1+1})s^{n_1}d_\Theta(x_0,x_1)\\
&\quad + \Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})(d_\Theta(x_{n_1+1},x_{n_1+2}),d_\Theta(x_{n_1+2},x_{n_2}))\\
&\le \Theta(x_{n_1},x_{n_2},x_{n_1+1})s^{n_1}d_\Theta(x_0,x_1)\\
&\quad + \Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})s^{n_1+1}d_\Theta(x_0,x_1) + \dots\\
&\quad + \Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})s^{n_2-1}d_\Theta(x_0,x_1)\\
&\le [\Theta(x_{n_1},x_{n_2},x_{n_1+1})s^{n_1} + \Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})s^{n_1+1} + \dots\\
&\quad + \Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})s^{n_2-1}]d_\Theta(x_0,x_1).
\end{aligned}
$$

Since $\lim_{n_1,n_2\to\infty}\Theta(x_{n_1},x_{n_2},x_{n_1+1})s<1$. We see that by ratio test the series $\sum_{n_1=1}^{\infty} s^{n_1}\prod_{j=1}^{n_1}\Theta(x_j,x_{n_2},x_{j+1})$ converges. Let $\mathcal{S}=\sum_{n_1=1}^{\infty} s^{n_1}\prod_{j=1}^{n_1}\Theta(x_j,x_{n_2},x_{j+1})$ and $\mathcal{S}_{n_1}=\sum_{i=1}^{n_1} s^{i}\prod_{j=1}^{i}\Theta(x_j,x_{n_2},x_{j+1})$. Therefore,

$$d_\Theta(x_{n_1},x_{n_2}) \le d_\Theta(x_1,x_0)[\mathcal{S}_{n_2-1}-\mathcal{S}_{n_1-1}]. \tag{4.4}$$

Letting $n_1\to\infty$, we obtain the desired result. □

Theorem 4.2. Let (X,d_Θ) be a complete extended cone b-metric space and P be a cone in E. Let $f:\mathsf{X}\to\mathsf{X}$ be a mapping that satisfies:

$$d_\Theta(f(x),f(y)) \le s[d_\Theta(x,f(x))+d_\Theta(y,f(y))]+td_\Theta(y,f(x)), \quad \text{forall } x,y\in\mathsf{X}, \tag{4.5}$$

where $s\in\left(0,\frac{1}{2}\right)$ and $t\in[0,1)$. Suppose that

$$\sup_{n_2\ge1}\lim_{n_1\to\infty}\Theta(x_{n_1},x_{n_2},x_{n_1+1}) < \frac{1-s}{s}, \quad \text{forall } x_0\in\mathsf{X}, \tag{4.6}$$

such that $x_{n_1}=f^{n_1}(x_0)$, $n_1\in\mathbb{N}$. Then f has a unique fixed point u in X. Moreover, for each $x\in\mathsf{X}$, the sequence $\{f^{n_1}(x)\}$ is convergent to u, and

$$d_\Theta(f(x_{n_1}),u) \le \frac{s}{1-t}\left(\frac{s}{1-s}\right)^{n_1} d_\Theta(f(x_0),x_0), \quad n_1=0,1,2,\dots. \tag{4.7}$$

Proof. Let $x_0\in\mathsf{X}$ be arbitrary. Define

$$x_{n_1+1}=f(x_{n_1})=f^{n_1+1}(x_0). \tag{4.8}$$

Clearly, x_{n_1} is a fixed point of f if $x_{n_1}=x_{n_1+1}$, for some $n_1\in\mathbb{N}$. If not, suppose that x_{n_1} and x_{n_1+1} are distinct points in X for each $n_1\ge0$. Since

$$d_\Theta(x_{n_1},x_{n_1+1}) = d_\Theta(f(x_{n_1-1}),f(x_{n_1})). \tag{4.9}$$

We have

$$d_\Theta(f(x_{n_1-1}),f(x_{n_1})) \le s[d_\Theta(x_{n_1-1},f(x_{n_1-1}))+d_\Theta((x_{n_1}),f(x_{n_1}))]+td_\Theta(x_{n_1},f(x_{n_1-1})). \tag{4.10}$$

Then

$$d_\Theta(x_{n_1},x_{n_1+1}) \le s[d_\Theta(x_{n_1-1},x_{n_1})+d_\Theta(x_{n_1},x_{n_1+1})]+td_\Theta(x_{n_1},x_{n_1}). \tag{4.11}$$

Therefore,

$$d_\Theta(x_{n_1},x_{n_1+1}) \le \left(\frac{s}{1-s}\right)d_\Theta(x_{n_1-1},x_{n_1}). \tag{4.12}$$

Proceeding in the similar manner, we see that

$$d_\Theta(x_{n_1},x_{n_1+1}) \leq \left(\frac{s}{1-s}\right)^n d_\Theta(x_0,x_1) \tag{4.13}$$

and

$$d_\Theta(f(x_{n_1-1}),f(x_{n_1}) \leq \left(\frac{s}{1-s}\right)^n d_\Theta(x_0,f(x_0)). \tag{4.14}$$

Suppose n_1,n_2 are natural numbers such that $n_2 > n_1$. By applying triangular inequality, we have

$$\begin{aligned} d_\Theta(x_{n_1},x_{n_2}) &\leq \Theta(x_{n_1},x_{n_2},x_{n_1+1})[d_\Theta(x_{n_1},x_{n_1+1})+d_\Theta(x_{n_1+1},x_{n_2})] \\ &\leq \Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1},x_{n_1+1}) \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})[d_\Theta(x_{n_1+1},x_{n_1+2})+d_\Theta(x_{n_1+2},x_{n_2})] \\ &\leq \Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1},x_{n_1+1}) \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})d_\Theta(x_{n_1+1},x_{n_1+2})+\dots \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2}) \\ &\quad \cdot\Theta(x_{n_1+2},x_{n_2},x_{n_1+3})\dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})d_\Theta(x_{n_2-1},x_{n_2}). \end{aligned} \tag{4.15}$$

Since

$$d_\Theta(x_{n_1},x_{n_1+1}) \leq \left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0,x_1), \quad n \geq 0, \tag{4.16}$$

we have

$$\begin{aligned} d_\Theta(x_{n_1},x_{n_2}) &\leq \Theta(x_{n_1},x_{n_2},x_{n_1+1})\left(\frac{s}{1-s}\right)^{n_1} d_\Theta(x_0,x_1) \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\left(\frac{s}{1-s}\right)^{n_1+1} d_\Theta(x_0,x_1)+\dots \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\Theta(x_{n_1+2},x_{n_2},x_{n_1+3}) \\ &\qquad \dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})\left(\frac{s}{1-s}\right)^{n_2-1} d_\Theta(x_0,x_1) \\ &\leq \Bigg[\Theta(x_{n_1},x_{n_2},x_{n_1+1})\left(\frac{s}{1-s}\right)^{n_1} \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\left(\frac{s}{1-s}\right)^{n_1+1}+\dots \\ &\quad +\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\Theta(x_{n_1+2},x_{n_2},x_{n_1+3}) \\ &\qquad \dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})\left(\frac{s}{1-s}\right)^{n_2-1}\Bigg]d_\Theta(x_0,x_1). \end{aligned} \tag{4.17}$$

Moreover,

$$\sup_{n_2\geq 1}\lim_{n_1,n_2\to\infty}\Theta(x_{n_1},x_{n_2},x_{n_1+1})\frac{s}{1-s} < 1. \tag{4.18}$$

We see that the series $\sum_{n_1=1}^{\infty}\left(\frac{s}{1-s}\right)^{n_1}\prod_{j=1}^{n_1}\Theta(x_j,x_{n_2},x_{j+1})$ is convergent for every natural number n_2 by ratio test.

Next suppose $\mathcal{S} = \sum_{n_1=1}^{\infty} \left(\frac{s}{1-s}\right)^{n_1} \prod_{j=1}^{n_1} \Theta(x_j, x_{n_2}, x_{j+1})$ and

$$\mathcal{S}_{n_1} = \sum_{i=1}^{n_1} \left(\frac{s}{1-s}\right)^i \prod_{j=1}^{i} \Theta(x_j, x_{n_2}, x_{j+1}). \tag{4.19}$$

Hence for $n_2 > n_1$, using above inequality we have

$$d_\Theta(x_{n_1}, x_{n_2}) \le d_\Theta(x_0, x_1)(\mathcal{S}_{n_2-1} - \mathcal{S}_{n_1-1}). \tag{4.20}$$

Then

$$\lim_{n\to\infty} d_\Theta(x_{n_1}, x_{n_2}) = 0. \tag{4.21}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in \mathsf{X}$ such that $x_{n_1} \to u$ as $n_1 \to \infty$.

Claim: u is a fixed point of f.

Since $d_\Theta(f(x_{n_1}), f(u)) \le s[d_\Theta(x_{n_1}, f(x_{n_1})) + d_\Theta(u, f(u))] + td_\Theta(u, f(x_{n_1}))$. In context of the previous supposition that d_Θ is continuous, taking limit $n_1 \to \infty$, we have

$$d_\Theta(u, f(u)) \le sd_\Theta(u, f(u)). \tag{4.22}$$

This is only possible when $d_\Theta(u, f(u)) = 0$. Hence $f(u) = u$.

Next we show that fixed point of f is unique. For this, let v be a fixed point of f distinct from u. Then

$$d_\Theta(u,v) = d_\Theta(f(u), f(v)) \le s[d_\Theta(u, f(u)) + d_\Theta(v, f(v))] + td_\Theta(u, f(v)). \tag{4.23}$$

We have

$$d_\Theta(u,v) \le td_\Theta(u,v). \tag{4.24}$$

This is only possible when $d_\Theta(u,v) = 0$. Hence u is the unique fixed point of f in X. Also, we have

$$d_\Theta(f(x_{n_1-1}), f(x_{n_1})) \le s[d_\Theta(f(x_{n_1-2}), f(x_{n_1-1})) + d_\Theta(f(x_{n_1-1}), f(x_{n_1}))] + td_\Theta(x_{n_1}, x_{n_1}). \tag{4.25}$$

This implies that

$$d_\Theta(f(x_{n_1-1}), f(x_{n_1})) \le \left(\frac{s}{1-s}\right) d_\Theta(f(x_{n_1-2}), f(x_{n_1-1})). \tag{4.26}$$

Further,

$$\begin{aligned} d_\Theta(f(x_{n_1}), u) &\le s[d_\Theta(f(x_{n_1-1}), f(x_{n_1})) + d_\Theta(u, f(u))] + td_\Theta(u, f(x_{n_1})) \\ &\le sd_\Theta(f(x_{n_1-1}), f(x_{n_1})) + td_\Theta(u, f(x_{n_1})) \end{aligned} \tag{4.27}$$

From (4.13), we have

$$d_\Theta(f(x_{n_1}), u) \le \frac{s}{1-t}\left(\frac{s}{1-s}\right)^{n_1} d_\Theta(f(x_0), x_0), \quad n \ge 0. \tag{4.28}$$

Hence the proof. □

Theorem 4.3. Let (X, d_Θ) be a complete extended cone b-metric space, d_Θ be a continuous functional and $\mathcal{N} \ne \phi$ be a closed set contained in X. Suppose $f : \mathcal{N} \to \mathcal{N}$ be a mapping that satisfies

$$d_\Theta(f(x), f(y)) \le s[d_\Theta(x, f(x)) + d_\Theta(y, f(y))] + td_\Theta(y, f(x)), \quad \text{for all } x, y \in \mathcal{N}, 0 \le s, t \le 1 \tag{4.29}$$

and there exist real numbers γ,δ where $\gamma\in(0,1)$ and $\delta>0$ such that for arbitrary $x\in\mathcal{N}$, there exists x^* in $\mathcal{N}$ satisfying

$$\begin{aligned} &d_\Theta(x^*,f(x^*))\le\gamma d_\Theta(x,f(x)),\\ &\quad d_\Theta(x^*,x)\le\delta d_\Theta(x,f(x)). \end{aligned}\tag{4.30}$$

Moreover, for an arbitrary $x_0\in\mathcal{N}$, suppose that $\{x_{n_1}=f^{n_1}(x_0)\}$ satisfies

$$\sup_{n_2\ge1}\lim_{n_1\to\infty}\Theta(x_{n_1},x_{n_1+1},x_{n_2})<\frac{1}{\gamma}.\tag{4.31}$$

Then f has a unique fixed point.

Proof. Consider an arbitrary element $x_0\in\mathcal{N}$. Let $\{x_{n_1}=f^{n_1}(x_0)\}$ be a sequence in $\mathcal{N}$. We see that

$$d_\Theta(f(x_{n_1+1}),x_{n_1+1})\le\gamma(d_\Theta(f(x_{n_1}),x_{n_1})),d_\Theta(f(x_{n_1+1}),x_{n_1+1})\le\delta d_\Theta(f(x_{n_1}),x_{n_1}),\quad n_1\ge0.\tag{4.32}$$

Moreover,

$$\begin{aligned} d_\Theta(x_{n_1+1},x_{n_1})&=d_\Theta(f(x_{n_1}),x_{n_1})\le\delta d_\Theta(f(x_{n_1}),x_{n_1}),\quad n_1\ge0,\\ \delta d_\Theta(f(x_{n_1}),x_{n_1})&\le\delta\gamma d_\Theta(f(x_{n_1-1}),x_{n_1-1})\\ &\le\delta\gamma^2 d_\Theta(f(x_{n_1-2}),x_{n_1-2})\\ &\ \ \vdots\\ &\le\delta\gamma^n d_\Theta(f(x_0),x_0). \end{aligned}\tag{4.33}$$

Hence

$$d_\Theta(x_{n_1+1},x_{n_1})\le\delta\gamma^n d_\Theta(f(x_0),x_0).\tag{4.34}$$

Let n_1,n_2 be two fixed natural numbers such that $n_2>n_1$. Using triangular inequality, we have

$$\begin{aligned} d_\Theta(x_{n_1},x_{n_2})&\le\Theta(x_{n_1},x_{n_2},x_{n_1+1})[d_\Theta(x_{n_1},x_{n_1+1})+d_\Theta(x_{n_1+1},x_{n_2})]\\ &\le\Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1},x_{n_1+1})\\ &\quad+\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})[d_\Theta(x_{n_1+1},x_{n_1+2})+d_\Theta(x_{n_1+2},x_{n_2})]\\ &\le\Theta(x_{n_1},x_{n_2},x_{n_1+1})d_\Theta(x_{n_1},x_{n_1+1})\\ &\quad+\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})d_\Theta(x_{n_1+1},x_{n_1+2})+\dots\\ &\quad+\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1},x_{n_2},x_{n_1+2})\\ &\quad\cdot\Theta(x_{n_1+2},x_{n_2},x_{n_1+3})\dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})d_\Theta(x_{n_2-1},x_{n_2}). \end{aligned}\tag{4.35}$$

From (4.34), we have

$$\begin{aligned} d_\Theta(x_{n_1},x_{n_2})&\le[\Theta(x_{n_1},x_{n_2},x_{n_1+1})\gamma^{n_1}\\ &\quad+\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\gamma^{n_1+1}d_\Theta(f(x_0,x_0))+\dots\\ &\quad+\Theta(x_{n_1},x_{n_2},x_{n_1+1})\Theta(x_{n_1+1},x_{n_2},x_{n_1+2})\\ &\quad\cdot\Theta(x_{n_1+2},x_{n_2},x_{n_1+3})\dots\Theta(x_{n_2-2},x_{n_2},x_{n_2-1})\gamma^{n_2-1}]\delta d_\Theta(f(x_0),x_0). \end{aligned}\tag{4.36}$$

Since $\sup_{n_2\ge1}\lim_{n_1,n_2\to\infty}\Theta(x_{n_1},x_{n_1+1},x_{n_2})\gamma<1$, the series $\sum_{n_1=1}^{\infty}\gamma^{n_1}\prod_{j=1}^{n_1}\Theta(x_j,x_{n_2},x_{j+1})$ is convergent for every natural number n_2 by ratio test.

Suppose $\mathcal{S} = \sum_{n_1=1}^{\infty} \gamma^{n_1} \prod_{j=1}^{n_1} \Theta(x_j, x_{n_2}, x_{j+1})$ and

$$\mathcal{S}_{n_1} = \sum_{i=1}^{n_1} \gamma^i \prod_{j=1}^{i} \Theta(x_j, x_{n_2}, x_{j+1}). \tag{4.37}$$

Hence for $n_2 > n_1$, using above inequality, we have

$$d_\Theta(x_{n_1}, x_{n_2}) \le d_\Theta(x_0, x_1)(\mathcal{S}_{n_2-1} - \mathcal{S}_{n_1-1})\delta. \tag{4.38}$$

Suppose $n_1 \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{N}$ is complete, there exists $u \in \mathcal{N}$ such that $x_{n_1} \to u$ as $n_1 \to \infty$.

We shall show that u is a fixed point of f. By (4.29), we see that

$$d_\Theta(f(x_{n_1}), f(u)) \le s[d_\Theta(x_{n_1}, f(x_{n_1})) + d_\Theta(u, f(u))] + t d_\Theta(u, f(x_{n_1})).$$

Therefore,

$$d_\Theta(x_{n_1+1}, f(u)) \le s[d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(u, f(u))] + t d_\Theta(u, x_{n_1+1}).$$

Letting $n \to \infty$ and using the continuity of d_Θ, we get

$$d_\Theta(u, f(u)) \le s d_\Theta(u, f(u)). \tag{4.39}$$

This is only possible when $f(u) = u$. Next, we shall show that f has a unique fixed point. If possible, suppose that v is a fixed point of f distinct from u. Then

$$\begin{aligned} 0 &< d_\Theta(u, v) \\ &= d_\Theta(f(u), f(v)) \\ &\le s[d_\Theta(u, f(u)) + d_\Theta(v, f(v))] + t d_\Theta(v, f(u)) \\ &= t d_\Theta(v, u). \end{aligned} \tag{4.40}$$

Now, $d_\Theta(u, v) \le t d_\Theta(u, v)$ is not possible. Therefore, f has a unique fixed point u in X. □

Remark 4.4. *Theorem* 4.2 *can be proved in extended cone b-metric space using the following:*

$$d_\Theta(u, f(u)) \le \beta d_\Theta(x, f(x)),$$
$$d_\Theta(u, x) \le \gamma d_\Theta(y, f(y)).$$

Proof. Let $x \in \mathsf{X}$ be arbitrary and $u = f(x)$. Then, we have

$$\begin{aligned} d_\Theta(u, f(u)) &= d_\Theta(f(x), f(u)) \\ &\le s[d_\Theta(x, f(x)) + d_\Theta(u, f(u))] + t d_\Theta(u, f(x)) \end{aligned}$$

$$\implies \quad d_\Theta(u, f(u)) \le \left(\frac{s}{1-s}\right) d_\Theta(x, f(x)),$$

where $\left(\frac{s}{1-s}\right) < 1$ and $d_\Theta(u, x) = d_\Theta(f(x), x)$. Let $x_0 \in \mathsf{X}$ be arbitrary. Next define a sequence $\{x_{n_1+1} = f(x_{n_1})\}$. Using Theorem 4.3, the above sequence is convergent. Hence $x_{n_1} \to u$ as $n_1 \to \infty$. Therefore, $f(u) = u$. Moreover for every $x \in \mathsf{X}$,

$$\begin{aligned} d_\Theta(f(x_{n_1-1}), f(x_{n_1})) &\le s[d_\Theta(f(x_{n_1-2}), f(x_{n_1-1})) + d_\Theta(f(x_{n_1-1}), f(x_{n_1}))] + t d_\Theta(x_{n_1}, f(x_{n_1-1})) \\ &\le \left(\frac{s}{1-s}\right) d_\Theta(f(x_{n_1-2}), f(x_{n_1-1})), \end{aligned}$$

$$\begin{aligned}d_\Theta(f(x_{n_1}),u) &\le s[d_\Theta(f(x_{n_1-1}),f(x_{n_1}))+d_\Theta(u,f(u))]+td_\Theta(u,f(x_{n_1}))\\ &\le sd_\Theta(f(x_{n_1-1}),f(x_{n_1}))+td_\Theta(u,f(x_{n_1}))\\ &\le \frac{s}{1-t}\left(\frac{s}{1-s}\right)^n d_\Theta(f(x),x)), \quad n_1\ge 0.\end{aligned}$$ □

Theorem 4.5. Let (X,d_Θ) be a complete extended cone b-metric space such that d_Θ is a continuous functional. Suppose that the map $f:\mathsf{X}\to\mathsf{X}$ satisfies

$$d_\Theta(f(x),f(y))\le \beta d_\Theta(x,f(x))+\gamma d_\Theta(y,f(y))+\delta d_\Theta(x,y)+td_\Theta(x,f(y)), \quad \text{for all } x,y\in\mathsf{X}, \tag{4.41}$$

where $\beta,\gamma,\delta,t\in\mathbb{R}_{\ge 0}$ such that $\beta+\gamma+\delta+t<1$ and $\gamma+\delta>0$. Suppose that for any $x_0\in\mathcal{N}$,

$$\sup_{n_2\ge 1}\lim_{n_1\to\infty}\Theta(x_{n_1},x_{n_2},x_{n_1+1})<\frac{1}{q}, \tag{4.42}$$

where $q=\left(\frac{\gamma+\delta}{1-\beta}\right)$ and $x_{n_1}=f^{n_1}(x_0)$. Then f has a unique fixed point.

Proof. Let $x_0\in\mathsf{X}$ be arbitrary. Consider the sequence $\{f^{n_1}(x_0)\}$. Put $x=f^{n_1-1}(x_0)=f(x_{n_1-1})=x_{n_1}$ and $y=f^{n_1-2}(x_0)=f(x_{n_1-2})=x_{n_1-1}$ in (4.41), we get

$$\begin{aligned}d_\Theta(f(x_{n_1}),f(x_{n_1-1}))&\le \beta d_\Theta(f(x_{n_1-1}),f(x_{n_1}))+\gamma d_\Theta(f(x_{n_1-2}),f(x_{n_1-1}))\\ &\quad+\delta d_\Theta(f(x_{n_1-1}),f(x_{n_1-2}))+td_\Theta(f(x_{n_1-1}),f(x_{n_1-1})).\end{aligned} \tag{4.43}$$

That is,

$$(1-\beta)d_\Theta(f(x_{n_1}),f(x_{n_1-1}))\le(\gamma+\delta)d_\Theta(f(x_{n_1-1}),f(x_{n_1-2})). \tag{4.44}$$

Therefore,

$$d_\Theta(f(x_{n_1}),f(x_{n_1-1}))\le\left(\frac{\gamma+\delta}{1-\beta}\right)d_\Theta(f(x_{n_1-1}),f(x_{n_1-2})). \tag{4.45}$$

Also,

$$\begin{aligned}d_\Theta(f(x_{n_1}),f(x_{n_1-1}))&\le qd_\Theta(f(x_{n_1-1}),f(x_{n_1-2}))\\ &\le q^2d_\Theta(f(x_{n_1-2}),f(x_{n_1-3}))\\ &\ \ \vdots\\ &\le q^{n-1}d_\Theta(f(x_1),f(x_{n_0})), \quad \text{for all } n_1>1.\end{aligned}$$

Hence we have

$$d_\Theta(f(x_{n_1}),f(x_{n_1-1}))\le q^nd_\Theta(x_0,x_1), \quad \text{for all } n_1\in\mathbb{N}.$$

By given hypothesis we see that $q=\left(\frac{\gamma+\delta}{1-\beta}\right)<1$. Proceeding similarly as in Theorem 4.3 we see that $\{x_{n_1}\}$ is a Cauchy sequence. Since X is complete. Then, there exists $u\in\mathsf{X}$ such that $f^{n_1}(x_0)\to u$ as $n_1\to\infty$. To show that u is fixed point of f, substitute $x=f^{n_1}(x_0)$ and $y=u$ in (4.41). We get

$$d_\Theta(f^{n_1+1}(x_0),f(u))\le\beta d_\Theta(f^{n_1}(x_0),f^{n_1+1}(x_0))+\gamma d_\Theta(u,f(u))+\delta d_\Theta(f^{n_1}(x_0),u)+td_\Theta(f^{n_1}(x_0),f(u)). \tag{4.46}$$

Hence

$$d_\Theta(x_{n_1+2},f(u))\le\beta d_\Theta(f^{n_1}(x_0),f^{n_1+1}(x_0))+\gamma d_\Theta(u,f(u))+\delta d_\Theta(x_{n_1},u)+td_\Theta(f(u),x_{n_1+1}), \tag{4.47}$$

that is,

$$\lim_{n\to\infty} d_\Theta(x_{n_1+2}, f(u)) \le \lim_{n\to\infty} \beta d_\Theta(f^{n_1}(x_0), f^{n_1+1}(x_0)) + \gamma d_\Theta(u, f(u)) + \delta d_\Theta(x_{n_1}, u) + t d_\Theta(f(u), x_{n_1+1}). \tag{4.48}$$

We get

$$d_\Theta(u, f(u)) \le (\gamma + t) d_\Theta(u, f(u)),$$

which is only possible when $u = f(u)$.

To prove that f has a unique fixed point let v be a fixed point of f distinct from u. Then by (4.41), we have

$$d_\Theta(f(v), f(u)) \le \beta d_\Theta(v, f(v)) + \gamma d_\Theta(u, f(u)) + \delta d_\Theta(v, u) + t d_\Theta(f(u), v)$$

$$\implies \quad d_\Theta(v, u) \le (t + \delta) d_\Theta(v, u) \tag{4.49}$$

which is not possible. Therefore, f has a unique fixed point. □

Definition 4.6. Let (X, d_Θ) be an extended cone b-metric space. A mapping $f : \mathsf{X} \to \mathsf{X}$ is said to be asymptotically regular if $d_\Theta(f^{n_1+1}(x), f^{n_1}(x)) \to 0$ as $n \to \infty$, for each $x \in \mathsf{X}$.

Theorem 4.7. Let (X, d_Θ) be a complete extended cone b-metric space such that d_Θ is a continuous functional. Let $f : \mathsf{X} \to \mathsf{X}$ be an asymptotically regular self mapping

$$d_\Theta(f(x), f(y)) \le s[d_\Theta(x, f(x)) + d_\Theta(y, f(y))], \quad \text{for all } x, y \in \mathsf{X}. \tag{4.50}$$

Then f has a unique fixed point in $u \in \mathsf{X}$.

Proof. Let $x \in \mathsf{X}$ and define $x_{n_1} = f^{n_1}(x)$. Let n_1 and n_2 be two fixed natural numbers such that $n_2 > n_1$, then by the definition of asymptotic regularity, we have

$$d_\Theta(f^{n_1+1}(x), f^{n_2+1}(x)) \le s[d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) + d_\Theta(f^{n_2}(x), f^{n_2+1}(x))] \to 0 \text{ as } n \to \infty. \tag{4.51}$$

Therefore, $\{f^{n_1}(x)\}$ is a Cauchy sequence. Since X is complete. Then, there is $u \in \mathsf{X}$ such that

$$\lim_{n_1\to\infty} f^{n_1}(x) = u. \tag{4.52}$$

Next we shall show that u is a fixed point of f in the following manner:

$$d_\Theta(f(x_{n_1}), f(u)) \le s(d_\Theta(x_{n_1}, f(x_{n_1}))) + d_\Theta(x_{n_2}, f(x_{n_2})), \tag{4.53}$$

that is,

$$d_\Theta(f(x_{n_1}), f(u)) \le s(d_\Theta(x_{n_1}, x_{n_1+1})) + d_\Theta(u, f(u)). \tag{4.54}$$

Let $n \to \infty$ and by using asymptotically regular of f, we have

$$d_\Theta(u, f(u)) \le s d_\Theta(u, f(u)) \tag{4.55}$$

which holds when $f(u) = u$. To prove that u is unique fixed point of f, let v be a fixed point of f distinct from u. We get

$$d_\Theta(u, v) = d_\Theta(f(u), f(v)) \tag{4.56}$$

$$\le s(d_\Theta(u, f(v)) + d_\Theta(v, f(v))) \tag{4.57}$$

which holds when $d_\Theta(u, v) = 0$. This implies that $u = v$. Therefore, u is the unique fixed point of f. Moreover, for each $x \in \mathsf{X}, \{f^{n_1}(x)\}$ is convergent to u. □

Remark 4.8. *The condition on* $\Theta(x_{n_1}, x_{n_1+1}, x_{n_2})$ *can be dropped if the map is asymptotically regular.*

Theorem 4.9. Let (X, d_Θ) be a complete extended cone b-metric space such that d_Θ is a continuous functional. Consider an asymptotically regular mapping $f : \mathsf{X} \to \mathsf{X}$ such that d_Θ is a continuous functional such that there exists $t \in (0,1)$ such that

$$d_\Theta(f(x), f(y)) \le t[d_\Theta(x, f(x)) + d_\Theta(y, f(y)) + d_\Theta(x,y)], \quad \text{for all } x, y \in \mathsf{X}. \tag{4.58}$$

Then f has a unique fixed point $u \in \mathsf{X}$ unless

$$\lim_{n\to\infty} \frac{t + t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})}{1 - t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})} \tag{4.59}$$

exists for $x_n = f^{n_1}(x), n_2 > n_1$ and $x \in \mathsf{X}$ is arbitrary.

Proof. Let $x \in \mathsf{X}$ and $x_{n_1} = f^{n_1}(x)$. Let n_1, n_2 be fixed natural numbers such that $n_2 > n_1$. Then by (4.58), we have

$$\begin{aligned}
d_\Theta(f^{n_1+1}(x), f^{n_2+1}(x) &\le t[d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) + d_\Theta(f^{n_2}(x), f^{n_2+1}(x)) + d_\Theta(f^{n_1}(x), f^{n_2}(x))] \\
&\le t[d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) + d_\Theta(f^{n_2}(x), f^{n_2+1}(x))] \\
&\quad + \Theta(x_{n_1}, x_{n_2}, x_{n_1+1})[d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) + d_\Theta(f^{n_1+1}(x), f^{n_2}(x))] \\
&\le t[d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) + d_\Theta(f^{n_2}(x), f^{n_2+1}(x))] \\
&\quad + t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1}) d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) \\
&\quad + t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})[d_\Theta(f^{n_1+1}(x), f^{n_2+1}(x)) \\
&\quad + d_\Theta(f^{n_2+1}(x), f^{n_2}(x))] d_\Theta(f^{n_1+1}(x), f^{n_2+1}(x)) \\
&\le \left(\frac{t + t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})}{1 - t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})} \right) d_\Theta(f^{n_1}(x), f^{n_1+1}(x)) \\
&\quad + \left(\frac{t + t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})}{1 - t\Theta(x_{n_1}, x_{n_2}, x_{n_1+1})\Theta(x_{n_1+1}, x_{n_2}, x_{n_2+1})} \right) d_\Theta(f^{n_1+1}(x), f^{n_2}(x)) \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, $\{f^{n_1}(x)\}$ is a Cauchy sequence. Since X is complete, there exists $u \in \mathsf{X}$ such that

$$\{f^{n_1}(x)\} \to u \text{ as } n \to \infty. \tag{4.60}$$

Next using triangular inequality and (4.58), we have

$$d_\Theta(f_{x_{n_1}}, f(u)) \le t[d_\Theta(x_{n_1}, f(x_{n_1})) + d_\Theta(u, f(u)) + d_\Theta(x_{n_1}, u)]. \tag{4.61}$$

Therefore,

$$d_\Theta(x_{n_1+1}, f(u)) \le t[d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(u, f(u)) + d_\Theta(x_{n_1}, u)]. \tag{4.62}$$

Taking limit $n_1 \to \infty$, we have

$$\lim_{n_1\to\infty} (d_\Theta(x_{n_1+1}, f(u)) \le \lim_{n\to\infty} t[d_\Theta(x_{n_1}, x_{n_1+1}) + d_\Theta(u, f(u)) + d_\Theta(x_{n_1}, u)]). \tag{4.63}$$

Therefore, $d_\Theta(u, f(u)) \le t d_\Theta(u, f(u))$. This implies that $u = f(u)$. To prove that f has a unique fixed point. Let v be a fixed point of f distinct from u. Then

$$d_\Theta(f(u), f(v)) \le t[d_\Theta(v, f(v)) + d_\Theta(u, f(u))] + d_\Theta(u,v)], \quad t < 1 \tag{4.64}$$

which is a contradiction. Therefore, f has a unique fixed point. Hence $\{f^{n_1}(x)\}$ is convergent for each $x \in X$. □

5. Conclusion

Examining specific topological properties and Kannan-type contractions in extended cone b-metric space is the main aim of this work. The idea of asymptotic regularity has also been applied to produce fixed point results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. Aghajani, M. Abbas and J. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Mathematica Slovaca* **64**(4) (2014), 941 – 960, DOI: 10.2478/s12175-014-0250-6.

[2] H. Aydi, A. Felhi, T. Kamran, E. Karapinar and M. Usman, On nonlinear contractions in new extended b-metric spaces, *Applications and Applied Mathematics: An International Journal (AAM)* **14**(1) (2019), Article 37, URL: https://digitalcommons.pvamu.edu/aam/vol14/iss1/37.

[3] A. Azam and M. Arshad, Kannan fixed point theorem on generalized metric spaces, *Journal of Nonlinear Sciences and Applications* **1**(1) (2008), 45 – 48, DOI: 10.22436/jnsa.001.01.07.

[4] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, *Functional Analysis* **30** (1989), 26 – 37.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae* **3** (1922), 133 – 181, DOI: 10.4064/fm-3-1-133-181.

[6] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti del Seminario Matematico e Fisico dell'Universita di Modena e Reggio Emilia* **46** (1998), 263 – 276.

[7] A. Das and T. Beg, Some fixed point theorems in extended cone b-metric spaces, *Communications in Mathematics and Applications* **13**(2) (2022), 647 – 659, DOI: 10.26713/cma.v13i2.1768.

[8] L.-G. Huang and X. Zhang, Cone metric space and fixed point theorems of contractive mapping, *Journal of Mathematical Analysis and Applications* **322**(2) (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[9] N. Hussian and M. H. Shah, KKM mappings in cone b-metric spaces, *Computers & Mathematics with Applications* **62**(4) (2011), 1677 – 1684, DOI: 10.1016/j.camwa.2011.06.004.

[10] T. Kamran, M. Samreen and A. Ul-Qurat, A generalization of b-metric space and some fixed point theorems, *Mathematics* **5**(2) (2017), 19, DOI: 10.3390/math5020019.

[11] R. Kannan, Some results on fixed points, *Bulletin of the Calcutta Mathematical Society* **60** (1968), 71 – 76.

[12] G. Rano, T. Bag and S. K. Samanta, Fuzzy metric space and generating space of quasi-metric family, *Annals of Fuzzy Mathematics and Informatics* **11**(2) (2016), 183 – 195, URL: http://www.afmi.or.kr/papers/2016/Vol-11_No-02/PDF/AFMI-11-2(183-195)-H-150506R1.pdf.

[13] Sh. Rezapour, M. Derafshpour and R. Hamlbarani, A review on topological properties of cone metric spaces, in: *Proceedings of the International Conference on Analysis, Topology and Applications 2008* (ATA'08), Technical Faculty, Cacak, University of Kragujevac Vrnjacka Banja, Serbia, May 30 - June 4, 2008, URL: http://at.yorku.ca/c/a/w/q/04.htm.