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Research Article

Roman Domination in the Shadow Distance Graphs

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Abstract. A function $\psi : \mathcal{V} \to \{0,1,2\}$ satisfying the requirement that each vertex x for which $\psi(x) = 0$ is adjacent to at least one vertex y for which $\psi(y) = 2$ is known as a *Roman dominating function* (Rdf) on a graph. A Rdf's weight is represented by the value $\psi(y) = \sum_{x \in \mathcal{V}} \psi(x)$. The *Roman domination number* (Rdn) of a graph $\mathcal{G}$ is the minimal weight of a Rdf on that graph. In this article, we establish Rdn for the shadow distance graph of the path, cycle, and star graphs with predetermined distance sets.

Keywords. Roman dominating function, Roman domination number, Shadow graph, Shadow distance graph

Mathematics Subject Classification (2020). 05C69

1. Introduction

An outstanding historical account about the military prowess of the Roman empire serves as the inspiration for the problem of Roman dominance (see ReVelle and Rosing [10], and Stewart [11]). Constantine the Great established a new, comprehensive defense strategy that involved search and motion of the legions throughout the Empire to defend the Roman empire's enclave. There has been a need to provide protection for areas vulnerable due of the Empire's fourth century energy reduction. The locality that had no legions had been considered insecure, whereas the locality that had at least one legion had been considered secure. Unsecured areas can be escorted by a legion deployed from a nearby location, however the motion of legions through new locality is only permitted if the old zone is still escorted, i.e., if any other legion still resides there, a legion may be transferred from one territory to a nearby one. For this reason, before a legion is deployed to any other zone, at least two legions must be stationed at the comfortable area. This motivated us to study the domination number in shadow distance graphs (Mekala *et al*. [9], and Kumar and Murali [7]). Motivated by the ancient Roman empire's fourth century proposal of this military tactic, The *Roman Domination Problem* (RDP) first formally introduced by Cockayne *et al*. [5] and more Roman domination was studied by Ahangar *et al*. [1,2], Chellali *et al*. [4], Henning and Hedetniemi [6], and Beeler *et al*. [3].

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directionless and simple graph, a function $\psi : \mathcal{V} \to \{0,1,2\}$ fulfilling the requirement that each vertex x for which $\psi(x) = 0$ is neighbor to a minimum of one vertex y for which $\psi(y) = 2$ is known as a *Roman dominating function* (Rdf). A Rdf's weight is represented by the value $\psi(y) = \sum_{x\in\mathcal{V}} \psi(x)$. The minimum weight of a Rdf on a graph $\mathcal{G}$ is called the *Roman domination number* (Rdn) of $\mathcal{G}$. The Rdn of $\mathcal{G}$, outlined $\gamma_R(\mathcal{G})$, is defined as the minimum value of a Rdf $\gamma_R(\mathcal{G}) = \min\limits_{\psi\in F} \psi(y)$, where F is the set of all Rdf.

The Roman Dominance Problem is a part of a major class of domination set problems which has recently been the focus of extensive research. If each vertex in $\mathcal{V}_{\mathcal{D}}$ has a minimum of one vertex in $\mathcal{D}$, the set $\mathcal{D} \subset \mathcal{V}$ is said to be dominating set. The domination number $\gamma(\mathcal{G})$ is defined as the minimum cardinality of the dominating set in $\mathcal{G}$. In their study of the fundamental characteristics of Roman dominant functions, Cockayne *et al*. [5] determined $\gamma_R(\mathcal{G})$ for few classes of graphs.

The shadow graph of $\mathcal{G}$ is created by taking two copies of $\mathcal{G}$, $\mathcal{G}$ itself and $\mathcal{G}'$, and attaching each vertex $x \in \mathcal{G}$ to its neighbour $x' \in \mathcal{G}'$, denoted by the symbol $\mathcal{D}(\mathcal{G})$. The total distances in $\mathcal{G}$ between unique twins of vertices are collected in $\mathcal{D}$, and let $\mathcal{D}_{\mathcal{S}} \subset \mathcal{D}$ (known as distance set). The distance graph of $\mathcal{G}$ is given by the symbol $\mathcal{D}(\mathcal{G}, \mathcal{D}_{\mathcal{S}})$, and it has the same vertex set as $\mathcal{G}$ with the vertices x and y being neighbors, whenever $d(x, y) \in \mathcal{D}_{\mathcal{S}}$. The shadow distance graph (sdG) $\mathcal{D}_{\mathcal{S}\mathcal{D}}(\mathcal{G}, \mathcal{D}_{\mathcal{S}})$ is created from $\mathcal{G}$ under mentioned constraints (Kumar and Murali [8]):

(i) There having 2 copies of $\mathcal{G}$, say $\mathcal{G}$ and $\mathcal{G}'$.

(ii) If $x \in \mathcal{V}(\mathcal{G})$, then $x' \in \mathcal{V}(\mathcal{G}')$ is used to represent the matching vertex.

(iii) The vertex set is $\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{G}')$.

(iv) The edge set is $\mathcal{E}(\mathcal{G}) \cup \mathcal{E}(\mathcal{G}') \cup \mathcal{E}_{\mathcal{D}\mathcal{S}}$ where $\mathcal{E}_{\mathcal{D}\mathcal{S}}$ is the set of all edges between two unique vertices $x \in \mathcal{V}(\mathcal{G})$ and $x' \in \mathcal{V}(\mathcal{G}')$ that fulfill the constraint $d(u, v) \in \mathcal{D}_{\mathcal{S}}$.

2. Main Results

The sdG associated with the path P_n, cycle C_n and star $K_{1,n}$ serves as the foundation for our results.

Theorem 2.1. *For $n \geq 3$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{P_n,\{1\}\}) = \begin{cases} n+1, & n \equiv 1 \pmod 3, \\ 3\lceil \frac{n}{3} \rceil, & n \equiv 0,2 \pmod 3. \end{cases}$*

Proof. Consider there are two instances of P_n, one of which is P_n directly and the other represented by P'_n. Let the vertices of P_n be $x_1,x_2,\dots,x_n$ and let the vertices of P'_n be $x'_1,x'_2,\dots,x'_n$. Let the edges of the first copy of P_n be $s_1,s_2,\dots,s_{n-1}$ and the edges of the second copy P'_n are $s'_1,s'_2,\dots,s'_{n-1}$, where $s_i = (x_i,x_{i+1})$, $s'_i = (x'_i,x'_{i+1})$ for $i = 1,2,\dots,n-1$. Let $\mathcal{G} = (\mathcal{D}_{\mathcal{SD}}\{P_n,\{1\}\})$. Then $\mathcal{E}(\mathcal{G}) = \{s_i\} \cup \{s'_i\} \cup \{s_{(j),(j+1)'}\} \cup \{s_{(k-1)',(k)}\}$ where $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, $2 \leq k \leq n$.

Let ψ be a γ_R function with $\psi = (X_0,X_1,X_2)$, by definition with each $u \in X_0$ will be adjacent to atleast one vertex $v \in X_2$.

Contemplate the following two cases:

Case I: $n \equiv 0,2 \pmod 3$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{3a-1}\}$, $1 \leq a \leq \lceil \frac{n}{3} \rceil$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 3\lceil \frac{n}{3} \rceil$.

Case II: $n \equiv 1 \pmod 3$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{3b-1}\}$, $1 \leq b \leq \lceil \frac{n}{3} \rceil - 1$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = n+1$. □

Theorem 2.2. *For $n \geq 3$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{P_n,\{2\}\}) = \begin{cases} \frac{4n}{5}, & n \equiv 0 \pmod 5, \\ \left(n - \lfloor \frac{n}{5} \rfloor\right) + 1, & n \equiv 1 \pmod 5, \\ 4\lceil \frac{n}{5} \rceil, & n \equiv 2,3,4 \pmod 5. \end{cases}$*

Proof. Consider there are two instances of P_n, one of which is P_n directly and the other represented by P'_n. Let the vertices of P_n be $x_1,x_2,\dots,x_n$ and let the vertices of P'_n be $x'_1,x'_2,\dots,x'_n$. Let the edges of the first copy of P_n be $s_1,s_2,\dots,s_{n-1}$ and the edges of the second copy P'_n are $s'_1,s'_2,\dots,s'_{n-1}$, where $s_i = (x_i,x_{i+1})$, $s'_i = (x'_i,x'_{i+1})$ for $i = 1,2,\dots,n-1$. Let $\mathcal{G} = (\mathcal{D}_{\mathcal{SD}}\{P_n,\{2\}\})$. Then $\mathcal{E}(\mathcal{G}) = \{s_i\} \cup \{s'_i\} \cup \{s_{(j),(j+2)'}\} \cup \{s_{(k-2)',(k)}\}$ where $1 \leq i \leq n-1$, $1 \leq j \leq n-2$, $3 \leq k \leq n$.

Let ψ be a γ_R function with $\psi = (X_0,X_1,X_2)$, by definition with each $u \in X_0$ will be adjacent to atleast one vertex $v \in X_2$.

Let $n \geq 8$, contemplate the following cases:

Case I: $n \equiv 0 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5a-2}\} \cup \{v'_{5a-2}\}$, $1 \leq a \leq \frac{n}{5}$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = \frac{4n}{5}$.

Case II: $n \equiv 1 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5b-2}\} \cup \{v'_{5b-2}\}$, $1 \leq b \leq \lfloor \frac{n}{5} \rfloor$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = (n - \lfloor \frac{n}{5} \rfloor) + 1$.

Case III: $n \equiv 2 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5c-2}\} \cup \{v'_{5c-2}\}$, $1 \leq c \leq \lfloor \frac{n}{5} \rfloor$ with $f\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 4\lceil \frac{n}{5} \rceil$.

Case IV: $n \equiv 3,4 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5d-2}\} \cup \{v'_{5d-2}\}$, $1 \leq d \leq \lceil \frac{n}{5} \rceil$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 4\lceil \frac{n}{5} \rceil$. □

Theorem 2.3. *For $n \geq 3$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{C_n,\{1\}\}) = \begin{cases} 3\lceil \frac{n}{3} \rceil, & n \equiv 0,2 \pmod 3, \\ n+1, & n \equiv 1 \pmod 3. \end{cases}$*

Proof. Consider there are two copies of C_n, one of which is C_n itself and the other indicated by C'_n. Let the vertices of C_n and C'_n be represented by $x_1, x_2, \dots, x_n$ and $x'_1, x'_2, \dots, x'_n$, respectively. Let the first copy C_n of edges be $s_1, s_2, \dots, s_n$. The edges of the second copy C'_n of are defined as $s'_1, s'_2, \dots, s'_n$, where $s_i = (x_i, x_{i+1})$ and $s'_i = (x'_i, x'_{i+1})$ for $i = 1, 2, \dots, n$, where calculation is under modulo n.

Let $\mathcal{G} = \mathcal{D}_{\mathcal{SD}}\{C_n,\{1\}\}$.

Let ψ be a γ_R function with $\psi = (X_0, X_1, X_2)$, by definition with each $u \in X_0$ will be adjacent to atleast one vertex $v \in X_2$.

Let $n \geq 5$, contemplate the following two cases:

Case I: $n \equiv 0,2 \pmod 3$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{3a-2}\}$, $1 \leq a \leq \lceil \frac{n}{3} \rceil$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 3\lceil \frac{n}{3} \rceil$.

Case II: $n \equiv 1 \pmod 3$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{3b-2}\}$, $1 \leq b \leq \lceil \frac{n}{3} \rceil - 1$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G})) = n+1$. □

Theorem 2.4. *For $n \geq 4$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{C_n,\{2\}\}) = \begin{cases} \frac{4n}{5}, & n \equiv 0 \pmod 5, \\ \left(n - \lfloor \frac{n}{5} \rfloor\right) + 1, & n \equiv 1 \pmod 5, \\ 4\lceil \frac{n}{5} \rceil, & n \equiv 2,3,4 \pmod 5. \end{cases}$*

Proof. Consider there are two copies of C_n, one of which is C_n itself and the other indicated by C'_n. Let the vertices of C_n and C'_n be represented by $x_1, x_2, \dots, x_n$ and $x'_1, x'_2, \dots, x'_n$, respectively. Let the first copy C_n of edges be $s_1, s_2, \dots, s_n$. The edges of the second copy C'_n of are defined as $s'_1, s'_2, \dots, s'_n$, where $s_i = (x_i, x_{i+1})$ and $s'_i = (x'_i, x'_{i+1})$ for $i = 1, 2, \dots, n$, where calculation is under modulo n.

Let $\mathcal{G} = \mathcal{D}_{\mathcal{SD}}\{C_n,\{2\}\}$.

Let ψ be a γ_R function with $\psi = (X_0, X_1, X_2)$, by definition with each $u \in X_0$ will be adjacent to atleast one vertex $v \in X_2$.

Let $n \geq 8$, contemplate the following cases:

Case I: $n \equiv 0 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5a-4}\} \cup \{v'_{5a-4}\}$, $1 \leq a \leq \frac{n}{5}$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = \frac{4n}{5}$.

Case II: $n \equiv 1 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5b-4}\} \cup \{v'_{5b-4}\}$, $1 \leq b \leq \lfloor \frac{n}{5} \rfloor$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = (n - \lfloor \frac{n}{5} \rfloor) + 1$.

Case III: $n \equiv 2 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5c-4}\} \cup \{v'_{5c-4}\}$, $1 \leq c \leq \lfloor \frac{n}{5} \rfloor$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 4\lceil \frac{n}{5} \rceil$.

Case IV: $n \equiv 3,4 \pmod 5$, there exist a minimal Roman dominating set $\mathcal{D} = \{v_{5d-4}\} \cup \{v'_{5d-4}\}$, $1 \leq d \leq \lceil \frac{n}{5} \rceil$ with $\psi(v_i) = 2$, $v_i \in \mathcal{D}$. Hence $\gamma_R(\mathcal{G}) = 4\lceil \frac{n}{5} \rceil$. □

Theorem 2.5. *For $n \geq 3$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{K_{1,n},\{1\}\}) = 3$.*

Proof. We skip since it is simple to prove. □

Theorem 2.6. *For $n \geq 3$, $\gamma_R(\mathcal{D}_{\mathcal{SD}}\{K_{1,n},\{2\}\}) = 4$.*

Proof. The proof is obvious. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] H. A. Ahangar, A. Bahremandpour, S. M. Sheikholeslami, N. D. Soner, Z. Tahmasbzadehbaee and L. Volkmann, Maximal Roman domination numbers in graphs, *Utilitas Mathematica* **103** (2017), 245 – 258.

[2] H. A. Ahangar, T. W. Haynes and J. C. Valenzuela-Tripodoro, Mixed Roman domination in graphs, *Bulletin of the Malaysian Mathematical Sciences Society* **40** (2017), 1443 – 1454, DOI: 10.1007/s40840-015-0141-1.

[3] R. A. Beeler, T. W. Haynes and S. T. Hedetniemi, Double Roman domination, *Discrete Applied Mathematics* **211** (2016), 23 – 29, DOI: 10.1016/j.dam.2016.03.017.

[4] M. Chellali, T. W. Haynes, S. T. Hedetniemi and A. A. McRae, Roman {2}-domination, *Discrete Applied Mathematics* **204** (2016), 22 – 28, DOI: 10.1016/j.dam.2015.11.013.

[5] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs, *Discrete Mathematics* **278**(1-3) (2004), 11 – 12, DOI: 10.1016/j.disc.2003.06.004.

[6] M. A. Henning and S. T. Hedetniemi, Defending the Roman empire — A new strategy, *Discrete Mathematics* **266**(1-3) (2003), 239 – 251, DOI: 10.1016/S0012-365X(02)00811-7.

[7] U. V. Kumar and R. Murali, Edge domination in shadow distance graphs, *International Journal of Mathematics and its Applications* **4**(2D) (2016), 125 – 130, URL: http://ijmaa.in/index.php/ijmaa/article/view/1075/1059.

[8] U. V. Kumar and R. Murali, s-Path domination in shadow distance graphs, *Journal of Harmonized Research in Applied Sciences* **6**(3) (2018), 194 – 199.

[9] A. Mekala, U. V. C. Kumar and R. Murali, Bi-domination in brick product graphs, *Journal of Algebraic Statistics* **13(2)** (2022), 1954 – 1960, URL: https://publishoa.com/index.php/journal/article/view/376/347.

[10] C. S. ReVelle and K. E. Rosing, Defendens imperium Romanum: A classical problem in military strategy, *The American Mathematical Monthly* **107**(7) (2000), 585 – 595, DOI: 10.1080/00029890.2000.12005243.

[11] I. Stewart, Defend the Roman empire!, *Scientific American* **281**(6) (1999), 136 – 138, URL: https://www.jstor.org/stable/26058532.