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Research Article

Star Rainbow Coloring in Graphs

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Abstract. In this paper, we define a new coloring parameter called the star rainbow connection number of a connected graph G, denoted by $st_{rc}(G)$ and determine this parameter for some standard graphs.

Keywords. Rainbow connection number, Edge coloring, Rainbow path, Star rainbow connection number

Mathematics Subject Classification (2020). 26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05

1. Introduction

The graphs that we consider in this paper are non-trivial, finite, undirected and connected. Rainbow coloring in a graph was introduced by Chartrand *et al*. [2]. For the results related to rainbow coloring, we refer [2].

A connected graph G is said to be star rainbow colored if every path on four vertices in it is rainbow colored. The minimum number of colors necessary to color G such that it exhibits a star rainbow coloring is called the star rainbow connection number of G, denoted by $st_{rc}(G)$.

To illustrate the concept of star rainbow coloring. Let us consider the Petersen graph G in Figure 1, where a star rainbow 5-coloring is shown.

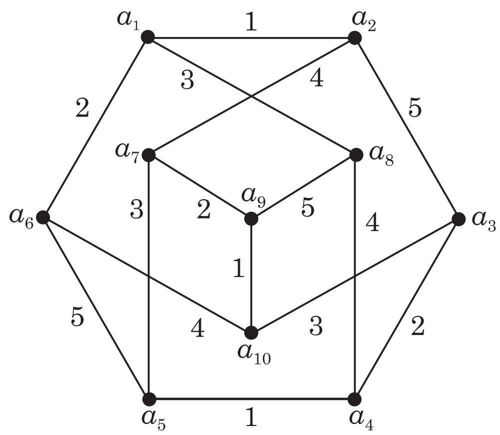

Figure 1. A star rainbow 5-coloring of the Peterson graph G

From this allocation of colors we have $st_{rc}(G) \le 5$. To show that $st_{rc}(G) \ge 5$. Let us assume that $st_{rc}(G) < 5$, i.e., let $st_{rc}(G) = 4$. Let us color the edges of G with these four colors so that G is star rainbow colored. Since the Petersen graph has a cycle of length six, minimum three colors are to be assigned to the edges of cycle of G for rainbow coloring. Now, if the remaining one color along with these colors are allocated to the remaining edges of G, then it can be easily verified that, adjacent edges of some of the paths on four vertices (such as the paths containing the vertices $\{a_1, a_2, a_7, a_5\}$, $\{a_1, a_8, a_9, a_{10}\}$, etc.) have the same color. This leads to a contradiction.
Hence, $st_{rc}(G) \ge 5$.
Therefore, $st_{rc}(G) = 5$.

Further, for the complete graph K_n on n vertices, $st_{rc}(K_n) = n_{C_2}$, also for the complete bipartite graph $K_{m,n}$, $st_{rc}(K_{m,n}) = mn$.

2. Main Results

In this section, we determine the parameter $st_{rc}(G)$ for some well-known graphs like the path graph, cycle graph and wheel graph.

We begin with the path P_n on n vertices.

Theorem 1. $st_{rc}(P_n) = 3$, *for* $n \ge 4$.

Proof. Let $G = P_n$, $V(G) = \{a_1, a_2, a_3, \cdots a_n\}$ and $E(G) = \{e_x : e_x = (a_x, a_{x+1})$ for $1 \le x \le n-1\}$.

Case 1: For $k \ge 1$, $n = 3k+1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le \frac{n-1}{3}$,
  $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$.

Therefore, $st_{rc}(G) \le 3$. To show that $st_{rc}(G) \ge 3$. Let us assume that $st_{rc}(G) < 3$, i.e., let $st_{rc}(G) = 2$.
Let us color the edges of G with these two colors so that G is star rainbow colored. In this case it is easy to verify that each path on four vertices contains at least two edges with the same color (like the paths connecting the vertices: $\{a_i : 1 \le i \le 4\}, \{a_j, 2 \le j \le 5\}$, etc.). This leads to a contradiction.
Hence, $st_{rc}(G) \ge 3$.
Therefore, $st_{rc}(G) = 3$.

Case 2: For $k \ge 1$, $n = 3k+2$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le \lfloor \frac{n-2}{3} \rfloor$,
  $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$ and $c(a_{n-1}, a_n) = 1$.

Therefore $st_{rc}(G) \le 3$. To show that $st_{rc}(G) \ge 3$. Let us assume that $st_{rc}(G) < 3$, i.e., let $st_{rc}(G) = 2$.
Let us color the edges of G with these two colors so that G is star rainbow colored. In this case it is easy to verify that each path on four vertices contains at least two edges with the same color (like the paths connecting the vertices: $\{a_i : 1 \le i \le 4\}$, $\{a_j, 2 \le j \le 5\}$, etc.). This leads to a contradiction.
Hence, $st_{rc}(G) \ge 3$.
Therefore, $st_{rc}(G) = 3$.

Case 3: For $k \ge 1$, $n = 3k+3$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le \frac{n-3}{3}$,
  $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$ and $c(a_{n-2}, a_{n-1}) = 1$, $c(a_{n-1}, a_n) = 2$.

Therefore, $st_{rc}(G) \le 3$. To show that $st_{rc}(G) \ge 3$. Let us assume that $st_{rc}(G) < 3$, i.e., let $st_{rc}(G) = 2$.
Let us color the edges of G with these two colors so that G is star rainbow colored. It is easy to verify that each path on four vertices contains at least two edges with the same color (like the paths containing the vertices: $\{a_i : 1 \le i \le 4\}$, $\{a_j, 2 \le j \le 5\}$, etc.). This leads to a contradiction.

Hence, $st_{rc}(G) \geq 3$.

Therefore, $st_{rc}(G) = 3$.

Since the inclusion of an edge to the end vertices of a path P_n will give the cycle C_n on n vertices, if we consider the vertex set and edge set of C_n as $V = \{a_1, a_2, a_3, \cdots a_n, a_{n+1} = a_1\}$ and $E = \{(a_x, a_{x+1})$ for $1 \leq x \leq n\}$ respectively, a star rainbow coloring for C_n can be immediately obtained from theorem 2.1 as follows:

(1) For $n = 3k+3$, $k \geq 1$, we consider the coloring $c : E(G) \to \{1,2,3\}$ as:

- For $1 \leq x \leq \frac{n}{3}$, $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$.

(2) For $n = 3k+4$, $k \geq 1$, we consider the coloring $c : E(G) \to \{1,2,3,4\}$ as:

- For $1 \leq x \leq \lfloor \frac{n}{3} \rfloor$, $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$ and $c(a_n, a_1) = 4$.

(3) For $n = 3k+5$, $k \geq 1$, we consider the coloring $c : E(C_n) \to \{1,2,3,4\}$ as:

- For $1 \leq x \leq \lfloor \frac{n}{3} \rfloor - 1$, $c(a_{3x-2}, a_{3x-1}) = 1$, $c(a_{3x-1}, a_{3x}) = 2$, $c(a_{3x}, a_{3x+1}) = 3$.
- For $x = \lfloor \frac{n}{3} \rfloor$, $c(a_{3x-1}, a_{3x}) = 1$, $c(a_{3x}, a_{3x+1}) = 2$, $c(a_{3x+1}, a_{3x+2}) = 3$ and $c(a_n, a_1) = c(a_{n-4}, a_{n-3}) = 4$. □

We state the above observations in Theorem 2.

Theorem 2. *For $k \geq 1$, $st_{rc}(C_n) = \begin{cases} 3 & \text{for } n = 3k+3, \\ 4 & \text{for } n = 3k+4 \text{ and } 3k+5. \end{cases}$*

The Bistar graph $B_{m,n}$ is the graph obtained by joining the central vertices of the two star graphs $K_{1,m}$ and $K_{1,n}$ by an edge.

By the graphs:

- $K_{1,n}(P_k)$, where $n \geq 2$ and $k \geq 3$, we mean the graph obtained from $K_{1,n}$ by attaching the end vertex of P_k to the central vertex of $K_{1,n}$.
- $K_{1,m}(P_k)K_{1,n}$, where $m, n \geq 2$ and $k \geq 3$, we mean the graph obtained from joining the central vertices of the two star graphs $K_{1,m}$ and $K_{1,n}$ by the path P_k.

Also, we denote the class of graphs containing the graphs $K_{1,n}(P_k)$, $K_{1,m}(P_k)K_{1,n}$ and the path P_n as $\{G_{KP}\}$.

In our next result, we characterize the graphs for which $st_{rc}(G) = 3$.

Theorem 3. *Let G be a connected star rainbow colorable graph of order at least 4. Then, $st_{rc}(G) = 3$ if and only if G is one of the following:*

(1) *the bistar $B_{m,n}$, $m, n \geq 2$,*

(2) *the cycle of length* $3k$, C_{3k}, $k \geq 2$,

(3) *the graphs in the class* $\{G_{KP}\}$.

Proof. *Case* 1: G be the bistar graph $B_{m,n}$, $m,n \geq 2$.

Let a and b be the central vertices of the star graphs $K_{1,m}$ and $K_{1,n}$, respectively.
Let $V(G) = V_1 \cup V_2$, where

- $V_1 = V(K_{1,m}) = \{a, a_1, a_2, \cdots, a_m\}$,
- $V_2 = V(K_{1,n}) = \{b, b_1, b_2, \cdots, b_n\}$.

Let $E(G) = \cup_{i=1}^{3} E_i$, where,

- $E_1 = E(K_{1,m}) = \{(a, a_x), 1 \leq x \leq m\}$,
- $E_2 = \{(a, b)\}$,
- $E_3 = E(K_{1,n}) = \{(b, b_x), 1 \leq x \leq n\}$.

For a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1, 2, 3\}$.

- For $1 \leq x \leq m$, $c(a, a_x) = 1$,
- $c(a, b) = 2$,
- For $1 \leq x \leq n$, $c(b, b_x) = 3$.

Therefore, $st_{rc}(G) \leq 3$. To show that $st_{rc}(G) \geq 3$. Let us assume that $st_{rc}(G) < 3$, i.e., let $st_{rc}(G) = 2$.

Let us color the edges of G with these two colors so that G is star rainbow colored. As G contains two star graphs $K_{1,m}$ and $K_{1,n}$, let us allocate any one color to the edges of $K_{1,m}$. Now, if the remaining one color is allocated to the remaining edges of G, it can be easily to verified that, every path on four vertices (like the paths containing the vertices: $\{a_x - a - b - b_y\}$ for $1 \leq x \leq m, 1 \leq y \leq n$) contains at least two edges with the same color. This leads to a contradiction.
Hence, $st_{rc}(G) \geq 3$.
Therefore, $st_{rc}(G) = 3$.

Case 2: The proof of this case follows from Theorem 2 and the converse part is obvious.

Case 3: G be the one of the graph of the class $\{G_{kP}\}$.

Subcase 1: $G = K_{1,n}(P_k)$ for $n \geq 2$, $k \geq 3$.
Let $V(G) = V_1 \cup V_2$, where

- $V_1 = V(K_{1,n}) = \{b_1, a_1, a_2, \cdots, a_n\}$,
- $V_2 = V(P_k) = \{b_1, b_2, \cdots, b_k\}$.

Let $E(G) = \cup_{i=1}^{2} E_i$, where

- $E_1 = E(K_{1,n}) = \{(b_1, a_x), 1 \le x \le n\}$,
- $E_2 = E(P_k) = \{(b_x, b_{x+1}), 1 \le x \le k-1\}$.

Depending on the value of k of P_k, we have the following subcases.

Sub-subcase 1: Let $k = 3i$ for $i \ge 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

For $i = 1$, we have the coloring as follows:

- For $1 \le x \le n$, $c(b_1, a_x) = 3$,
- $c(b_1, b_2) = 1$,
- $c(b_2, b_3) = 2$.

For $i \ge 2$, we have the following coloring.

- For $1 \le x \le n$, $c(b_1, a_x) = 3$,
- For $1 \le x \le \frac{k-3}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$ and $c(b_{k-2}, b_{k-1}) = 1$, $c(b_{k-1}, b_k) = 2$.

Sub-subcase 2: Let $k = 3i+1$ for $i \ge 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le n$, $c(b_1, a_x) = 3$,
- For $1 \le x \le \frac{k-1}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$.

Sub-subcase 3: Let $k = 3i+2$ for $i \ge 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le n$, $c(b_1, a_x) = 3$,
- For $1 \le x \le \frac{k-2}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$ and $c(b_{k-1}, b_k) = 1$.

Therefore, $st_{rc}(G) \le 3$. To show that $st_{rc}(G) \ge 3$. Let us assume that $st_{rc}(G) < 3$, i.e., let $st_{rc}(G) = 2$.

Let us color the edges of G with these two colors so that G is star rainbow colored. As G contains a star graph $K_{1,n}$, let us allocate any one color to the edges of $K_{1,n}$. Now, if the remaining one color is allocated to the remaining edges of G, it can be easily to verified that, every path on four vertices (like the paths containing the vertices: $\{(a_1, b_i) : 1 \le i \le 3\}$, $\{b_i : 1 \le i \le 4\}$) are such that at least two of its edges have the same color. This leads to a contradiction.
Hence, $st_{rc}(G) \ge 3$.

Therefore, $st_{rc}(G) = 3$.

Subcase 2: $G = K_{1,m}(P_k)K_{1,n}$ for $m, n \geq 2$, $k \geq 3$.
Let $V(G) = V_1 \cup V_2 \cup V_3$ where

- $V_1 = V(K_{1,m}) = \{b_1, a_1, a_2, \cdots, a_m\}$,
- $V_2 = V(P_k) = \{b_1, b_2, \cdots, b_k\}$,
- $V_3 = V(K_{1,n}) = \{b_k, c_1, c_2, \cdots, c_n\}$.

Let $E(G) = \cup_{i=1}^{3} E_i$, where

- $E_1 = E(K_{1,m}) = \{(b_1, a_x), 1 \leq x \leq m\}$,
- $E_2 = E(P_k) = \{(b_x, b_{x+1}), 1 \leq x \leq k-1\}$,
- $E_3 = E(K_{1,n}) = \{(b_k, c_x), 1 \leq x \leq n\}$.

Depending on the value of k of P_k, we have the following subcases.

Sub-subcase 1: Let $k = 3i$ for $i \geq 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.
For $i = 1$, we have the coloring as follows:

- For $1 \leq x \leq m$, $c(b_1, a_x) = 3$,
- For $1 \leq x \leq n$, $c(b_3, c_x) = 3$,
- $c(b_1, b_2) = 1$ and $c(b_2, b_3) = 2$.

For $i \geq 2$,we have the coloring as follows:

- For $1 \leq x \leq m$, $c(b_1, a_x) = 3$,
- For $1 \leq x \leq n$, $c(b_k, c_x) = 3$,
- For $1 \leq x \leq \frac{k-3}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$ and $c(b_{k-2}, b_{k-1}) = 1$, $c(b_{k-1}, b_k) = 2$.

Sub-subcase 2: Let $k = 3i+1$ for $i \geq 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \leq x \leq m$, $c(b_1, a_x) = 3$,
- For $1 \leq x \leq n$, $c(b_k, c_x) = 3$,
- For $1 \leq x \leq \frac{k-1}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$.

Sub-subcase 3: Let $k = 3i+2$ for $i \geq 1$.
In this case, for a star rainbow coloring, we consider the following coloring $c : E(G) \to \{1,2,3\}$.

- For $1 \le x \le m$, $c(b_1, a_x) = 3$,
- For $1 \le x \le n$, $c(b_k, c_x) = 2$,
- For $1 \le x \le \frac{k-2}{3}$, $c(b_{3x-2}, b_{3x-1}) = 1$, $c(b_{3x-1}, b_{3x}) = 2$, $c(b_{3x}, b_{3x+1}) = 3$ and $c(b_{k-1}, b_k) = 1$.

Therefore, $st_{rc}(G) \le 3$. To show that $st_{rc}(G) \ge 3$. Let us assume that $st_{rc}(G) < 3$, i.e., $st_{rc}(G) = 2$.

Let us color the edges of G with these two colors so that G star rainbow colored. As G contains two star graphs $K_{1,m}$ and $K_{1,n}$, let us allocate any one color to the edges of $K_{1,m}$. Now, if the remaining one color is allocated to the remaining edges of G, it can be easily to verified that, every path on four vertices (like the paths containing the vertices: $\{a_x, b_1, b_2, b_3\}$, for $1 \le x \le m$ etc.) are such that at least two of its edges have the same color. This leads to a contradiction.

Hence, $st_{rc}(G) \ge 3$.

Therefore, $st_{rc}(G) = 3$.

Subcase 3: $G = P_n$ for $n \ge 3$.

The proof follows by Theorem 1.

The converse part of *Case* 3 is obvious.

Consider any tree that is non-isomorphic to the class of graphs $\{G_{KP}\}$. Let P be a path of maximum length in it. Then, if we color P with the colors 1,2 and 3 (as in Theorem 1), it is easy to verify that for a star rainbow coloring of this tree, the total number of colors required will be between one and $\lfloor \frac{n}{2} \rfloor + 1$. Also, from Theorem 3, the graphs that are isomorphic to the class of graphs $\{G_{KP}\}$ have star rainbow connection number $= 3$. □

These observations lead to the following simple result, which we state as our next theorem. It should be noted that both upper and lower bounds mentioned in this theorem are sharp.

Theorem 4. *For $n \ge 4$, $3 \le st_{rc}(T_n) \le \lfloor \frac{n}{2} \rfloor + 1$.*

If G contains a cycle, then it is easy to verify that at least one edge in a cycle in G should be colored with a color 4 or higher for a proper star rainbow coloring. We state this observation in our next result.

Theorem 5. *Let G be a connected star rainbow colorable graph of order at least 4. Then $st_{rc}(G) \ge 3$ if G contains a cycle.*

Another well-known class of graphs constructed from cycles are the wheels. For $n \ge 3$, the wheel $W_{1,n}$ is defined as the graph $C_n + K_1$, the join of C_n and K_1, constructed by joining a new vertex to every vertex of C_n.

In our next result, we determine $st_{rc}(W_{1,n})$.

Theorem 6. *For $k \geq 2$,*

$$st_{rc}(W_{1,n}) = \begin{cases} 2n & \text{for } n = 3,4,5, \\ n+3 & \text{for } n = 3k, \\ n+4 & \text{for } n = 3k+1 \text{ and } 3k+2. \end{cases}$$

Proof. Suppose that $W_{1,n}$ consists of an n-cycle $C_n = \{a_1, a_2, a_3, \cdots a_n, a_{n+1} = a_1\}$ and another vertex b joined to every vertex of C_n.

Let $G = W_{1,n}$ and let $E(G) = \cup_{i=1}^{2} E_i$, where

- $E_1 = \{e_x : e_x = (a_x, a_{x+1}), 1 \leq x \leq n\}$,
- $E_2 = \{e'_x : e'_x = (b, a_x), 1 \leq x \leq n\}$.

Case 1: $n = 3,4,5$.
In this case, for a star rainbow coloring, we consider the following $(2n)$-coloring $c : E(W_{1,n}) \to \{1,2,3,\cdots 2n\}$.

- For $1 \leq x \leq n$, $c(b, a_x) = x$,
- For $1 \leq x \leq n$, $c(a_x, a_{x+1}) = n + x$.

Therefore, $st_{rc}(G) \leq 2n$. To show that $st_{rc}(G) \geq 2n$. Let us assume that $st_{rc}(G) < 2n$, i.e., let $st_{rc}(G) = 2n-1$. Let us color the edges of G with these $2n-1$ colors so that G is star rainbow colored.

Since G has n edges connecting (b, a_x) for $1 \leq x \leq n$, we allocate n colors to these edges. Now, if the remaining $n-1$ colors are allocated to the edges of C_n in G, it can be easily to verified that, there exists at least one path on four vertices (such as the paths containing the vertices: $\{a_n, a_1, a_2, b\}$) are such that two of its edges have the same color. This leads to a contradiction.
Hence, $st_{rc}(G) \geq 2n$.
Therefore, $st_{rc}(G) = 2n$.

Case 2: $n = 3k$, $k \geq 2$.
In this case, for a star rainbow coloring, we consider the following $(n+3)$-coloring $c : E(W_{1,n}) \to \{1,2,3,\cdots n+3\}$.

- For $1 \leq x \leq n$, $c(b, a_x) = x$,
- For $1 \leq x \leq \frac{n}{3}$, $c(a_{3x-2}, a_{3x-1}) = n+1$, $c(a_{3x-1}, a_{3x}) = n+2$, $c(a_{3x}, a_{3x+1}) = n+3$.

Therefore, $st_{rc}(G) \leq n+3$. To show that $st_{rc}(G) \geq n+3$. Let us assume that $st_{rc}(G) < n+3$, i.e., let $st_{rc}(G) = n+2$. Let us color the edges of G with these $n+2$ colors so that G is star rainbow colored.

Since G has n edges connecting (b, a_x) for $1 \leq x \leq n$, we allocate n colors to these edges. Now, if the remaining two colors are assigned to the edges of C_n in G, it is easy to verify that, some paths on four vertices (such as the paths containing the vertices: $\{a_i : 1 \le i \le 4\}$, $\{a_j, 2 \le j \le 5\}$, etc.). are such that at least two of its edges have the same color. This leads to a contradiction.

Hence, $st_{rc}(G) \ge n+3$.

Therefore, $st_{rc}(G) = n+3$.

Case 3: $n = 3k+1$, $k \ge 2$.
In this case, for a star rainbow coloring, we consider the following $(n+4)$-coloring $c : E(W_{1,n}) \to \{1,2,3,\cdots n+4\}$.

- For $1 \le x \le n$, $c(b,a_x) = x$,
- For $1 \le x \le \lfloor \frac{n}{3} \rfloor$, $c(a_{3x-2},a_{3x-1}) = n+1$, $c(a_{3x-1},a_{3x}) = n+2$, $c(a_{3x},a_{3x+1}) = n+3$ and $c(a_n,a_1) = n+4$.

Therefore, $st_{rc}(G) \le n+4$. To show that $st_{rc}(G) \ge n+4$. Let us assume that $st_{rc}(G) < n+4$, i.e., let $st_{rc}(G) = n+3$. Let us color the edges of G with these $n+3$ colors so that G is star rainbow colored.

Since G has n edges connecting (b,a_x) for $1 \le x \le n$, we assign n colors to these edges. Now, if the remaining three colors allocated to the edges of C_n in G, it is easy to verify that, some paths on four vertices (such as the paths containing the vertices: $\{a_{n-1},a_n,a_1,a_2\}$, $\{a_n,a_1,a_2,b\}, etc.$) are such that at least two of its edges have the same color. This leads to a contradiction.

Hence, $st_{rc}(G) \ge n+4$.

Therefore, $st_{rc}(G) = n+4$.

Case 4: $n = 3k+2$, $k \ge 2$.
In this case, for a star rainbow coloring, we consider the following $(n+4)$-coloring $c : E(W_{1,n}) \to \{1,2,3,\cdots n+4\}$.

- For $1 \le x \le n$, $c(b,a_x) = x$,
- For $1 \le x \le \lfloor \frac{n}{3} \rfloor - 1$,
  $c(a_{3x-2},a_{3x-1}) = n+1$, $c(a_{3x-1},a_{3x}) = n+2$, $c(a_{3x},a_{3x+1}) = n+3$.
- For $x = \lfloor \frac{n}{3} \rfloor$,
  $c(a_{3x-1},a_{3x}) = n+1$, $c(a_{3x},a_{3x+1}) = n+2$, $c(a_{3x+1},a_{3x+2}) = n+3$ and $c(a_n,a_1) = c(a_{n-4},a_{n-3}) = n+4$.

Therefore, $st_{rc}(G) \le n+4$. To show that $st_{rc}(G) \ge n+4$. Let us assume that $st_{rc}(G) < n+4$, i.e., let $st_{rc}(G) = n+3$. Let us color the edges of G with these $n+3$ colors so that G is star rainbow colored.

Since G has n edges connecting (b,a_x) for $1 \le x \le n$, we allocate n colors to these edges. Now, if the remaining three colors are assigned to the edges of C_n in G, it is easy to verify that, some paths on four vertices (such as the paths containing the vertices: $\{a_{n-1},a_n,a_1,a_2\}$,

$\{a_n, a_1, a_2, a_3\}, etc.$) are such that at least two of its edges have the same color. This leads to a contradiction.

Hence, $st_{rc}(G) \geq n+4$.

Therefore, $st_{rc}(G) = n+4$. □

If G is either a complete graph or the wheel graph $W_{1,5}$, we have the following result.

Theorem 7. *Let G be a connected star rainbow colorable graph of order $n \geq 4$. Then $st_{rc}(G) = \frac{n(n-1)}{2}$, $st_{rc}(G) = n_{C_2}$ if and only if G is either complete graph or $W_{1,5}$.*

Proof. If G is a complete graph or $W_{1,5}$, then the coloring i for $1 \leq i \leq \frac{n(n-1)}{2}$ to each edge of G gives $st_{rc}(G) \leq \frac{n(n-1)}{2}$.

Conversely, if G is not a complete graph or $W_{1,5}$, then G contains two non-adjacent vertices. Clearly, a star rainbow coloring of this graph requires at most $\leq \frac{n(n-1)}{2} - 1$ colors.

Hence, $st_{rc}(G) \geq \frac{n(n-1)}{2}$, $st_{rc}(G) = n_{C_2}$.

Therefore, $st_{rc}(G) = \frac{n(n-1)}{2}$, $st_{rc}(G) = n_{C_2}$. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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