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Research Article

Some Fixed Point Theorems for Generalized α-Admissible $\mathcal{Z}$-Contraction via Simulation Function

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Abstract. In this paper, we prove some fixed point theorems in metric-like space by using generalized α-admissible mapping embedded in the simulation function. Our results generalize and extend several known results on literature.

Keywords. Metric-like space, Fixed point, Generalized α-admissible mapping, Simulation function, $\mathcal{Z}$-contractions

Mathematics Subject Classification (2020). 54H25, 47H10

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1. Introduction

Amini-Harandi [2] considered the concept of metric-like spaces and established some fixed point results in the class of metric-like space. Very recently, several fixed point results on metric-like space have been provided, for example see Alsamir *et al.* [1], Aydi *et al.* [4,5], and Mishra *et al.* [11]. In 2012, Samet *et al.* [16] introduced the concept of α-contraction and α-admissible mappings and proved various fixed point theorem in complete metric spaces. For other results using different concepts of α-admissible mappings, see Aydi *et al.* [5], Cho [6], Dewangan *et al.* [7], Felhi *et al.* [8], Mishra *et al.* [11], and Qawaqneh *et al.* [13]. In 2015, Khojasteh *et al.* [10] introduced the notion of $\mathcal{Z}$-contraction by using a new class of auxiliary functions called simulation functions. Argoubi *et al.* [3] modified the definition [10] and proved some fixed point theorems with nonlinear contractions.

There are many fixed point results in the setting of simulation function (for instance, Cho [6], Dewangan *et al.* [7], Felhi *et al.* [8], Karapınar [9], Mishra *et al.* [11], Rus [12], and Roldán-López-de-Hierro *et al.* [14]). Padcharoen *et al.* [12] introduced the notion of generalized α-admissible $\mathcal{Z}$-contraction and established various fixed point theorems for such mappings in complete metric spaces by using the concepts of Khojasteh *et al.* [10], Rus [15] and Samet [16].

In this paper, we consider some simulation functions to show the existence of fixed points of generalized α-admissible $\mathcal{Z}$-contraction mappings in metric-like spaces. Our results generalize and extend some existing results in the literature. We modify and generalize the results of Padcharoen *et al.* [12], Dewangan *et al.* [7], and Cho [6].

2. Preliminaries

Throughout this article, we assume the symbols $\mathbb{R}$ and $\mathbb{N}$ as a set of real numbers and set of natural number, respectively.

Definition 2.1 ([2])**.** Let X be a non empty set. A function $\sigma : X \times X \to [0,\infty)$ is said to be a metric-like (or a dislocated metric) on X, if for any $x,y,z \in X$ the following conditions hold:

(σ_1): $\sigma(x,y) = 0 \Rightarrow x = y$;

(σ_2): $\sigma(x,y) = \sigma(y,x)$;

(σ_3): $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.

The pair (X,σ) is called a metric-like space. Then a metric-like on X satisfies all conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. Following [2], we have the following topological concepts.

Each metric-like σ on X generates a topology τ_σ on X, whose base is the family of open σ-balls, then for all $x \in X$ and $\epsilon > 0$,

$$B_\sigma(X,\epsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \epsilon\}.$$

Now, let (X,σ) be a metric-like space. A sequence $\{x_n\}$ in the metric-like space (X,σ) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x).$$

Let (X,σ) be metric-like space, and let $T : X \to X$ be a continuous mapping. Then

$$\lim_{n\to\infty} x_n = x \quad \Longrightarrow \quad \lim_{n\to\infty} T(x_n) = T(x).$$

A sequence $\{x_n\}$ is Cauchy in (X,σ), if and only if $\lim_{n,m\to\infty} \sigma(x_m,x_n)$ exists and is finite. Moreover, the metric-like space (X,σ) is called complete, if and only if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n\to+\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m).$$

It is clear that every metric space and partial metric space is a metric-like space but the converse is not true.

Example 2.2. Let $X = \{0,1\}$ and $\sigma(x,y) = \begin{cases} 2, & \text{if } x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$

Then (X,σ) is a metric-like space. It is neither a partial metric space ($\sigma(0,0) \not\leq \sigma(0,1)$) nor a metric space ($\sigma(0,0) = 2 \neq 0$).

The following lemma is useful to prove our results.

Lemma 2.3 ([2]). *Let (X,σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$, where $x \in X$ and $\sigma(x,y) = 0$. Then, for all $y \in X$ we have*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x,y).$$

Definition 2.4 ([10]). A function $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ is called a simulation function if ζ satisfies the following conditions:

(ζ_1): $\zeta(0,0) = 0$.

(ζ_2): $\zeta(t,s) < s - t$, for all $t,s > 0$.

(ζ_3): If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation function by $\mathcal{Z}$.

The following unique fixed point theorem is established by Khojasteh *et al.* [10].

Theorem 2.5. *Let (X,d) be a metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a simulation function ζ, that is*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0, \quad \text{for all } x,y \in X.$$

Then T has a unique fixed point.

It is worth mentioning that the Banach contraction is an example of z-contractions by defining $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ via

$$\zeta(t,s) = \lambda s - t, \quad \text{for all } s,t \in [0,\infty),$$

where $\lambda \in [0,1)$.

Argoubi *et al.* [3] modified Definition 2.4 as follows.

Definition 2.6 ([3]). A simulation function is a function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies the following conditions:

(i) $\zeta(t,s) < s - t$, for all $s,t > 0$.

(ii) If $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

It is clear that any simulation function in the sense of Khojasteh *et al.* [10] (Definition 2.4) is also a simulation function in the sense of Argoubi *et al.* [3] (Definition 2.6). The converse is not true.

Remark 2.7 ([3, 10])**.** It is clear from the definition of simulation function that $\zeta(t,s)<0$, for all $t\geq s>0$. Therefore, if T is a $\mathcal{Z}$-contraction with respect to ζ, then $d(Tx,Ty)<d(x,y)$, for all distinct $x,y\in X$.

Example 2.8 ([3])**.** Define a function $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ by

$$\zeta(t,s)=\begin{cases}1, & \text{if } (s,t)=(0,0),\\ \lambda s-t, & \text{otherwise,}\end{cases}$$

where $\lambda\in(0,1)$. Then ζ is a simulation function in the sense of Argoubi *et al*. [3].

Let Ψ be the family of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(i) ψ is non decreasing,

(ii) there exists $n_0\in\mathbb{N}$ and $a\in(0,1)$ and a convergent series of non-negative terms $\sum\limits_{n=1}^{\infty}v_n$ such that $\psi^{n+1}(t)\leq a\psi^n(t)+v_n$, for $n\geq n_0$ and any $t\in\mathbb{R}^+$.

Lemma 2.9 ([15])**.** *If $\psi\in\Psi$, then the following hold:*

(i) *$(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$, for all $t\in\mathbb{R}^+$,*

(ii) *$\psi(t)<t$, for any $t\in\mathbb{R}^+$,*

(iii) *ψ is continuous at 0,*

(iv) *the series $\sum\limits_{k=1}^{\infty}\psi^k(t)$ converges for any $t\in\mathbb{R}^+$.*

Definition 2.10 ([9])**.** Let T be a self mapping defined on a metric space (X,d). If there exist $\zeta\in\mathcal{Z}$ and $\alpha:X\times X\to[0,\infty)$ such that

$$\zeta(\alpha(x,y)d(Tx,Ty),d(x,y))\geq 0,\quad \text{for all } x,y\in X,$$

then we say that T is an α-admissible $\mathcal{Z}$-contraction with respect to ζ.

3. Main Results

Definition 3.1 ([12])**.** Let (x,σ) be a metric-like space and $T:X\to X$ be a self mapping. If there exist $\zeta\in\mathcal{Z}$ and $\alpha:X\times X\to[0,\infty)$ such that

$$\zeta(\alpha(x,Tx)\alpha(y,Ty)\sigma(Tx,Ty),M(x,y))\geq 0, \tag{3.1}$$

for all distinct $x,y\in X$, where

$$M(x,y)=\max\left\{\sigma(x,y),\frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}, \tag{3.2}$$

then T is called generalized α-admissible $\mathcal{Z}$-contraction with respect to ζ.

Remark 3.2. It is clear from the definition of simulation function that $\zeta(t,s)<0$, for all $t\geq s>0$. Therefore, T is a generalized α-$\mathcal{Z}$-contraction with respect to ζ, then

$$\alpha(x,Tx)\alpha(y,Ty)\sigma(Tx,Ty)<M(x,y),$$

for all distinct $x,y\in X$.

Theorem 3.3. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a generalized α-admissible $\mathcal{Z}$-contraction with respect to a ζ simulation function if there exist $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that*

$$\zeta(\psi(\alpha(x,Tx)\alpha(y,Ty)\sigma(Tx,Ty)),\psi(M(x,y))) \geq 0, \tag{3.3}$$

for all distinct $x,y \in X$, where

$$M(x,y) = \max\left\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}.$$

Assume that,

(i) *T is admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq 1$,*

(iii) *for every sequence $\{x_n\}$ in X such that $\alpha(x_n,Tx_n) \geq 1$, for all $n \in \mathbb{N}\cup\{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x,Tx) \geq 1$,*

(iv) *$\alpha(x,Tx) \geq 1$, for all $x \in Fix(T)$.*

Then T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$.

Proof. By (ii) of this theorem, there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$. Since T is α-admissible, we obtain $\alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1$ implies $\alpha(Tx_1,Tx_2) = \alpha(x_2,x_3) \geq 1$.

By induction, we get

$$\alpha(x_n,x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{3.4}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}\cup\{0\}$, then $x_n = x_{n+1} = Tx_n$ and hence x_n is a fixed point of T. Therefore, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, we get $\sigma(x_n,x_{n+1}) > 0$, so by equations (3.1), (3.2) and (3.3), we have

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n,Tx_n)\alpha(x_{n-1},Tx_{n-1})\sigma(Tx_n,Tx_{n-1})),\psi(M(x_n,x_{n-1}))) \\ &= \zeta(\psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n)),\psi(M(x_n,x_{n-1}))). \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} M(x_n,x_{n-1}) &= \max\left\{\sigma(x_n,x_{n-1}), \frac{[1+\sigma(x_n,Tx_n)]\sigma(x_{n-1},Tx_{n-1})}{1+\sigma(x_n,x_{n-1})}\right\} \\ &= \max\left\{\sigma(x_n,x_{n-1}), \frac{[1+\sigma(x_n,x_{n+1})]\sigma(x_{n-1},x_n)}{1+\sigma(x_n,x_{n-1})}\right\} \\ &= \max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\}. \end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned} 0 &\leq \zeta(\psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n)),\psi(\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\})) \\ &< \psi(\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\}) - \psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n)). \end{aligned} \tag{3.7}$$

Consequently, we obtain that for all $n = 0,1,2,3,\ldots,$

$$\psi(\sigma(x_n,x_{n+1})) < \psi(\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\}).$$

If $\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\} = \sigma(x_n,x_{n+1})$ for some n, then $\psi(\sigma(x_n,x_{n+1})) < \psi(\sigma(x_n,x_{n+1}))$, which is contradiction. Hence $\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\} = \sigma(x_n,x_{n-1})$, for all $n \ge 0$, and hence from (3.7),

$$0 < \psi(\sigma(x_n,x_{n-1})) - \psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n))$$

or

$$\psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n)) < \psi(\sigma(x_n,x_{n-1})), \tag{3.8}$$

using the property of ψ, we get

$$\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_{n+1},x_n) < \sigma(x_n,x_{n-1}), \tag{3.9}$$

for all $n \ge 0$. Thus, we conclude that the sequence $\{\sigma(x_n,x_{n-1})\}$ is monotonically decreasing sequence of non-negative reals and bounded from below by zero. So there is some $r \ge 0$ such that $\lim_{n\to\infty} \sigma(x_n,x_{n-1}) = r$. We will show that

$$\lim_{n\to\infty} \sigma(x_n,x_{n-1}) = 0. \tag{3.10}$$

Suppose that $r > 0$ and since T is a generalize α-admissible $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, therefore by the properties of Ψ, (3.5), (3.8), (3.9) and the condition (ζ_3), we have

$$0 \le \limsup_{n\to\infty} \zeta(\psi(\alpha(x_n,x_{n+1})\alpha(x_{n-1},x_n)\sigma(x_n,x_{n+1})),\psi(\sigma(x_n,x_{n-1}))) < 0.$$

This is a contradiction. Then we conclude that $r = 0$, that is $\lim_{n\to\infty} \sigma(x_n,x_{n-1}) = 0$.

Now, we will show that sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Thus, for all $\epsilon > 0$, and subsequences $\{x_{m_{(k)}}\}$ and $\{x_{n_{(k)}}\}$ of $\{x_n\}$ with for all $m(k) > n(k) > k$ such that for every k,

$$\sigma(x_{n_{(k)}},x_{m_{(k)}}) \ge \epsilon, \tag{3.11}$$

that is

$$\sigma(x_{n_{(k)}},x_{m_{(k)-1}}) < \epsilon, \tag{3.12}$$

for all $m,n,k \in \mathbb{N}$. Therefore, by the triangular inequality and using (3.11) and (3.12), we get

$$\begin{aligned}\epsilon < \sigma(x_{n_{(k)}},x_{m_{(k)}}) &\le \sigma(x_{(n_{(k)})},x_{m_{(k)}-1}) + \sigma(x_{m_{(k)}-1},x_{m_{(k)}})\\ &< \epsilon + \sigma(x_{m_{(k)}-1},x_{m_{(k)}}).\end{aligned}$$

Letting $k \to \infty$ in the above inequalities and by using (3.10) and (3.11), we have

$$\lim_{k\to\infty} \sigma(x_{n_{(k)}},x_{m_{(k)}}) = \epsilon. \tag{3.13}$$

Now from the triangular inequality, we have

$$\sigma(x_{n_{(k)}},x_{m_{(k)}}) \le \sigma(x_{n_{(k)}},x_{n_{(k)+1}}) + \sigma(x_{n_{(k)+1}},x_{m_{(k)}}),$$

$$|\sigma(x_{n_{(k)+1}},x_{m_{(k)}}) - \sigma(x_{n_{(k)}},x_{m_{(k)}})| \le \sigma(x_{n_{(k)}},x_{n_{(k)+1}}).$$

On taking limit as $k \to \infty$ on both sides of above inequality and using (3.10) and (3.13), we get

$$\lim_{k\to\infty} \sigma(x_{n_{(k)}+1},x_{m_{(k)}}) = \epsilon. \tag{3.14}$$

Similarly, it is easy to show that

$$\lim_{k\to\infty} \sigma(x_{n_{(k)}+1},x_{m_{(k)}+1}) = \lim_{k\to\infty} \sigma(x_{n_{(k)}},x_{m_{(k)}+1}) = \epsilon. \tag{3.15}$$

Moreover, T is a generalized α-admissible $\mathcal{Z}$-contraction with respect to ζ, we have

$$\alpha(x_{n_{(k)}},x_{n_{(k)}+1})\geq 1 \quad \text{and} \quad \alpha(x_{m_{(k)}},x_{m_{(k)}+1})\geq 1. \tag{3.16}$$

We deduce

$$M(x_{n_{(k)}},x_{m_{(k)}})=\max\left\{\sigma(x_{n_{(k)}},x_{m_{(k)}}),\frac{[1+\sigma(x_{n_{(k)}},x_{n_{(k)}+1})]\sigma(x_{m_{(k)}},x_{m_{(k)}+1})}{1+\sigma(x_{n_{(k)}},x_{m_{(k)}})}\right\}.$$

Taking $k\to\infty$ and using (3.10), (3.13) and (3.14), we obtain

$$\lim_{k\to\infty}\psi(M(x_{n_{(k)}},x_{m_{(k)}}))=\epsilon. \tag{3.17}$$

By (3.13), (3.17) and the condition (ζ_3), we get

$$0\leq\limsup_{k\to\infty}\zeta(\psi(\alpha(x_{n_{(k)}},x_{n_{(k)}+1})\alpha(x_{m_{(k)}},x_{m_{(k)}+1})\sigma(x_{n_{(k)}+1},x_{m_{(k)}+1})), \quad \psi(M(x_{n_{(k)}},x_{m_{(k)}})))<0,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Thus, $\lim_{n,m\to\infty}\sigma(x_n,x_m)$ exists and is equal to 0. Since (X,σ) is a complete metric-like space, there exists $u\in X$ such that

$$\lim_{n\to\infty}\sigma(x_n,u)=\sigma(u,u)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0, \tag{3.18}$$

and $\alpha(u,Tu)\geq 1$. Moreover,

$$\begin{aligned}0&\leq\zeta(\psi(\alpha(x_n,Tx_n)\alpha(u,Tu)\sigma(Tx_n,Tu)),\psi(M(x_n,u)))\\&=\zeta(\psi(\alpha(x_n,x_{n+1})\alpha(u,Tu)\sigma(x_{n+1},Tu)),\psi(M(x_n,u)))\\&<\psi(M(x_n,u))-\psi(\alpha(x_n,x_{n+1})\alpha(u,Tu)\sigma(x_{n+1},Tu)),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}M(x_n,u)&=\max\left\{\sigma(x_n,u),\frac{[1+\sigma(x_n,x_{n+1})]\sigma(u,Tu)}{1+\sigma(x_n,u)}\right\}\\&\leq\max\left\{\sigma(x_n,u),\frac{[1+\sigma(x_n,u)+\sigma(u,x_{n+1})]\sigma(u,Tu)}{1+\sigma(x_n,u)}\right\}\\&=\sigma(u,Tu), \quad \text{for large } n.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\sigma(x_{n+1},Tu)&=\sigma(Tx_n,Tu)\\&\leq\alpha(x_n,Tx_n)\alpha(u,Tu)\sigma(Tx_n,Tu)\\&<\sigma(u,Tu).\end{aligned} \tag{3.20}$$

By (3.19), (3.20) and the condition (ζ_3), we get

$$0\leq\limsup_{n\to\infty}\zeta(\psi(\alpha(x_n,Tx_n)\alpha(u,Tu)\sigma(Tx_n,Tu)),\psi(M(x_n,u)))<0.$$

This is a contradiction. Hence, therefore u is a fixed of T. To prove the uniqueness of the fixed point, suppose that there exists $w\in X$ such that $Tw=w$ and $w\neq u$ that is $u,w\in Fix(T)$. By (3.3), we have

$$0\leq\zeta(\psi(\alpha(u,Tu)\alpha(w,Tw)\sigma(Tu,Tw)),\psi(M(u,w))), \tag{3.21}$$

where

$$\begin{aligned}M(u,w)&=\max\left\{\sigma(u,w),\frac{[1+\sigma(u,Tu)]\sigma(w,Tw)}{1+\sigma(u,w)}\right\}\\&=\sigma(u,w)\end{aligned} \tag{3.22}$$

from (3.21), (3.22) and (ζ_2), we have

$$\begin{aligned}0 &\le \zeta(\psi(\alpha(u,u)\alpha(w,w)\sigma(u,w)),\psi(\sigma(u,w)))\\ &< \psi(\sigma(u,w)) - \psi(\alpha(u,u)\alpha(w,w),\sigma(u,w)).\end{aligned} \tag{3.23}$$

By using the property of ψ, we have

$$0 < \sigma(u,w) - \alpha(u,u)\alpha(w,w)\sigma(u,w).$$

This is contradiction. Thus, we have $u = w$. Hence T has a unique fixed point $u \in X$ with $\sigma(u,u) = 0$. This completes the proof. □

Theorem 3.4. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a generalized α-admissible Z-contraction with respect to ζ simulation function, if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(t) < t$ such that*

$$\zeta(\psi(\alpha(x,Tx)\alpha(y,Ty),\sigma(Tx,Ty)),\psi(M(x,y))) \ge 0,$$

for all distinct $x,y \in X$, where $M(x,y) = \max\left\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}$.
Assume that

(i) *T is admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$,*

(iii) *X is a regular and for every sequence $\{x_n\}$ in X such that $\alpha(x_n,x_{n+1}) \ge 1$, for all $n \in \mathbb{N}\cup\{0\}$, and we have $\alpha(x_m,x_n) \ge 1$, for all $m,n \in \mathbb{N}$ with $m < n$,*

(iv) *$\alpha(x,y) \ge 1$, for all $x,y \in Fix(T)$.*

Then T has a unique fixed point u in X.

Proof. By (ii), let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$. There exists $x_n \in X$ such that $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. We have by Theorem 3.3, $\{x_n\}$ is a Cauchy sequence such that $\lim_{n\to\infty} \sigma(x_n,x_{n+1}) = 0$. Thus, $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is equal to 0. Since (X,σ) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,u) = 0, \tag{3.24}$$

then

$$\lim_{n,m\to\infty} \sigma(x_m,x_n) = \lim_{n\to\infty} \sigma(x_n,u) = \sigma(u,u) = 0. \tag{3.25}$$

Since X is regular, therefore there exists a subsequence $\{x_{n_{(k)}}\}$ of $\{x_n\}$ such that $\alpha(x_{n_{(k)}},u) \ge 1$, for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}0 &\le \zeta(\psi(\alpha(x_{n_{(k)}},Tx_{n_{(k)}})\alpha(u,Tu)\sigma(Tx_{n_{(k)}},Tu)),\psi(M(x_{n_{(k)}},u)))\\ &= \zeta(\psi(\alpha(x_{n_{(k)}},x_{n_{(k)}+1})\alpha(u,Tu)\sigma(x_{n_{(k)}+1},Tu)),\psi(M(x_{n_{(k)}},u)))\\ &< \psi(M(x_{n_{(k)}},u)) - \psi(\alpha(x_{n_{(k)}},x_{n_{(k)}+1})\alpha(u,Tu)\sigma(x_{n_{(k)}+1},Tu)),\end{aligned}$$

using the property of ψ, we get

$$= M(x_{n_{(k)}},u) - \alpha(x_{n_{(k)}},x_{n_{(k)}+1})\alpha(u,Tu)\sigma(x_{n_{(k)}+1},Tu), \tag{3.26}$$

where

$$M(x_{n_{(k)}},u) = \max\left\{\sigma(x_{n_{(k)}},u), \frac{[1+\sigma(x_{n_{(k)}},Tx_{n_{(k)}})]\sigma(u,Tu)}{1+\sigma(x_{n_{(k)}},u)}\right\}$$

$$\leq \max\left\{\sigma(x_{n_{(k)}},u), \frac{[1+\sigma(x_{n_{(k)}},u)+\sigma(u,x_{n_{(k)}+1})]\sigma(u,Tu)}{1+\sigma(x_{n_{(k)}},u)}\right\}$$
$$= \sigma(u,Tu), \quad \text{for large } k.$$

Consequently, we have

$$\begin{aligned}\sigma(x_{n_{(k)}+1},Tu) &= \sigma(Tx_{n_{(k)}},Tu)\\ &\leq \alpha(x_{n_{(k)}},Tx_{n_{(k)}})\alpha(u,Tu)\sigma(Tx_{n_{(k)}},Tu)\\ &< \sigma(u,Tu), \quad \text{for all } k \in \mathbb{N}.\end{aligned} \tag{3.27}$$

By (3.19), (3.27) and the condition (ζ_3), we get

$$0 \leq \limsup_{k\to\infty} \zeta(\psi(\alpha(x_n,Tx_n)\alpha(u,Tu)\sigma(Tx_n,Tu)),\psi(M(x_n,u))) < 0.$$

This is a contradiction. Hence, therefore u is a fixed point of T. Suppose that u and u^* be two fixed points of T and hence, $u,u^* \in Fix(T)$ which is a generalized α-admissible $\mathcal{Z}$-contraction self-mappings of a metric-like space (X,σ). By (3.3), we have that

$$0 \leq \zeta(\psi(\alpha(u,Tu)\alpha(u^*,Tu^*)\sigma(Tu,Tu^*)),\psi(M(u,u^*))), \tag{3.28}$$

where

$$M(u,u^*) = \max\left\{\sigma(u,u*), \frac{[1+\sigma(u,Tu)]\sigma(u^*,Tu^*)}{1+\sigma(u,u^*)}\right\} = \sigma(u,u^*). \tag{3.29}$$

From (3.28) and (3.29), we have

$$\begin{aligned}0 &\leq \zeta(\psi(\alpha(u,Tu)\alpha(u^*,Tu^*)\sigma(Tu,Tu^*)),\psi(M(u,u^*)))\\ &= \zeta(\psi(\alpha(u,u),\alpha(u^*,u^*)\sigma(u,u^*)),\psi(\sigma(u,u^*))).\end{aligned}$$

This is a contradiction. Thus, we have $u = u^*$. Hence T has a unique fixed point. □

Corollary 3.5. *Let (X,σ) be a complete metric-like space and $T : X \to X$ be a self-mapping, there exist $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to [0,\infty)$ be a function with $\alpha(x,y) = 1$ for all $x,y \in X$ such that $\zeta(\sigma(Tx,Ty),M(x,y)) \geq 0$ for all distinct $x,y \in X$, where*

$$M(x,y) = \max\left\{\sigma(x,y), \frac{[1+\sigma(x,Tx)]\sigma(y,Ty)}{1+\sigma(x,y)}\right\}.$$

Then T has a unique fixed point $u \in X$.

4. Conclusion

In this attempt, we studied generalized α-admissible mappings embedded in the simulation function and proved some fixed point theorems in metric-like spaces. Our results are generalized and extended form of recent results in the literature.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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