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A First Digit Theorem for Powers of Perfect Powers

Research Article

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Abstract. For any fixed power exponent, it is shown that the first digits of powers from perfect power numbers follow a *generalized Benford law* (GBL) with size-dependent parameter that converges asymptotically to a GBL with half of the inverse power exponent. In particular, asymptotically as the power goes to infinity these first digit sequences obey Benford's law. Moreover, we show the existence of a one-parameter size-dependent function that converges to the parameter of these GBL's and determine an optimal value that minimizes its deviation to two minimum estimators of the size-dependent parameter over the finite range of powers from perfect power numbers less than $10^{5m\cdot s}$, $m=2,\dots,6$, where $s=1,2,3,4,5$ is a fixed power exponent.

Keywords. First digit; Perfect power number; Asymptotic counting function; Probabilistic number theory; Mean absolute deviation; Probability weighted least squares

MSC. 11A25; 11K36; 11N37; 11Y55; 62E20; 62F12

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1. Introduction

It is well-known that the first digits of many numerical data sets are not uniformly distributed. Newcomb [1], and Benford [2] observed that the first digits of many series of real numbers obey *Benford's law*

$$P^B(d)=\log_{10}(1+d)-\log_{10}(d), \quad d=1,2,\dots,9. \tag{1.1}$$

The increasing knowledge about Benford's law and its applications has been collected in various bibliographies, the most recent being Beebe [3], and Berger and Hill [4]. It is also known that for

**Mailing address*: Feldstrasse 145, CH-8004 Zürich, Switzerland any fixed power exponent $s \geq 1$, the first digits of some integer sequences, like integer powers and square-free integer powers, follow asymptotically a *Generalized Benford law* (GBL) with exponent $\alpha = s^{-1} \in (0,1)$ (see Hürlimann [5], [6]) such that

$$P_{\alpha}^{GB}(d) = \frac{(1+d)^{\alpha} - d^{\alpha}}{10^{\alpha} - 1}, \quad d = 1,2,\ldots,9. \tag{1.2}$$

Clearly, the limiting case $\alpha \to 0$ respectively $\alpha \to 1$ of (1.2) converges weakly to Benford's law respectively the uniform distribution.

We study the first digits of powers from perfect power numbers along the line of [6]. The method consists to fit the GBL to appropriate samples of first digits using two goodness-of-fit measures, namely the MAD measure (mean absolute deviation) and the WLS measure (probability weighted least square or chi-square divided by sample size). In Section 2, we determine the minimum MAD and WLS estimators of the GBL over finite ranges of powers up to $10^{s \cdot m}$, $m \geq 10$, $s \geq 1$ a fixed power exponent. Calculations illustrate the convergence of the size-dependent GBL with minimum MAD and WLS estimators to the GBL with exponent $(2s)^{-1}$. Moreover, we show the existence of a one-parameter size-dependent function that converges to the parameter of these GBL's and determine an optimal value that minimizes its deviation to the minimum MAD and WLS estimators. A mathematical proof of the asymptotic convergence of the finite sequences to the GBL with exponent $(2s)^{-1}$ follows in Section 3.

2. Size-Dependent Generalized Benford Law for Powers of Perfect Powers

A *perfect power number* is a positive integer that can be expressed as an integer power of another positive integer. It is of the form $n = m^k$ for some natural numbers $m > 1$, $k > 1$. The number $1 = 1^k$, for any $k > 1$, is also counted as perfect number (sequence A001597 in Sloane's OEIS, URL: `https://oeis.org/`). To investigate the optimal fitting of the GBL to first digit sequences of powers from perfect powers, it is necessary to specify goodness-of-fit (GoF) measures according to which optimality should hold. For this purpose, we use here the following two GoF measures. Let $\{x_n\} \subset [1,\infty)$, $n \geq 1$, be an integer sequence, and let d_n be the (first) significant digit of x_n. The number of x_n's, $n = 1,\ldots,N$, with significant digit $d_n = d$ is denoted by $X_N(d)$. The *MAD measure* or *mean absolute deviation* measure for the GBL is defined to be

$$MAD_N(\alpha) = \frac{1}{9} \cdot \sum_{d=1}^{9} \left| P_{\alpha}^{GB}(d) - \frac{X_N(d)}{N} \right|. \tag{2.1}$$

This measure has been used to assess conformity to Benford's law by Nigrini [7] (see also Nigrini [8, Table 7.1, p. 160]). The *WLS measure* for the GBL is defined by

$$WLS_N(\alpha) = \sum_{d=1}^{9} \frac{\left(P_{\alpha}^{GB}(d) - \frac{X_N(d)}{N}\right)^2}{P_{\alpha}^{GB}(d)}. \tag{2.2}$$

In the context of first digit distributions, this chi-square divided by sample size has been used by Leemis et al. [9] (see also [6], [10]). Consider now the sequence of integer powers $\{n_{pp}^s\}$,

$n_{pp}^s < 10^{s\cdot m}$, for a fixed exponent $s = 1,2,3,\ldots$, and arbitrary perfect power numbers n_{pp} below 10^m, $m \geq 10$. Denote by $I_k^s(d)$ the number of powers from perfect power numbers below 10^k, $k \geq 1$, with first digit d. This number is defined recursively by the relationship

$$I_{k+1}^s(d) = S\left(\sqrt[s]{(d+1)\cdot 10^k}\right) - S\left(\sqrt[s]{d\cdot 10^k}\right) + I_k^s(d), \quad k = 1,2,\ldots, \tag{2.3}$$

with $S(x)$ the counting function given by (see Nyblom [11, Theorem 3.1])

$$S(x) = \lfloor x \rfloor - \sum_{d|P_x} \mu(d)\cdot \left\lfloor \sqrt[d]{x} \right\rfloor. \tag{2.4}$$

In (2.4) the sum is taken over all divisors of $P_x = \prod_{p\leq \lfloor \log_2 x\rfloor} p$ (p a prime number), $\mu(k)$ is the Möbius function such that $\mu(k) = 0$ if the prime square p^2 divides k and $\mu(k) = (-1)^e$ if k is a square-free number with e distinct prime factors, and $\lfloor\cdot\rfloor$ denotes the integer-part function. Alternatively, one has (see Nyblom [12, equation (1)])

$$S(x) = \sum_{k=1}^{m} (-1)^{k+1} \sum_{1\leq i_1<\ldots<i_k\leq m} \left\lfloor \sqrt[p_{i_1}\cdots p_{i_k}]{x} \right\rfloor, \tag{2.5}$$

where the sum is taken over all ordered k-element subsets $\{i_1,\ldots,i_k\}$ of the set $\{1,2,\ldots,m\}$, and $p_1, p_2, \ldots, p_m$ are the prime numbers less than or equal to $\lfloor \log_2 x \rfloor$. Another more recent exact recursion formula is (see Jakimczuk [13, Theorem 2.2])

$$S(x+1) = S(x) + 1 - \sum_{j=2}^{x+1}\left(1 - \left\lfloor \frac{\lfloor \sqrt[j]{x+1} \rfloor}{\sqrt[j]{x+1}} \right\rfloor\right), \quad S(1) = 1. \tag{2.6}$$

However, simple efficient algorithms to compute these arithmetic functions do not seem to be known so far. At the cost of some loss in accuracy, one can overcome computational difficulties by using appropriate approximation formulas for $S(x)$. Since we are mostly interested in the asymptotic behaviour of the first digits, we replace the exact value of the counting function by an asymptotic formula $S_{as}(x)$. For simplicity we use the formula of Jakimczuk [14] defined by

$$S_{as}(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x}, \tag{2.7}$$

which implies that $\sqrt{x} + \sqrt[3]{x} < S(x) < \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}$ for all sufficiently large x.

In general, with $N = S(10^m)$ one has $X_N(d) = I_{s\cdot m}^s(d)$ in (2.1)-(2.2). Based on (2.7) a list of approximate values for $I_{5m\cdot s}^s(d)$, $m = 2,\ldots,6$, $s = 1,2,3,4,5$, together with approximate sample sizes $N = S_{as}(10^{5m})$, is provided in Table A.1 of the Appendix. Based on this, we have determined the so-called minimum MAD and minimum WLS estimators of the GBL. Together with their GoF measures, these optimal estimators are reported in Table 1 below. Note that the minimum WLS is a critical point of the equation

$$\frac{\partial}{\partial \alpha} WLS_N(\alpha) = \sum_{d=1}^{9} \frac{\partial P_\alpha^{GB}(d)}{\partial \alpha} \cdot \frac{P_\alpha^{GB}(d)^2 - \left(\frac{X_N(d)}{N}\right)^2}{P_\alpha^{GB}(d)^2} = 0,$$

Table 1. GBL fit for first digits of powers from perfect powers: MAD vs WLS criterion

$s = 1$	Parameters		Δ to LL estimate		MAD GoF measures			WLS GOF measure		
$m =$	WLS	MAD	WLS	MAD	LL	WLS	MAD	LL	WLS	MAD
2	0.49630632	0.49627797	0.131	0.128	8232.7	1515.9	1480	7122.9	250.04	253.28
3	0.49945998	0.49945671	0.040	0.044	266.15	162.32	161.4	89.403	24.946	25.677
4	0.49992119	0.49992070	0.288	0.293	173.54	23.411	23.31	339.23	5.2022	5.2978
5	0.49998845	0.49998838	0.655	0.662	39.643	3.4253	3.409	173.56	1.1135	1.1352
6	0.49999831	0.49999830	1.194	1.204	7.2438	0.50235	0.5002	57.600	0.23950	0.24335
$s = 2$	Parameters		Δ to LL estimate		MAD GoF measures			WLS GOF measure		
$m =$	WLS	MAD	WLS	MAD	LL	WLS	MAD	LL	WLS	MAD
2	0.24822460	0.24823214	0.072	0.073	4706.8	1115.9	1100	24272	2177.9	21803
3	0.24973362	0.24973013	0.016	0.020	107.73	40.429	38.46	127.92	15.100	20.232
4	0.24996110	0.24996059	0.139	0.144	87.199	5.8216	5.773	816.00	3.2209	4.3031
5	0.24999430	0.24999425	0.320	0.325	19.983	0.85191	0.848	431.29	0.69086	0.79669
6	0.24999916	0.24999916	0.586	0.595	3.6553	0.12497	0.1242	144.68	0.14860	0.17876
$s = 3$	Parameters		Δ to LL estimate		MAD GoF measures			WLS GOF measure		
$m =$	WLS	MAD	WLS	MAD	LL	WLS	MAD	LL	WLS	MAD
2	0.16542250	0.16534139	0.042	0.034	3236.3	2282.8	2133	164899	89214	92003
3	0.16648931	0.16648716	0.011	0.013	70.793	22.471	20.9	545.45	58.088	77.623
4	0.16664079	0.16664057	0.092	0.094	58.350	2.5917	2.51	3602.6	6.4122	8.5418
5	0.16666288	0.16666287	0.212	0.123	13.422	0.37850	0.376	1914.8	1.3598	1.3760
6	0.16666611	0.16666611	0.390	0.393	2.45986	0.05552	0.0544	643.54	0.29242	0.32967
$s = 4$	Parameters		Δ to LL estimate		MAD GoF measures			WLS GOF measure		
$m =$	WLS	MAD	WLS	MAD	LL	WLS	MAD	LL	WLS	MAD
2	0.12416155	0.12419543	0.041	0.045	3557.5	2443.9	2352	1641403	921650	926528
3	0.12486723	0.12486632	0.008	0.009	52.043	12.007	10.1	2762.6	196.79	232.09
4	0.12498061	0.1298050	0.069	0.070	43.823	1.4391	1.38	20210	19.995	25.038
5	0.12499716	0.12499714	0.159	0.161	10.096	0.21270	0.203	10767	4.2918	5.8817
6	0.12499958	0.12499958	0.292	0.294	1.8503	0.03119	0.0299	3620.8	0.92328	1.1744
$s = 5$	Parameters		Δ to LL estimate		MAD GoF measures			WLS GOF measure		
$m =$	WLS	MAD	WLS	MAD	LL	WLS	MAD	LL	WLS	MAD
2	0.09907068	0.09910273	0.007	0.010	2743.7	2527.2	2429	10763907	10551418	10595115
3	0.09989330	0.09989271	0.007	0.007	41.837	10.789	10.1	20491	1415.4	1566.1
4	0.09998449	0.09993444	0.055	0.056	0.056	35.083	0.92565	129296	80.820	92603
5	0.09999773	0.09999772	0.127	0.128	8.0887	0.13604	0.129	68899	17.549	23.318
6	0.09999967	0.09999966	0.233	0.235	1.4828	0.01996	0.0188	23176	3.7765	5.3048

with

$$\frac{\partial P_{\alpha}^{GB}(d)}{\partial \alpha} = \frac{(1+d)^{\alpha}\{\ln\left(\frac{1+d}{10}\right)10^{\alpha} - \ln(1+d)\} - d^{\alpha}\{\ln\left(\frac{d}{10}\right)10^{\alpha} - \ln(d)\}}{(10^{\alpha}-1)^2}, \quad d = 1, 2, \ldots, 9. \quad (2.8)$$

For comparison, the MAD and WLS measures for the following size-dependent GBL exponent

$$\alpha_{LL}(5m \cdot s) = (2s)^{-1} \cdot \{1 - c \cdot 10^{-m}\}, \quad (2.9)$$

with $c = 1$, called LL estimator, are listed. This type of estimator is named in honour of Luque and Lacasa [15] who introduced it in their GBL analysis for the prime number sequence.

Through calculation one observes that the LL estimator minimizes the absolute deviations between the LL estimator and the MAD (resp. WLS) estimators over the finite ranges of powers $[1, 10^{5m \cdot s}]$, $m = 2, \dots, 6$, $s = 1, 2, 3, 4, 5$. In fact, if one denotes the MAD and WLS estimators of the sequence $\{n_{pp}^s\}$, $n_{pp}^s < 10^{5m \cdot s}$, by $\alpha_{MAD}(5m \cdot s)$ and $\alpha_{WLS}(5m \cdot s)$, then one has uniformly over the considered finite ranges (columns "Δ to LL estimate" in Table 1 in units of 10^{-m})

$$|\alpha_{WLS}(5m \cdot s) - \alpha_{LL}(5m \cdot s)| \le 1.195 \cdot 10^{-m},$$

$$|\alpha_{MAD}(5m \cdot s) - \alpha_{LL}(5m \cdot s)| \le 1.205 \cdot 10^{-m}. \tag{2.10}$$

Table 1 below displays our results. The MAD (respectively WLS) measures are given in units of 10^{-8} (respectively $10^{-(7+m+s)}$). The optimal MAD and WLS measures decrease with increasing sample size as should be.

3. Asymptotic Counting Function for Powers of Perfect Powers

The following mimics [6], Section 3, through extension of Luque and Lacasa [15, Section 5(a)]. It is well-known that a random process with uniform density x^{-1} generates data that are Benford distributed. Similarly, a sequence of numbers generated by a power-law density $x^{-\alpha}$, $\alpha \in (0,1)$, has a GBL first-digit distribution $P_{1-\alpha}^{GB}(d)$ with exponent $1-\alpha$ (e.g. Pietronoro et al. [16, equation (3)]). From such a density it is possible to derive a counting function $C(N)$ that yields the number of elements of that sequence in the interval $[1, N]$. However, assuming a local density of the form $x^{-\alpha(x)}$ such that $C(N) \sim \int_2^N x^{-\alpha(x)} dx$ is not appropriate in general. Indeed, the power relation for perfect power numbers over an interval $[1, N^s]$ that belongs to (2.9), namely

$$\alpha(N^s) = \frac{2s - 1 + \alpha(N)}{2s}, \quad \alpha(N) = \frac{c}{\sqrt[5]{N}}, \tag{3.1}$$

does not vary smoothly in $[1, N^s]$, which should be the case for such an approximation. However, this drawback can be overcome as follows. Denote by $Q_s(N^s)$ the counting function for powers of perfect power numbers in the interval $[1, N^s]$. Instead of $\int_2^{N^s} x^{-\alpha(N^s)} dx$ define

$$Q_s(N^s) = (2s)^{-1} \cdot \int_2^{N^s} x^{-\alpha(N^s)} dx, \tag{3.2}$$

where the integral pre-factor is chosen to fulfil the asymptotic limiting value for the perfect power number counting function, that is (note that $n_{pp}^s < N^s$ if, and only if, one has $n_{pp} < N$)

$$\lim_{N \to \infty} \frac{Q_s(N^s)}{\sqrt{N}} = 1. \tag{3.3}$$

In fact, by Jamkiczuk [14, Theorem 5], an infinite sequence of asymptotic expansions for the counting function is known, one for each odd prime number. However, it suffices to use the simple asymptotic estimate (3.3) that has been proved in Nyblom [12, Theorem 2.1]. From (3.2) one gets for arbitrary $s = 1, 2, \dots$

$$Q_s(N^s) = (2s)^{-1} \cdot \int_2^{N^s} x^{-\alpha(N^s)} dx = \frac{1}{2s \cdot (1 - \alpha(N^s))} \cdot N^{s \cdot (1 - \alpha(N^s))}. \tag{3.4}$$

With (3.1) this transforms to

$$Q_s(N^s) = \frac{1}{1-\alpha(N)} \cdot N^{0.5(1-\alpha(N))} = \sqrt{N} \cdot \frac{\sqrt[5]{N}}{\sqrt[5]{N}-c} \cdot \exp\left(-\frac{1}{2}c\frac{\ln(N)}{\sqrt[5]{N}}\right), \tag{3.5}$$

which is independent of s and simply denoted by $Q(N)$. The equality $Q_s(N^s) = Q(N)$ reflects the fact that there are as many powers of perfect power numbers in $[1,N^s]$ as there are perfect power numbers in $[1,N]$. Now, what is a good value of $c \in [1, \sqrt[5]{N})$? Clearly, the factor

$$f_N(c) = \frac{\sqrt[5]{N}}{\sqrt[5]{N}-c} \cdot \exp\left(-\frac{1}{2}c\frac{\ln(N)}{\sqrt[5]{N}}\right) \tag{3.6}$$

converges to 1 as $N \to \infty$ for any fixed c. Its derivative with respect to c satisfies the property

$$\frac{\partial}{\partial c} f_N(c) < 0, \quad \text{for all } c \in \left[1, \frac{\ln(N)-2}{\ln(N)}\sqrt[5]{N}\right) \subseteq [1, \sqrt[5]{N}), \quad \text{for all } N \ge 44, \tag{3.7}$$

which implies the following min-max property of (3.7) at $c = 1$:

$$\min_{N \ge 10^{20}} \left\{ \max_{c \in [1, \frac{\ln(N)-2}{\ln(N)} \cdot \sqrt[5]{N})} f_N(c) \right\} = f_{10^{20}}(1) = 0.9978. \tag{3.8}$$

Therefore, the size-dependent exponent (3.1) with $c = 1$ not only minimizes the absolute deviations between the LL estimator and the MAD (resp. WLS) estimators over the finite ranges of powers from perfect power numbers $[1, 10^{5m\cdot s}]$, $m = 2,\dots,6$, $s = 1,2,3,4,5$, as shown in Section 2, but it turns out to be uniformly best with maximum error less than $2.2 \cdot 10^{-3}$ against the asymptotic estimate, at least if $N \ge 10^{20}$. Moreover, one has the following limiting result.

First Digit Theorem for Powers of Perfect Powers (*GBL for powers of perfect powers*). The asymptotic distribution of the first digit of power sequences from perfect power numbers $n_{pp}^s < 10^{5m\cdot s}$, $m \ge 2$, for fixed $s = 1,2,3,\dots$, as $m \to \infty$, is given by

$$\lim_{m\to\infty} \frac{I_{5m\cdot s}^s(d)}{S(10^{5m})} = \lim_{m\to\infty} P_{\alpha(5m\cdot s)}^{GB}(d) = P_{(2s)^{-1}}^{GB}(d), \quad d = 1,\dots,9, \ \alpha(5m\cdot s) = \frac{1}{2s}\left(1 - \frac{1}{10^m}\right). \tag{3.9}$$

It is important to note that the size-dependent GBL parameter (3.9) is proportional to half of the inverse power. This contrasts with [5], [6], [17], where the size-dependent GBL parameters are proportional to the inverse power. Finally, the next Table 2 compares the new counting function $Q(N) = Q_s(N^s)$, for all $s = 1,2,\dots$, with the asymptotic counting functions $S_{as}(N)$ in (2.7) and $\sqrt{N}$ in (3.3). While $S_{as}(N)$ converges to $\sqrt{N}$ from above the function $Q(N)$ does the same from below.

Let us conclude and present a brief outlook on future work in this area. Departures from Benford's law occur quite frequently. For the sequences of integer powers, square-free integer powers, powers of perfect powers, and prime numbers (see [17]), the observed discrepancies can be explained in a non-trivial way. More precisely, the first significant digits of these sequences obey a generalized Benford law with size dependent parameter proportional to the inverse of a multiple of the power exponent. In future work, we intend to pursue the present approach and analyse along the same line other important number theoretical integer sequences.

Table 2. Comparison of perfect power number counting functions for $N = 10^{5m}$

m	$S_{as}(N)$	$Q(N)$	$S_{as}(N)/\sqrt{N}$	$Q(N)/\sqrt{N}$
2	102208	90025	1.022080	0.900250
3	30723460	31112471	1.003184	0.983863
4	10'004'649'434	9'977'998'438	1.000465	0.997800
5	3'162'493'188'959	3'161'399'228'458	1.000068	0.999722
6	1'000'010'000'900'000	999'966'9461'786'523	1.000010	0.999966

Appendix. Tables of First Digits for Powers of Perfect Power Numbers

Based on the recursive relation (2.3)-(2.4), the calculation of $I^s_{5m\cdot s}(d)$, $m = 2,\dots,6$, is straightforward. These numbers are listed in Table A.1. The entry $s \to \infty$ corresponds to the limiting Benford law as the power goes to infinity.

Table A.1. First digit distribution of powers from perfect powers up to $10^{5m\cdot s}$, $m = 2,\dots,6$, $s = 1,2,3,4,5,\infty$

1st digit / $s = 1$	102'207	31'723'459	10'004'649'433	3'162'493'188'958	1'000'010'000'900'000
1	19'655	6'080'457	1'916'682'428	605'825'630'753	191'565'791'418'106
2	15'047	4'664'197	1'470'653'116	464'863'159'841	146'993'453'440'479
3	12'669	3'931'349	1'239'783'414	391'896'173'692	123'921'139'420'180
4	11'153	3'463'126	1'092'249'719	345'266'469'200	109'176'669'536'772
5	10'074	3'130'581	987'454'422	312'144'062'359	98'703'222'173'932
6	9'259	2'878'628	908'047'094	287'045'582'050	90'766'956'805'042
7	8'614	2'679'178	845'181'122	267'175'119'461	84'483'805'200'026
8	8'088	2'516'191	793'804'639	250'936'042'187	79'348'909'957'272
9	7'648	2'379'725	750'793'479	237'340'949'415	75'050'052'948'190
1st digit / $s = 2$	102'206	31'723'458	10'004'649'432	3'162'493'188'957	1'000'010'000'900'000
1	24'887	7'714'161	2'432'312'211	768'836'150'790	243'112'113'929'420
2	16'671	5'171'702	1'630'874'475	515'517'295'553	163'011'126'882'218
3	12'891	4'000'271	1'261'567'173	398'784'358'135	126'099'353'989'648
4	10'648	3'306'928	1'042'969'199	329'687'776'396	104'250'494'249'911
5	9'151	2'842'164	896'426'207	283'366'645'443	89'603'385'678'048
6	8'066	2'506'067	790'450'257	249'868'230'361	79'010'911'161'490
7	7'240	2'250'226	709'775'218	224'367'151'316	70'947'253'653'262
8	6'592	2'048'100	646'036'130	204'219'370'152	64'576'349'944'280
9	6'060	1'883'839	594'238'562	187'846'210'811	59'399'011'435'721

Table Contd.

$s = 3$ / 1st digit	102'207	31'723'459	10'004'649'433	3'162'493'188'958	1'000'010'000'900'000
1	26'787	8'305'972	2'619'110'080	827'890'749'711	261'786'072'543'005
2	17'151	5'322'015	1'678'327'501	530'519'549'944	167'755'084'207'379
3	12'886	3'999'937	1'261'469'949	398'753'886'392	126'089'730'542'031
4	10'425	3'237'252	1'020'978'124	322'735'604'409	102'052'117'937'611
5	8'810	2'735'780	862'847'106	272'750'895'697	86'246'525'098'448
6	7'663	2'378'751	750'260'382	237'162'441'870	74'993'144'233'644
7	6'790	2'110'488	665'666'911	210'422'511'539	66'537'739'600'101
8	6'116	1'900'934	599'581'714	189'532'962'548	59'932'274'803'397
9	5'579	1'732'330	546'407'666	172'724'586'848	54'617'311'934'381

$s = 4$ / 1st digit	102'206	31'723'458	10'004'649'432	3'162'493'188'957	1'000'010'000'900'000
1	27'755	8'609'781	2'715'007'714	858'208'082'123	271'372'876'206'818
2	17'381	5'392'856	1'700'689'802	537'589'354'833	169'990'670'581'712
3	12'868	3'995'294	1'260'006'484	398'291'313'448	125'943'462'072'868
4	10'309	3'199'359	1'009'016'850	318'954'173'825	100'856'373'709'173
5	8'635	2'681'053	845'573'653	267'290'022'564	84'519'713'596'426
6	7'456	2'314'907	730'109'173	230'791'755'873	72'978'634'509'405
7	6'571	2'041'601	643'919'475	203'547'159'979	64'363'645'729'936
8	5'886	1'829'290	576'966'124	182'383'133'528	57'671'386'034'917
9	5'346	1'659'318	523'360'158	165'438'192'785	52'313'238'458'744

$s = 5$ / 1st digit	102'207	31'723'459	10'004'649'433	3'162'493'188'958	1'000'010'000'900'000
1	28'353	8'794'459	2'773'299'821	876'636'724'353	277'200'295'549'123
2	17'510	5'433'880	1'716'641'490	541'648'003'862	171'285'464'226'364
3	12'859	3'991'079	1'258'675'176	397'870'469'182	125'810'386'560'405
4	10'233	3'175'683	1'001'543'867	316'591'659'559	100'109'311'304'827
5	8'523	2'647'777	835'071'901	263'969'968'083	83'469'861'426'531
6	7'334	2'276'601	718'017'241	226'968'959'988	71'769'807'116'412
7	6'440	2'000'621	630'983'100	199'457'387'662	63'070'395'764'702
8	5'756	1'786'960	563'602'327	178'158'225'583	56'335'403'693'624
9	5'199	1'616'399	509'814'510	161'155'790'686	50'959'075'258'012

$s = \infty$ / 1st digit	102'207	31'723'459	10'004'649'433	3'162'493'188'958	1'000'010'000'900'000
1	30'767	9'549'713	3'011'699'575	952'005'310'959	301'033'006'234'865
2	17'998	5'586'224	1'761'731'315	556'887'407'399	176'093'020'123'754
3	12'770	3'963'489	1'249'968'260	395'117'903'561	124'939'986'108'111
4	9'905	3'074'321	969'550'707	306'477'256'080	96'910'982'195'406
5	8'093	2'511'903	792'180'608	250'410'151'319	79'182'037'931'348
6	6'842	2'123'784	669'779'161	211'718'766'229	66'947'459'158'762
7	5'927	1'839'705	580'189'099	183'399'137'331	57'992'526'949'349
8	5'228	1'622'735	511'763'055	161'769'503'838	51'153'034'018'643
9	4'677	1'451'536	457'787'652	144'707'752'242	45'757'948'176'763

Competing Interests

The author declare that they have no competing interests.

Authors' Contributions

The author contributed significantly in writing this article. The author read and approved the final manuscript.

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