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On the Gibbs Phenomenon for $|N,p_n,q_n|$-Summability Method

Research Article

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Abstract. In this paper we have proved a theorem on the Gibbs phenomenon for $|N,p_n,q_n|$-summability method which gives some new results and generalizes some previous known results.

Keywords. Approximation; Convergence; Fourier series; (C,α) summability and $|N,p_n,q_n|$-summability method

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1. Introduction

In the theory of approximation, it is important to study about the limit of convergence of approximating function and the limit of approximant. The relating study for a discontinuous function $\phi(x)$, defined as $\phi(x) = (\pi - x)/2$, $0 < x < 2\pi = 0$, $x = 0$, 2π, has been firstly investigated by J.W. Gibbs by taking partial sums $\{s_n(x)\}$ of the Fourier series of $\phi(x)$ in the neighborhood of a point of discontinuity of $\phi(x)$. Since

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} = \phi(x), \quad 0 < x < 2\pi. \tag{1.1}$$

Here we see that the series is not uniformly convergent in the neighbourhood of $x = 0$. Let $x > 0$, we have

$$s_n(n) = \left(\frac{-x}{2}\right) + \int_0^x D_n(t)dt \tag{1.2}$$

when $D_n(t) = \sin\left(\frac{n+1}{2}\right) t \sin\left(\frac{t}{2}\right)$.

Since the integral

$$\frac{2}{\pi}\int_0\left(\frac{\sin nt}{t}\right)$$

is uniformly bounded in n and ξwe have

$$s_n(x)+\left(\frac{x}{2}\right)=\int_0^{nx}\left(\frac{\sin t}{t}\right)dt+O(1). \tag{1.3}$$

Uniformly in $0 \le x \le \pi$. Thus $s_n(x)$are uniformly bounded, but the curve of approximation overshoot the mark in the neighborhood of $x = 0$ in the interval $(0, \pi]$ (Knopp [5]).

The smoothing of convergence of Fourier series is quite important for filter design (Hamming [3]). More precisely, we consider the integral of $(\sin t)/t$ over the intervals $(k\pi, (k+1)\pi)$, $k = 0, 1, 2, \dots$ We know that these integrals decrease in absolute value and are of alternating sign (Zygmund [7]) for $k = 0, 1, 2, \dots$, the curve

$$y=\int_0^{x}\left(\frac{\sin t}{t}\right)dt=G(x), \quad \text{say}.$$

Takes maxima with $M_1 > M_3 > M_5 \dots$ at the points $\pi, 3\pi, 5\pi, \dots$ and minima $M_2 < M_4 < M_6 < \dots$ at points $2\pi, 4\pi, 6\pi, \dots$ from (1.3), we have

$$s_n(\pi/n)\to\int_0^{\pi}\left(\frac{\sin t}{t}\right)dt>\pi/2.$$

Thus though $s_n(x)$ tends to $\phi(x)$ at every fixed x, $0 < x < 2\pi$, the curve $y = s_n(x)$, which passes through the point $(0, 0)$ condense to the interval $0 \le y \le G(\pi)$ of the y-axis, the ratio of whose length to that of interval $0 \le y \le \phi(+O) = \pi/2$ is

$$\left(\frac{2}{\pi}\right)\int_0^{\pi}\left(\frac{\sin t}{t}\right)dt=1.179.$$

Similarly, to the left of $x = 0$, the curve $y = s_n(x)$, condense to the interval $-G(\pi) \le y \le 0$ this behaviour is called Gibbs phenomenon i.e. if the ratio $[s_n(+0) - s_n(0)]/[\phi(+0) - \phi(0)] > 1$, then $s_n(x)$ show Gibbs phenomenon in the right of $x = 0$.

Let $\{p_n\}$ and $\{q_n\}$ be any two non-negative and non increasing sequences with P_n and Q_n as their n-th partial sums respectively and let

$$R_n=(p*q)_n=\sum_{k=0}^{n}p_{n-k}\cdot q_k=\sum_{k=0}^{n}P_k q_{n-k}, \text{ tends to infinity as } n\to\infty.$$

The sequence to sequence transformation for the $\{s_n\}$ sequence of partial sums (Borwein [1])

$$t_n^{p,q}=\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k s_k.$$

If $t_n^{p,q} \to s$, as $n \to \infty$ then $\{s_n\}$ is (N, p_n, q_n) summable to s.

2. Known Results

Zygmund [7] has proved the following theorem on Gibbs phenomenon for (C, α) method of summability.

Theorem 2.1. *There is an absolute constant* α_0, $0 < \alpha_0 < 1$, *with the following property: if* $f(x)$ *has a simple discontinuity at a point* ξ, *the mean* $\sigma_n^\alpha(x; f)$ *shows Gibbs phenomenon at* ξ *for* $\alpha < \alpha_0$ *but not for* $\alpha \geq \alpha_0$.

Later, Hille and Tamarkin [4] have proved the following theorem for (N, p_n) method of summability.

Theorem 2.2. *Let* $\{p_n\}$ *be a non negative, non increasing sequence and let* $t_n(x)$*denote the* (N, p_n) *means of* $\{s_n(x)\}$. *Then for* $[f(x+t) + f(x-t) - \{f(x+0) + f(x-0)\}] = 0(1)$ *as* $t \to 0$ *then* $t_n(x) \to \frac{1}{2}[f(x+0) + f(x-0)]$ *iff*

$$\sum_{k=1}^{n} \frac{P_k}{k} \leq m P_n, \quad n = 1, 2, \ldots \tag{2.1}$$

where M *is some positive constant.*

We know that the condition for sequence $\{p_n\}$ is equivalent to Dikshit and Kumar [2],

$$K \geq P_m \sum_{n=m}^{\infty} \left(\frac{1}{n P_n} \right) \tag{2.2}$$

where k is some positive constant.

Hence $\left(\frac{P_k}{P_n}\right) \leq \left(\frac{k}{n}\right)^\alpha$, $1 \leq k \leq n$, for some α in $(0,1)$.

Later on Singh [6] have proved the following theorem.

Theorem 2.3. *Let* $\{p_n\}$*be a non negative and non increasing sequence. Let* α *be a number such that* $\left(\frac{P_k}{P_n}\right) \leq \left(\frac{k}{n}\right)^\alpha$, $1 \leq k \leq n$ *then there exist a constant* α_0, $0 < \alpha_0 < 1$, *such that* (N, p_n) *method shows Gibbs phenomenon for* $\alpha < \alpha_0$, *but not for* $\alpha \geq \alpha_0$ *at a point of simple discontinuity* ξ *of* $f(x)$.

3. Main Result

In this paper we have proved the following theorem, on the Gibbs Phenomenon for $|N, p_n, q_n|$-summability method.

Theorem 3.1. *Let* $\{p_n\}$ *and* $\{q_n\}$ *are non negative and non increasing sequence with convoluted product* (R_n). *Let* α *be a number such that,* $\frac{R_k}{R_n} \leq \left(\frac{k}{n}\right)^\alpha$, $1 \leq k \leq n$, *then there exist a constant* α_0, $0 < \alpha_0 < 1$, *such that the* $|N, p_n, q_n|$ *method shows Gibbs phenomenon for* $\alpha < \alpha_0$ *but not for* $\alpha \geq \alpha_0$ *at a point of simple discontinuity* ξ *of* $f(x)$.

4. Lemmas

We have needed the following lemmas for the proof of our theorem.

Lemma 4.1. *Let* $\{p_n\}$ *and* $\{q_n\}$ *are two be any non negative and non increasing sequences with convoluted product* R_n *and let*

$$(R_m)\sum_{n=m}^{\infty}\frac{1}{n}R_n \le K, \quad m = 1,2\dots \tag{4.1}$$

where k *is some positive constant, then*

$$\frac{R_m}{R_n} \le \left(\frac{m}{n}\right)^{\delta}, \text{ for some } 0<\delta\le 1,\ 1\le m\le [n/c], \tag{4.2}$$

where c *is some fixed positive integer.*

Proof. For any integer K, we have

$$K \ge R_m\sum_{n=m}^{\infty}\frac{1}{n}R_n \ge R_m\sum_{n=m}^{km}\frac{1}{n}R_n$$

$$\ge \frac{R_m}{R_{km}}\log k\,. \tag{4.3}$$

That is

$$\frac{R_{km}}{R_m} \ge \left(\log_4 k \log 4^e/k\right)$$

$$> 4, \quad \text{for large } \ k \ge k_0\,. \tag{4.4}$$

We take for convenience $k_0 \ge 4$. For a given sufficiently large n, we can find a fixed integer $c \ge k_0$, and b such that

$$c^{b+(1/2)}m \le n < c^{b+1}m$$

we have

$$\frac{R_n}{R_m} = \left(\frac{R_n}{R_c b_m}\right)\left(\frac{R_e b_m}{R_m}\right) \ge \left(\frac{R_n}{R_c b_m}\right)4^b \tag{4.5}$$

by a repeated application of the fact we have

$$R_{km}/R_m > 4.$$

we can find a number μ, $(1/2)\le\mu$ such that $n = c^{b+\mu}m$.
We have,

$$b = \log_4(n/m)^{\delta} - \mu \tag{4.6}$$

where $\delta = (1/\log_4 c)$, obviously $\delta \le 1$.

From (4.5) and (4.6), we get

$$\left(\frac{R_n}{R_m}\right) \geq \frac{R_c^{b+\mu} m}{R_c^b m}(4)^{\log_4 (n/m)^\delta - \mu}$$

$$= \frac{R_c^{b+\mu} m}{R_c^b m}(4)^{-\mu}\left(\frac{n}{m}\right)^\delta . \tag{4.7}$$

Again from (4.1), we have

$$k \geq R_c^b m \sum_{n=c^b m}^{c^{b+\mu} m} \frac{1}{n} R_n \geq \frac{R_c^b m}{R_c^{b+\mu} m} \log c^\mu ,$$

$$\frac{R_c^{b+\mu} m}{R_c^b m} \geq (\log c^\mu / k). \tag{4.8}$$

Now from (4.7) and (4.8), we obtain

$$\frac{R_n}{R_m} \geq \frac{\log c^\mu}{k} 4^{-\mu}(n/m)^\delta$$

$$\geq 4\mu 4^{-\mu}(n/m)^\delta$$

by the fact, $4^\mu \leq 4^\mu$, for $\frac{1}{2} \leq \mu < 1$.

Thus $\dfrac{R_m}{R_n} \leq \left(\dfrac{m}{n}\right)^\delta$, $0 < \delta \leq 1$, $1 \leq m \leq [n/c]$.

This proves the lemma. □

Lemma 4.2 (Zygmund [7]). *Given any $m > 0$, there exist a $\delta(m) > 0$ and $n_0(m)$ such that*

$$\delta_n(x) < \left(\frac{\pi}{2}\right) - \delta \quad \textit{for } 0 \leq x \leq \left(\frac{m}{n}\right),\ n > n_0.$$

5. Proof of the Theorem

We prove it, for the function

$$f(x) \sin x + (\sin 2x/2) + (\sin 3x/3) + \ldots \quad \text{at } \xi = 0$$

Observing that, $s_n = \cos x + \cos 2x + \ldots$, we get

$$s_n(x) = \int_0^x \left(\sum_{k=1}^n \cos kt\right) dt = \pi - \int_\pi^x D_n(t) dt,$$

and

$$t_n^{p,q}(x) = ((\pi - x)/2) - (1/2R_n) \sum_{k=0}^n \left(\int_x^\pi q_k p_{n-k} \frac{\sin(k+\frac{1}{2})t}{\sin t/2} dt\right)$$

$$= ((\pi - x)/2) - \frac{1}{2R_n}\left\{\left(\sum_{k=0}^{n/2} + \sum_{k=\frac{n}{2}+1}^{n}\right) \int_x^\pi p_{n-k} q_k \frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}} dt\right\}$$

$$= \frac{\pi - x}{2} + \Sigma_1 + \Sigma_2 \quad (\text{say}).$$

Applying Abel's Lemma, we find that

$$|\Sigma_1| \le \frac{1}{n}(\sin(t/2))^2. \tag{5.1}$$

Hence

$$\left|\int_x^\pi \Sigma_1 dt\right| \le \frac{2}{2}\cot(x/2). \tag{5.2}$$

Again using mean value theorem, we have for some $x<\xi<\pi$

$$\left|\int_x^\pi \epsilon_2 dt\right| \le k\left(\frac{R_{1/x}}{nR_n\sin(x/2)}\right). \tag{5.3}$$

Since $R_{1/\xi} \le R_{1/x}$ for $x<\xi$.
Combining (5.1), (5.2) and (5.3), we get that

$$t_n^{p,q}(x) \le \frac{(\pi-x)}{2}+\frac{2}{n}\cot x/2+k\left(\frac{R_{1/k}}{nR_n\sin x/2}\right). \tag{5.4}$$

By the hypothesis that $\left(\frac{R_k}{R_n}\right) \le \left(\frac{k}{n}\right)^\alpha$, $0<\alpha<1$, we see that the second term in (5.4) dominate the last term thus if, nx is sufficiently large, say $nx>m$, $n \ge n_1$ and $nx^2>1$, we find that,

$$\left|t_n^{p,q}\right| \le \pi/2 \quad \text{for } \left(\frac{n}{m} \le x \le \pi\right). \tag{5.5}$$

Now consider $t_n^{p,q}-\sigma(x)$, where $\sigma(x)$ denote the $(C,1)$ mean of $s_n(x)$, we have,

$$\begin{aligned}\left|t_n^{p,q}-\sigma(x)\right| &= \left|\sum_{k=0}^{n}\frac{P_{n-k}q_k}{R_n}\frac{\sin kx}{k}-\sum_{k=0}^{n}\frac{n-k+1}{n+1}\frac{\sin nkx}{k}\right| \\ &\le x\sum_{k=0}^{n}\frac{n-k+1}{R_n}\left(\frac{P_{n-k}q_k}{n-k+1}-\frac{R_n}{n+1}\right).\end{aligned}$$

Since $(R_n/n)$is non increasing for $\{p_n\}$and $\{q_n\}$, we have

$$\begin{aligned}\left|t_n^{p,q}(x)-\sigma(x)\right| &\le x\left[\frac{(n+1)^{\alpha+1}}{(\alpha+1)n^\alpha}-\frac{n+2}{2}\right] \\ &= \frac{nx(1-\alpha)}{2(\alpha+1)}+x\left[\frac{(x+1)^{\alpha+1}-(n)^{\alpha+1}}{(\alpha+1)n^\alpha}-1\right]\end{aligned}$$

since $(n+1)^{\alpha+1}-n^{\alpha+1} \le (2n)^\alpha$ and $2^\alpha \le \alpha+1$ for $0 \le \alpha \le 1$, we have

$$\left|t_n^{p,q}(x)-\sigma_n(x)\right| \le \left[\frac{nx(1-\alpha)}{2(\alpha+1)}\right],$$

that is,

$$t_n^{p,q}(x) \le \sigma_n(x)+\left(\frac{nx}{2}\right)(1-\alpha)$$

By Lemma 4.2, we have that,

$$t_n^{p,q} \le \frac{\pi}{2} - \delta(m) + \frac{m_1}{2}(1-\alpha), \quad 0 \le nx \le m_1.$$

Now if we take α such that, $(1-\alpha)\frac{m_1}{2} - \delta(m_1) < 0$, then $t_n^{p,q}(x) \le \frac{\pi}{2}$, for $0 \le nx \le m_1$.
In order to show that for positive and small enough α, the Gibbs phenomenon occurs, and it does not occur for $\alpha \ge 1$. We consider the difference $t_n^{p,q} - \delta n(x)$.
We have,

$$\left|t_n^{p,q}(x) - s_n(x)\right| \le x(n - R_n^{'}/R_n) < nx\alpha.$$

Thus,

$$\left|t_n^{p,q}(\pi/n) - s_n(\pi/n)\right| \le \pi\alpha, \quad \text{for } 0 < \alpha < 1.$$

Consequently, $s_n(\pi/n) - \pi\alpha \le t_n^{p,q}(\pi/n) \le \pi\alpha + s_n(\pi/n))$ from the above inequality, we see that for small α,

$$\liminf_{n\to\infty} t_n^{p,q}(\pi/n) > \pi/2$$

by the fact that $s_n(\pi/n)$ tends to a limit greater than $(\pi/2)$.
Hence the Gibbs phenomenon occurs for small value of α. This proves that there exist α_0, $0 < \alpha_0 < 1$, such that for $\alpha < \alpha_0$, the Gibbs phenomenon exist while for $\alpha > \alpha_0$, it does not exist.

6. Corollaries

Our theorem have the following results:

Corollary 6.1. *If we take $p_n = 1$ then our theorem reduces to Theorem* 2.3.

Corollary 6.2. *If we take $Q_n = 1$ then our theorem shows the Gibbs Phenomenon for $|\bar{N},p_n|$-summbility method.*

7. Conclusion

The theorem which has proved in this research article have more general results rather than some previous non results on the Gibbs Phenomenon for summability methods. Hence this will be enrich the literature on the Gibbs Phenomenon for summability methods.

Competing Interests

The author declare that they have no competing interests.

Authors' Contributions

The author contributed significantly in writing this article. The author read and approved the final manuscript.

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