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Research Article

Weak Zero-Divisor Graphs of Finite Commutative Rings

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Abstract. This paper introduces weak zero-divisor graphs of finite commutative rings. The concept of zero-divisor graphs over rings has been extensively investigated for decades. These graphs are constructed with the zero-divisors of rings as their vertex set. We define a weak zero-divisor graph as a graph whose vertex set consists of nonzero elements u and v of a ring R and such that the vertices u and v are adjacent if and only if $(uv)^n = 0$ for some positive integer n. This article will study parameters of weak zero-divisor graphs of commutative rings, including their diameter, their girth, their radius, their center, and their domination number. We also determine whether weak zero-divisor graphs of the ring of integers $\mathbb{Z}_m$ are Eulerian and Hamiltonian.

Keywords. Weak zero-divisor graphs, Zero-divisor graphs, Ring of integers modulo n, Weak-nil ring, Nilpotent elements

Mathematics Subject Classification (2020). 13A99, 05C99, 05C12, 05C45

1. Introduction

The interplay between algebraic structures and graphs started with Arthur Cayley [7]. In 1988, Beck [4] published an article on graph coloring relating graphs to rings. Subsequently, Anderson and Livingston [2] gave a definition of the zero-divisor graph of a commutative ring. Progress on the construction of zero-divisor graphs of commutative rings can be found in the article, "*Zero-divisor graphs of finite commutative rings: a survey*" [13] where Singh and Bhat present a thorough overview. Various authors extended this concept to construct zero-divisor graphs of non-commutative rings. Redmond [12], and Božic and Petrović [6] studied these graphs of non-commutative rings, and of matrices over commutative rings, respectively. Recently, the extended zero-divisor graph of commutative rings were introduced by Bennis *et al*. [5].

The purpose of this article is to introduce weak zero-divisor graphs of finite commutative rings and study their structural properties. The weak zero-divisor graph of a commutative ring R is the undirected graph where two distinct vertices u and v form an edge if and only $(uv)^n = 0$ for u and v in R and n a positive integer. We note that if $n = 1$, then the vertices u and v are zero-divisors and the corresponding graph is the zero-divisor graph of R. Recall that a nonzero element a in a ring R is called a left zero-divisor of R if there exists a nonzero element b in R such that $ab = 0$. Right zero-divisors are defined similarly. We note that a weak zero-divisor graph has order greater than or equal to the order of a classical or an extended zero-divisor graph.

A nonzero element c in R is called a nilpotent element of R if $c^n = 0$ for some positive integer n. We represent the set of zero-divisors and the set of nilpotent elements of R by $Z(R)$ and $Nil(R)$, respectively.

Following the notion introduced in [10], a nonzero element w in R is called a weak zero-divisor of R if $(xw)^k = 0$ or $(wx)^k = 0$ for some nonzero element x in R and a positive integer k; i.e, if xw or wx is a nilpotent element of R. As in [10], we denote by $W_z(R)$ the set of weak zero-divisors of R and recall that a ring R is called a weak nil-ring if every element of R is a weak zero-divisor. Also, as observed in [10], we reiterate that $Nil(R) \subseteq Z(R) \subseteq W_z(R)$. A ring R is called reduced if it has no nonzero nilpotent elements; otherwise, it's a not-reduced ring.

We recall some rudimentary graph theory notions which can be found in [8]. The degree of a vertex u in a graph G, denoted by $\deg(u)$, is the number of vertices in the graph that are adjacent to u. The maximum degree of a graph G, denoted by $\Delta(G)$, is the largest degree among the vertices of the graph. Two vertices u and v of G are connected is there is a path from u to v in G. The graph G is said to be connected if any two distinct vertices of G are connected. The distance $d(u,v)$ from u to v in a connected graph is the minimum of all lengths of the paths from u to v. The eccentricity of a vertex v is the distance between v and the vertex that is farthest from v in the graph. The radius of a graph G, denoted by $\text{rad}(G)$, is the smallest eccenticity among all the vertices in the graph. A vertex v is in the center of G if its eccentricity is the radius of G. The diameter of a connected graph is the greatest distance among any pair of vertices of the graph. The girth of a graph is the length of the smallest cycle in the graph.

The following definitions can also be found in [8]. For a vertex u in a graph G, the neighborhood of u, denoted by $N(u)$, is the set vertices in G adjacent to u. The set $N[u] = N(u) \cup \{u\}$ is called the closed neighborhood of u. A vertex u is said to dominate the vertices in $N[u]$. A set A is called a dominating set of a graph G if every vertex of G is dominated by at least one vertex of A. The dominating number of a graph G, denoted by $\gamma(G)$, is the minimum cardinanity of all the dominating sets of G. A minimum dominating set is accordingly defined.

It is worth noting a few well-know theorems regarding zero-divisor graphs. It was shown in [2] and [3] that the zero-divisor graph $\Gamma(R)$ of a commutative ring R is connected, its diameter is less than or equal to 3, and its girth is less than or equal to 4.

This rest of this paper is organized as follows. In Section 2, we investigate the structural properties of weak zero-divisor graphs of commutative rings with unity. We particularly look at the order, diameter, radius, center, and girth of their weak zero-divisor graphs. Section 3 explores weak zero-divisor graphs of the ring of integers $\mathbb{Z}_m$. We evaluate their domination number, and determine whether the weak zero-divisor graph of $\mathbb{Z}_m$ are Hamiltonian and Eulerian.

In this article, R will be assumed to be a commutative ring with unity. We will also represent the a zero-divisor graph of a ring R by $\Gamma(R)$ and its weak zero-divisor graph by $\Omega(R)$. We will also denote by M^* any set $M-\{0\}$.

2. Weak Zero-Divisor Graphs of Commutative Rings

First, we invoke the following theorem which will be useful in the sequel.

Theorem 2.1 ([10, Proposition 2.3]). *If R is commutative and not reduced, then R is weak-nil.*

In this paper, R will be assumed to be a commutative ring. We now arrive at our first results. The first three statements are readily verified.

Theorem 2.2. *The vertex set of a zero-divisor graph $\Gamma(R)$ of a commutative ring R is a subset of the vertex set of the weak zero-divisor graph $\Omega(R)$ of R.*

Corollary 2.1. *The zero-divisor graph of a commutative ring R is isomorphic to a subgraph of the weak zero-divisor graph of R.*

Theorem 2.3. *Let R be a reduced ring. Then the zero-divisor graph of R is isomorphic to the weak zero-divisor graph of R.*

Theorem 2.4. *If a finite commutative ring R has a nilpotent element (if R is not reduced), then R^* and the weak zero-divisor graph $\Omega(R)$ have the same order.*

Proof. As shown in [10], let a be a nonzero nilpotent element of a commutative ring R. For any element x of R, we see that $(ax)^n = a^n x^n = 0$. Therefore, the vertex represensted by a is adjacent to every nonzero element of R; i.e, the order of the weak zero-divisor graph $\Omega(R)$ is equal to the order of R^*. □

By the proof of Theorem 2.4, we see that the vertex represented by a nilpotent element of the ring is adjacent to every vertex of the weak zero-divisor graph. This brings us to our next result.

Theorem 2.5. *Let $\Omega(R)$ be the weak zero-divisor graph of a not-reduced commutative ring R, then $diam(\Omega(R)) \leq 2$.*

Proof. Let R be a not-reduced commutative ring. Since $\Omega(R)$ is a weak zero-divisor graph, there exists an element a in R such that $(ax)^n = 0$ for each x in R; i.e, $d(a,x) = 1$ for every vertex x in R. Choose any two distinct vertices u and v in R that are not equal to a. Then $d(a,u) = d(a,v) = 1$. If u and v are zero-divisors or weak zero-divisors, then u and v are adjacent, and so $d(u,v) = 1$. Otherwise, if $uv \neq 0$, we have $d(u,v) = d(u,a) + d(a,v) = 1+1 = 2$. In any case, $d(u,v) \leq 2$ for any u and v in $\Omega(R)$. Therefore, $diam(\Omega(R) \leq 2$. □

Theorem 2.6. *Let $\Omega(R)$ be the weak zero-divisor graph of a commutative ring R, then $rad(\Omega(R)) = 1$.*

Proof. This holds since, for every vertex x in $\Omega(R)$, $d(a,x) = 1$ for every nilpotent element a of R. □

Corollary 2.2. *Let $\Omega(R)$ be the weak zero-divisor graph of a commutative ring R, then $Cen(\Omega(R)) = Nil(R)$.*

Theorem 2.7. *Let $\Omega(R)$ be the weak zero-divisor graph of a commutative ring R. If $\Omega(R)$ contains a cycle, then the girth $g(\Omega(R)) \leq 3$.*

Proof. By Corollary 2.1, we know that the zero-divisor graph $\Gamma(R)$ is isomorphic to a subgraph of $\Omega(R)$. It is also known that, if $\Gamma(R)$ has a cycle, then $g(\Gamma(R)) \leq 4$. Thus, $\Omega(R)$ has a cycle of smallest length less than or equal to 4. Pick that cycle in $\Omega(R)$. If that cycle has length 3, then we are done. Assume, on the other hand, that the cycle in $\Omega(R)$ has length 4, say $\Omega(R)$ has a cycle C: u_1-u_2-u_3-u_4-u_0. Now, pick a vertex u_0 in $\Gamma(R)$ represented by a nilpotent element of R. Then u_0 is adjacent to every vertex in $\Gamma(R)$. We see that the cycles $u_0 - u_1 - u_2 - u_0$, $u_0 - u_2 - u_3 - u_0$, and $u_0 - u_3 - u_4 - u_0$ are all cycles of length 3 in $\Omega(R)$ since u_0 is adjacent to $_u2, u_3$, and u_4. Therefore, $g(\Omega(R)) \leq 3$. □

We see that if $\Omega(R)$ has order $n \geq 3$, then $g(\Omega(R)) = 3$.

3. Weak Zero-Divisor Graphs of $\mathbb{Z}_n$

We now consider weak zero-divisor graphs of the rings of integers modulo m. First, we take a look at the vertices. For zero-divisor graphs of $\mathbb{Z}_m$, the degree of a vertex in $\Gamma(Z_m)$ is known. In [11], Philips showed that, for any vertex u in $\Gamma(\mathbb{Z}_m)$,

$$\deg(u) = \begin{cases} \gcd(u,m) - 1, & \text{if } u^2 \neq 0, \\ \gcd(u,m) - 2, & \text{if } u^2 = 0. \end{cases}$$

For weak zero-divisor graphs, we find:

Theorem 3.1. *Let a be a vertex of $\Omega(\mathbb{Z}_m)$. If a is a nonzero nilpotent element of $\mathbb{Z}_m$, then $deg(a) = m - 2$.*

Proof. As shown in Theorem 2.4, if a is a nilpotent element of a commutative ring, then a is adjacent to every vertex in $\Gamma(\mathbb{Z}_m^*)$, and hence $\deg(a) = m-2$. □

Theorem 3.2. *If p is prime, then $\Gamma(\mathbb{Z}_{p^m})$, then the vertices in $\mathbb{Z}_{p^m}$ represented by kp^t, where $1 \le t \le m-1$ and $1 \le k \le p-1$, form the center of $\Gamma(\mathbb{Z}_{p^m})$.*

Figure 1 and Figure 2 are examples of the zero-divisor graph of $\mathbb{Z}_{12}$ and the weak zero-divisor graph of $\mathbb{Z}_{12}$, respectively.

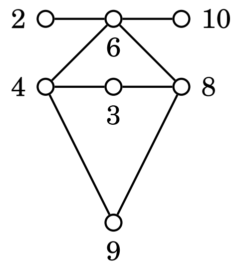

Figure 1. Zero-divisor graph of $\mathbb{Z}_{12}$

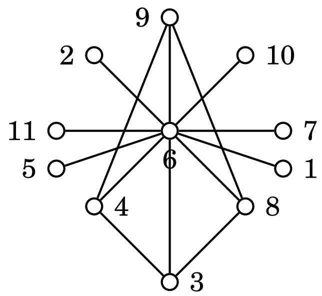

Figure 2. Weak zero-divisor graph of $\mathbb{Z}_{12}$

The domination number of a zero divisor graph $\Gamma(\mathbb{Z}_{p^m})$ has been found in [1, Theorem 1]. There, it was shown that the domination number of a zero-divisor graph is 1 for $m = 2p$ and for $m = p^k$. It was shown that the domination number for a zero-divisor graph is k for $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where α_i are positive integers.

We now determine the domination number of a weak zero-divisor graph of the ring of integers $\mathbb{Z}_m$.

Theorem 3.3. *Let $m = p_1^{t_1} p_2^{t_2} \dots p_t^{t_k}$. Let $\Omega(\mathbb{Z}_m)$ be the weak zero-divisor graph of $\mathbb{Z}_m$ and $\gamma(\Omega(\mathbb{Z}_m))$ be the domination number of $\Omega(\mathbb{Z}_m)$:*

(a) *if $m = p^t$, then $\gamma(\Omega(\mathbb{Z}_m)) = 1$;*

(b) *if $m = 2p$, then $\gamma(\Omega(\mathbb{Z}_m)) = 1$;*

(c) *if $m = p_1^{t_1} p_2^{t_2} \dots p_t^{t_k}$, where $t_j > 1$ for $1 \le j \le k$, then $\gamma(\Omega(\mathbb{Z}_m)) = 1$;*

(d) *if $m = p_1 p_2 \dots p_t$, then $\gamma(\Omega(\mathbb{Z}_m)) = t$.*

Proof. (a): This holds since every nilpotent element in $\mathbb{Z}_{p^m}$ dominates every vertex of its weak zero-divisor graph.

(b) and (d): The weak zero-divisor graph of $\mathbb{Z}_m$ are isomorphic to the zero-divisor graph for $m = 2p$ and $m = p_1 p_2 \dots p_t$. The results hold by [1, Theorem 1].

(c) Since $\mathbb{Z}_m$ is not square-free, it has nilpotent elements. Each of the nonzero nilpotent elements represents a vertex of degree $p^m - 2$. Therefore, the domination number is 1. □

We recall additional definitions. A path in a graph G that contains every vertex of G is called a Hamiltonian path. A Hamiltonian cycle contains every vertex of G. A graph G is said to be Hamiltonian if it has a Hamiltonian cycle. A connected graph G is called an Eulerian graph if it contains an Eulerian circuit; i.e if it has a circuit that contains every edge of G with no repeated edge.

From [9, Theorems 7.1 and 7.2], we know that the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is Hamiltonian for $n = p_1^{e_1} p_2^{e_2}$ and $n = p_1^2 p_2 \dots p_r$. It is also known from [8] that the zero-divisor graph of $\mathbb{Z}_m$ is Eulerian if and only if m is odd and square-free or $n = 4$. Below are some results on Hamiltonian and Eulerian structures of weak zero-divisor graphs of $\mathbb{Z}_m$.

Theorem 3.4. *The weak zero-divisor graph $\gamma(\Omega(Z_{2^m}))$ contains a Hamiltonian path, but is not Hamiltonian.*

Proof. The ring of integers $\mathbb{Z}_{2^m}$ has 2^{m-1} units and $2^{m-1} - 1$ nonzero nilpotents. Let u_i be the units and z_j be the nilpotents, where $1 \le i \le 2^m$ and $1 \le j \le 2^{m-1} - 1$. Note that each nilpotent is adjacent to all the units, and the units are not adjacent to each other. Form the path $u_1 - z_1 - u_2 - z_2 - \dots - z_{2^m - 1} - u_{2^m}$ alternating vertices between units and nilpotents. This path is Hamiltonian, but the graph is not. □

We invoke the following theorem for our result on the Eulerian structure of weak zero-divisor graphs.

Theorem 3.5 ([9, Theorem 3.1])**.** *$\Gamma(\mathbb{Z}_m)$ is Eulerian if and only if m is odd and square-free or $m = 4$.*

From this theorem, we conclude that the weak zero-divisor graph of $\mathbb{Z}_m$ is Eulerian if and only if $m = p_1 p_2 p_3 \dots p_k$, where p_i for $1 \le i \le k$ are all odd prime and $p_i \ne p_j$ for $i \ne j$.

We also note that, if $m = 4$, the weak zero-divisor graph of $\mathbb{Z}_m$ has an Eulerian trail but is not Eulerian.

We close this section with this result on $\mathbb{Z}_m$ for odd m.

Theorem 3.6. *Let $\mathbb{Z}_m$ be the ring of integers where m is odd. If $\mathbb{Z}_m$ has exactly two nonzero nilpotent elements and the nilpotent elements are the only zero-divisors, then the weak zero-divisor graph of $\mathbb{Z}_m^*$ has an Eulerian trail.*

Proof. We note that since m is odd, the vertices represented by the two nonzero nilpotent elements have odd degrees. Also, since there are no other zero-divisors, the remaining elements are all units with degree 2 being each adjacent to the two nilpotent vertices. Therefore, by [8, Theorem 3.2], since the two nonzero nilpotent elements are the only vertices with odd degrees, the weak zero-divisor graph has an Eulerian trail. □

An example of a weak zero-divisor graph containing an Eulerian trail is given in Figure 3.

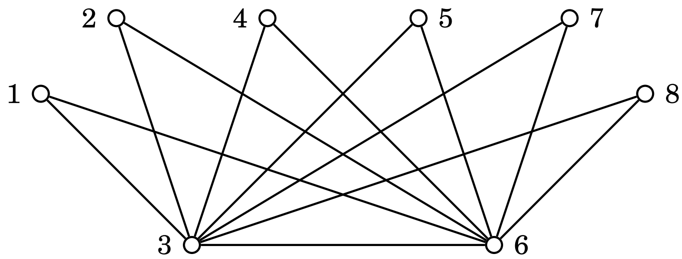

Figure 3. Weak Zero-Divisor Graph of $\mathbb{Z}_9$

4. Conclusion

In this paper, we described weak zero-divisor graphs of finite commutative rings with unity and studied some of their structural properties. We also investigated certain properties of weak zero-divisor graphs of rings of integers $\mathbb{Z}_m$ as examples of finite commutative rings with unity. Among all the results found, the diameter of a weak zero-divisor graph is particularly interesting. We observe that it is less than the diameter of a zero-divisor graph. We also found that the graph of a weak zero-divisor graph is not Eulerian, but it does have an Eulerian trail. Further research on this topic will include the structure of weak zero-divisor graphs of non-commutative rings.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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