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Research Article

A Novel Approach to Solve Nonlinear Higher Order VFIDE Using the Laplace Transform and Adomian Decomposition Method

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Abstract. This study explores the application of a novel approach called the *Laplace Discrete Modified Adomian Decomposition Method* (LDMADM) to solve non-homogeneous higher-order nonlinear VFIDEs. LDMADM is an extension of the *Laplace Modified Adomian Decomposition Method* (LMADM) and combines it with quadrature integration criteria to improve accuracy. The proposed method is evaluated by comparing its results with exact solutions and calculating absolute error measurements. The study establishes the existence of unique solutions and presents experimental, numerical findings that demonstrate the high accuracy and effectiveness of the LDMADM approach. This method offers a promising alternative to analytical approaches for solving higher-order nonlinear *Volterra Fredholm-type Integro Differential Equations* (VFIDEs).

Keywords. Integro differential equation (IDE), Volterra Fredholm-type integro differential equation (VFIDE), Modified Adomian Decomposition Method (MADM), Laplace Discrete Adomian Decomposition Method (LDADM)

Mathematics Subject Classification (2020). 45J05, 65R20

1. Introduction

Functional equations, such as integral and *integro-differential equations* (IDEs), *partial differential equations* (PDEs), stochastic equations, and others are typically produced when real-world issues are mathematically modeled. The IDEs have piqued the interest of physicists and mathematicians more than other types of equations because they are effective at describing a variety of real-world dynamical phenomena that arise in scientific and engineering fields like biology, physics, electrochemistry, economy, chemistry, control theory, electromagnetic, viscoelasticity, and chemical kinetics (Alkan and Hatipoglu [3], Hamoud and Ghadle [11], and Saha *et al*. [16]). Since it is frequently challenging to solve integro-differential equations analytically, it is necessary to find an effective approximation.

The Chebyshev collocation method, BEM with piecewise linear approximation, Runge-Kutta method, Galerkin method, Taylor collection method, Galerkin methods with hybrid functions, rationalised Haar functions method, and ADM can all be used to solve IEs and IDEs (Bakodah *et al*. [5], and Hamoud and Ghadle [10]) with some basic functions. Khuri [13] employed the Laplace transform numerical scheme in addition to these numerical methods. Additionally, several authors have explored the characteristics of the IDEs.

The ADM and LADM methods have a variety of applications, including solving differential equations(DEs), PDEs, IEs, and IDEs (Dawood *et al*. [6], Daoud and Khidir [7], and Sarkar and Sen [17]).

The LMADM is renowned for its quick convergence of solutions and for using few iterations, as effectively demonstrated. The MADM and LMADM methods are used to solve DEs [4], [12], PDEs, IEs, and IDEs, nonlinear boundary value problems (Abbasbandy [1], Ahmed *et al*. [2], Daoud and Khidir [7], Duan *et al*. [8], Kumar and Singh [14], and Ramana and Prasad [15]).

It appears that there is always room for improvement in the LMADM approach, particularly in discretizing the MADM.

The goal of this study is to extend the LMADM approach for solving nonlinear higher-order VFIDEs by discretizing the MADM first, then connecting various numerical integration schemes or quadrature rules. This paper will focus on the higher-order nonlinear VFIDE of second kind of the form:

$$v^{(n)}(x)=g(x)+\int_0^x \Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt+\int_0^a \Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt, \tag{1.1}$$

with the initial conditions:

$$v^{(k)}(0)=\alpha_k, \quad \text{for } 0\le k\le (n-1), \tag{1.2}$$

where L_1, L_2 are the linear functions of $v(t)$ and N_1, N_2 are nonlinear functions of $v(t)$.

2. Preliminaries

2.1 Definition [9]

The Laplace transform of a function $v(t)$ is denoted by $\mathcal{L}\{v(t)\}$ or $V(s)$ and it is defined by the integral

$$V(s)=\mathcal{L}\{v(t)\}=\int_0^\infty e^{-st}v(t)dt, \tag{2.1}$$

for those s where the integral converges. Here s is allowed to take complex values.

2.2 Adomian Decomposition Method (ADM) [8]

We provide some basic information regarding the Adomian decomposition method in this section.

Consider the DEs of the form:

$$LV + RV + NV = h(x), \tag{2.2}$$

where L is the highest order derivative of the linear operator, R is the remainder of the linear operator, which includes derivatives of lower order than L, and NV denotes the non-linear terms and h is the source term. Equation (2.2) can be rewritten as:

$$LV = h(x) - RV - NV. \tag{2.3}$$

Using the above conditions and the inverse operator L^{-1} on both sides of equation (2.3), we obtain

$$V = L^{-1}\{h(x)\} - L^{-1}(RV) - L^{-1}(NV). \tag{2.4}$$

A function $g(x)$ is defined in the equation after integrating the source term and adding it to the terms resulting from the problem's stated conditions

$$V = g(x) - L^{-1}(RV) - L^{-1}(NV). \tag{2.5}$$

For linear part of V, we put

$$V(x) = \sum_{k=0}^{\infty} v_k(x). \tag{2.6}$$

An infinite series of Adomian polynomials created specifically for the given non linearity serve as the representation for the nonlinear operator $FV = NV$, assuming NV is analytic

$$F(v) = \sum_{k=0}^{\infty} A_k, \tag{2.7}$$

where A_k's are given by:

$$\begin{aligned}
A_0 &= F(v_0),\\
A_1 &= v_1 F'(v_0),\\
A_2 &= v_2 F'(v_0) + \frac{1}{2} v_1^2 F''(v_0),\\
A_3 &= v_3 F'(v_0) + v_1 v_2 F''(v_0) + \frac{1}{3!} v_1^3 F'''(v_0),\\
A_4 &= v_4 F'(v_0) + \left(\frac{1}{2!} v_2^2 + v_1 v_3\right) F''(v_0) + \frac{1}{2!} v_1^2 v_2 F'''(v_0) + \frac{1}{4!} v_1^4 F^{(iv)}(v_0),\\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

The following approach generates the polynomial A_k's for all types of non-linearity so that they only depend on v_0 to v_k's:

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[F\left(\sum_{i=0}^{k} \lambda^i v_i \right) \right]_{\lambda=0}. \tag{2.8}$$

3. Results

Integrating n-times of eqn. (1.1) in the interval $[0,x]$ with respect to x we obtain,

$$v(x)=D^{-1}g(x)+\sum_{r=0}^{n-1}\frac{x^r}{r!}\alpha_r+D^{-1}\int_0^x\Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt$$
$$+D^{-1}\int_0^a\Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt, \tag{3.1}$$

where D^{-1} is the multiple integration operator given as follows:

$$D^{-1}(\cdot)=\int_0^x\int_0^x\int_0^x\cdots\int_0^x(\cdot)dx\,dx\,dx\cdots dx(n\text{-times}). \tag{3.2}$$

Let $\psi(x)=D^{-1}g(x)+\sum_{r=0}^{n-1}\frac{x^r}{r!}\alpha_r$, thus eqn. (3.1) becomes

$$v(x)=\psi(x)+D^{-1}\int_0^x\Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt+D^{-1}\int_0^a\Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt, \tag{3.3}$$

where

$$D^{-1}\int_0^x\Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt=\int_0^x\frac{(x-t)^n}{n!}\Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt. \tag{3.4}$$

To establish the result for the existence of unique solution to the considered type problem we are using the following assumptions:

(A1) There exists four constants $\beta_1,\beta_2,\beta,\gamma,\gamma_1$ and γ_2 such that for any $v_1,v_2\in C([0,a],\mathbb{R})$

$$|L_1(v_1)-L_1(v_2)|\le\beta_1|v_1-v_2|,\quad |N_1(v_1)-N_1(v_2)|\le\beta_2|v_1-v_2|,$$
$$|L_2(v_1)-L_2(v_2)|\le\gamma_1|v_1-v_2|,\quad |N_2(v_1)-N_2(v_2)|\le\gamma_2|v_1-v_2|$$

and $\beta=\beta_1+\beta_2$, $\gamma=\gamma_1+\gamma_2$.

(A2) Suppose for all $x\in[0,a]$

$$\left|\frac{(x-t)^n}{n!}\Omega_1(x,t)\right|\le\theta_1 \quad\text{and}\quad |D^{-1}\Omega_2(x,t)|\le\theta_2.$$

(A3) $\psi(x)$ is bounded function for all $x\in[0,a]$.

Theorem 3.1. *Suppose that assumptions* (A1)-(A3) *hold. If*

$$\lambda=(\theta_1\beta+\theta_2\gamma)a<1, \tag{3.5}$$

then there exists a unique solution $v(x)\in C([0,a])$ to the IVP (1.1) *and* (1.2).

Proof. Let v_1 and v_2 be two different solutions of the IVP (1.1) and (1.2), then

$$|v_1-v_2|=\left|\int_0^x\frac{(x-t)^n}{n!}\Omega_1(x,t)[L_1(v_1(t))-L_1(v_2(t))+N_1(v_1(t))-N_1(v_2(t))]dt\right.$$
$$\left.+D^{-1}\int_0^a\Omega_2(x,t)[L_2(v_1(t))-L_2(v_2(t))+N_2(v_1(t))-N_2(v_2(t))]dt\right|$$
$$\le\int_0^x\left|\frac{(x-t)^n}{n!}\Omega_1(x,t)\right|[|L_1(v_1(t))-L_1(v_2(t))|+|N_1(v_1(t))-N_1(v_2(t))|]dt$$

$$+\int_0^a |D^{-1}\Omega_2(x,t)|[|L_2(v_1(t))-L_2(v_2(t))|+|N_2(v_1(t))-N_2(v_2(t))|]dt$$
$$\le [\theta_1(\beta_1+\beta_2)x+\theta_2(\gamma_1+\gamma_2)a]|v_1-v_2|$$
$$\le [\theta_1(\beta_1+\beta_2)+\theta_2(\gamma_1+\gamma_2)]a|v_1-v_2|$$
$$\le (\theta_1\beta+\theta_2\gamma)a|v_1-v_2|.$$

This implies,

$$(1-\lambda)|v_1-v_2|\le 0. \tag{3.6}$$

Since $1-\lambda>0$, so $|v_1-v_2|=0$. Therefore, $v_1=v_2$ and this completes the proof. □

4. Description of the Method

The development of more advanced and effective approaches for Higher-order nonlinear VFIDE, such as the LDMADM, has received significant attention. In this part, we will explain this technique.

4.1 Laplace Discrete Modified Adomian Decomposition Method

We know,

$$\mathcal{L}\{v'(x)\}=s\mathcal{L}\{v(x)\}-v(0), \tag{4.1}$$

more generally,

$$\mathcal{L}\{v^{(n)}(x)\}=s^n\mathcal{L}\{v(x)\}-s^{n-1}v(0)-s^{n-2}v'(0)-\cdots-v^{(n-1)}(0). \tag{4.2}$$

Write $g(x)$ as a sum of two functions, say $g_1(x)$ and $g_2(x)$. Then, the eqns. (1.1) and (1.2) becomes

$$v^{(n)}(x)=g_1(x)+g_2(x)+\int_0^x \Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt+\int_0^a \Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt, \tag{4.3}$$

with the initial conditions:

$$v^{(k)}(0)=\alpha_k,\quad \text{for } 0\le k\le (n-1). \tag{4.4}$$

Thus, on applying the Laplace transform to both sides of eqn. (4.3), we obtain

$$\mathcal{L}\{v^{(n)}(x)\}=\mathcal{L}\Big\{g_1(x)+g_2(x)+\int_0^x \Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt$$
$$+\int_0^a \Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt\Big\}. \tag{4.5}$$

Using (4.2), we have

$$s^n\mathcal{L}\{v(x)\}-s^{n-1}v(0)-s^{n-2}v'(0)-\cdots-v^{(n-1)}(0)=\mathcal{L}\{g_1(x)\}+\mathcal{L}\{g_2(x)\}$$
$$+\mathcal{L}\Big\{\int_0^x \Omega_1(x,t)[L_1(v(t))+N_1(v(t))]dt$$
$$+\int_0^a \Omega_2(x,t)[L_2(v(t))+N_2(v(t))]dt\Big\}$$

$$\implies \quad \mathcal{L}\{v(x)\} = \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2} - \frac{\alpha_2}{s^3} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{\alpha_{n-1}}{s^n} + \frac{1}{s^n}\mathcal{L}\{g_1(x)\} + \frac{1}{s^n}\mathcal{L}\{g_2(x)\} + \frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)[L_1(v(t)) + N_1(v(t))]dt + \int_0^a \Omega_2(x,t)[L_2(v(t)) + N_2(v(t))]dt\right\}. \tag{4.6}$$

In the decomposition approach, the solution $v(x)$ is represented as a series of the form:

$$v(x) = \sum_{m=0}^{\infty} v_m(x), \tag{4.7}$$

and the nonlinear term $N_1(v(t))$ and $N_2(v(t))$ are decomposed into an infinite series of the form

$$N_1(v(t)) = \sum_{i=o}^{\infty} A_i(t) \text{ and } N_2(v(t)) = \sum_{i=o}^{\infty} B_i(t), \tag{4.8}$$

where A_i and B_i are the Adomian polynomials of $v_0, v_1, v_2, \dots, v_i$, given by the formula

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_1\left(\sum_{i=0}^{k} \lambda^i v_i\right)\right]_{\lambda=0} \quad \text{and} \quad B_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_2\left(\sum_{i=0}^{k} \lambda^i v_i\right)\right]_{\lambda=0}. \tag{4.9}$$

By using eqns. (4.7), (4.8) and (4.9) in eqn. (4.6), we get

$$\mathcal{L}\left\{\sum_{m=0}^{\infty} v_m(x)\right\} = \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2} - \frac{\alpha_2}{s^3} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{\alpha_{n-1}}{s^n} + \frac{1}{s^n}\mathcal{L}\{g_1(x)\} + \frac{1}{s^n}\mathcal{L}\{g_2(x)\} + \frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)\left[L_1\left(\sum_{m=0}^{\infty} v_m(t)\right) + \sum_{i=o}^{\infty} A_i(t)\right]dt + \int_0^a \Omega_2(x,t)\left[L_2\left(\sum_{m=0}^{\infty} v_m(t)\right) + \sum_{i=o}^{\infty} B_i(t)\right]dt\right\}. \tag{4.10}$$

On comparing between the right and left hand sides of the eqn. (4.10) we thus obtain:

$$\mathcal{L}\{v_0(x)\} = \frac{\alpha_0}{s} + \frac{\alpha_1}{s^2} - \frac{\alpha_2}{s^3} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{\alpha_{n-1}}{s^n} + \frac{1}{s^n}\mathcal{L}\{g_1(x)\}, \tag{4.11}$$

$$\mathcal{L}\{v_1(x)\} = \frac{1}{s^n}\mathcal{L}\{g_2(x)\} + \frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)[L_1(v_0(t)) + A_0(t)]dt\right\} + \frac{1}{s^n}\mathcal{L}\left\{\int_0^a \Omega_2(x,t)[L_2(v_0(t)) + B_0(t)]dt\right\}, \tag{4.12}$$

and for $m \geq 1$,

$$\mathcal{L}\{v_{m+1}(x)\} = \frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)[L_1(v_m(t)) + A_m(t)]dt + \int_0^a \Omega_2(x,t)[L_2(v_m(t)) + B_m(t)]dt\right\}. \tag{4.13}$$

By using the inverse laplace transform of eqn. (4.11) we may obtain $v_0(x)$, and consequently A_0, B_0 will be obtained. Also, using A_0, B_0 we can evaluate $v_1(x)$. The obtained values of $v_0(x)$ and $v_1(x)$ will helps to determine of A_1, B_1 that will allow to find $v_2(x)$, and so on. The recursive relation is defined by

$$v_0(x) = \mathcal{L}^{-1}\left\{\frac{\alpha_0}{s} + \frac{\alpha_1}{s^2} - \frac{\alpha_2}{s^3} + \cdots + \frac{\alpha_{n-2}}{s^{n-1}} + \frac{\alpha_{n-1}}{s^n}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{g_1(x)\}\right\}, \tag{4.14}$$

$$v_1(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\{g_2(x)\}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)[L_1(v_0(t)) + A_0(t)]dt\right\}\right\}$$
$$+\mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^a \Omega_2(x,t)[L_2(v_0(t)) + B_0(t)]dt\right\}\right\}, \tag{4.15}$$

and for $m \geq 1$,

$$v_{m+1}(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\mathcal{L}\left\{\int_0^x \Omega_1(x,t)[L_1(v_m(t)) + A_m(t)]dt\right.\right.$$
$$\left.\left.+\int_0^a \Omega_2(x,t)[L_2(v_m(t)) + B_m(t)]dt\right\}\right\}. \tag{4.16}$$

Hence the solution of the given problem is

$$v(x) = v_0(x) + v_1(x) + v_2(x) + \cdots + v_m(x) + \cdots. \tag{4.17}$$

5. Numerical Examples

In this section, we discussed some numerical example based on the ADM and LDMADM.

Example 5.1. Consider the VFIDE,

$$v^{(v)}(x) = xe^{-x} - 2\cosh x - \int_0^x e^{t-x}v(t)dt + \int_0^1 e^{x+3t}v^3(t)dt, \tag{5.1}$$

with the initial conditions:

$$v(0) = 1, v'(0) = -1,\ v''(0) = 1,\ v'''(0) = -1 \text{ and } v^{(iv)}(0) = 1. \tag{5.2}$$

where $v(x) = e^{-x}$ is the exact solution.

Here, $g(x) = xe^{-x} - 2\cosh x$, $\Omega_1(x,t) = -e^{t-x}$, $\Omega_2(x,t) = e^{x+3t}$, $L_1(v(t)) = v(t)$, $L_2(v(t)) = 0$, $N_1(v(t)) = 0$ and $N_2(v(t)) = v^3(t)$. So, $g(x) = xe^{-x} - 2\cosh x = xe^{-x} - e^x - e^{-x} = -e^{-x} + xe^{-x} - e^x$.

Choose, $g_1(x) = xe^{-x}$ and $g_2(x) = -2\cosh(x) = -e^x - e^{-x} = -(e^x + e^{-x})$.

Applying Laplace transform of equation (5.1),

$$\mathcal{L}\{v^{(v)}(x)\} = \mathcal{L}\{xe^{-x}\} - \mathcal{L}\{e^x + e^{-x}\} - \mathcal{L}\left\{\int_0^x e^{t-x}v(t)dt\right\} + \mathcal{L}\left\{\int_0^1 e^{x+3t}v^3(t)dt\right\}$$
$$\implies \mathcal{L}\{v(x)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^5(s+1)^2} - \frac{1}{s^5(s-1)} - \frac{1}{s^5(s+1)}$$
$$-\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x}v(t)dt\right\} + \frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t}v^3(t)dt\right\}. \tag{5.3}$$

Now applying Modified Adomian decomposition on (5.3), we get

$$\mathcal{L}\left\{\sum_{m=0}^{\infty} v_m(x)\right\} = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5} + \frac{1}{s^5(s+1)^2} - \frac{1}{s^5(s-1)} - \frac{1}{s^5(s+1)}$$
$$-\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x}\sum_{m=0}^{\infty} v_m(t)dt\right\} + \frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t}\sum_{m=0}^{\infty} B_m(t)dt\right\}, \tag{5.4}$$

where

$$v(x) = \sum_{m=0}^{\infty} v_m(x) \quad \text{and} \quad B_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_2\left(\sum_{i=0}^{k}\lambda^i v_i\right)\right]_{\lambda=0}. \tag{5.5}$$

Taking inverse Laplace transform of equation (5.4), we obtain

$$\sum_{m=0}^{\infty} v_m(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s^4}+\frac{1}{s^5}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)^2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^5(s-1)}\right\}$$
$$-\mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x}\sum_{m=0}^{\infty} v_m(t)dt\right\}\right\}$$
$$+\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t}\sum_{m=0}^{\infty} B_m(t)dt\right\}\right\}. \tag{5.6}$$

On comparing both sides of equation (5.6), we get

$$v_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s^3}-\frac{1}{s^4}+\frac{1}{s^5}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)^2}\right\}$$
$$= 6-5x+2x^2-\frac{x^3}{2}+\frac{x^4}{24}-5e^{-x}-xe^{-x}, \tag{5.7}$$
$$v_1(x) = -\mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^5(s-1)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x}v_0(t)dt\right\}\right\}$$
$$+\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t}B_0(t)dt\right\}\right\} \tag{5.8}$$

and

$$v_{m+1}(x) = -\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x}v_m(t)dt\right\}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t}B_m(t)dt\right\}\right\}, \quad \text{for } m \geq 1. \tag{5.9}$$

The numerical results are given in Table 1.

Table 1. Numerical results of Example 5.1

Value of x	Exact value of $v(x)$	ADM	LDMADM	error (ADM)	error (LDMADM)
0	1.0000000000000	0.999998999980947	0.999999999984941	1.00×10^{-5}	1.50×10^{-11}
0.1	0.904837418035960	0.904837518017604	0.904837418020688	9.99×10^{-8}	1.52×10^{-11}
0.2	0.818730753077982	0.818731753069754	0.818730753062487	9.99×10^{-7}	1.54×10^{-11}
0.3	0.740818220681718	0.740815220650577	0.740818220665667	3×10^{-6}	1.60×10^{-11}
0.4	0.670320046035639	0.670311046101893	0.670320046017837	8.99×10^{-6}	1.78×10^{-11}
0.5	0.606530659712633	0.606532759643347	0.606530659690178	2.09×10^{-6}	2.24×10^{-11}
0.6	0.548811636094026	0.548811605943781	0.548811636061119	3.08×10^{-8}	3.29×10^{-11}
0.7	0.496585303791410	0.496586313480114	0.496585303737787	1×10^{-6}	5.36×10^{-11}
0.8	0.449328964117222	0.449325963554682	0.449328964026174	3×10^{-6}	9.10×10^{-11}
0.9	0.406569659740599	0.406569858637504	0.406569659586523	1.98×10^{-7}	1.54×10^{-10}
1	0.367879441171442	0.367875419296401	0.367879440916897	4.02×10^{-6}	2.54×10^{-10}

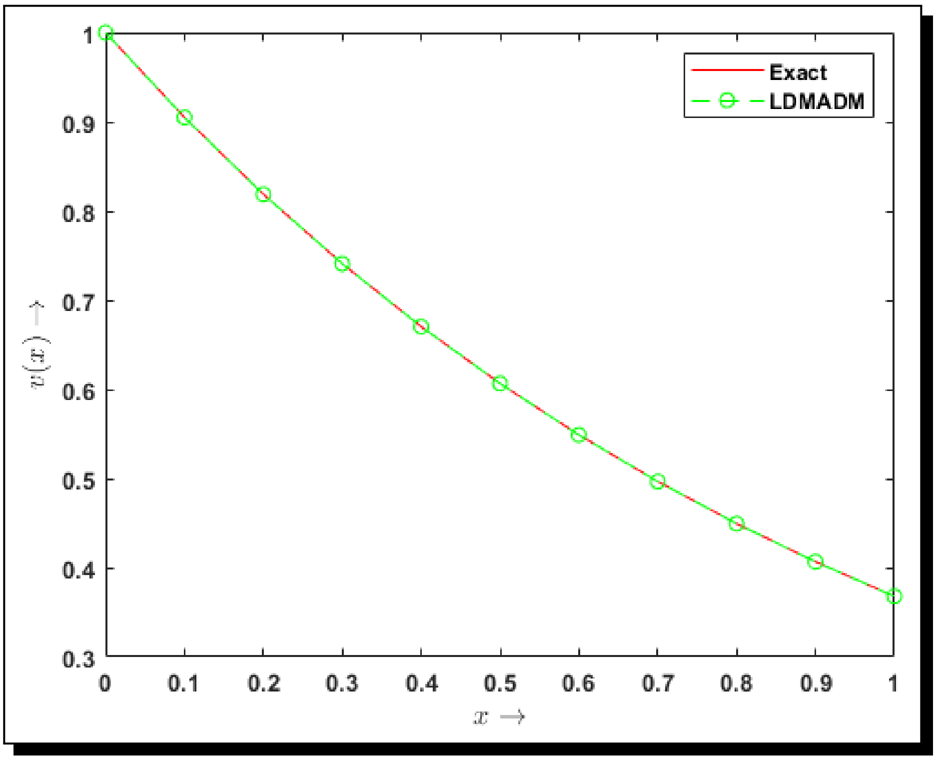

Figure 1. Numerical results of Example 5.1

But, if we choose $g_1(x) = -e^{-x}$ and $g_2(x) = xe^{-x} - e^{x}$, then from equation (5.6) formula for the recursive relationship is

$$v_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \frac{1}{s^5}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)}\right\} = e^{-x},$$

$$\begin{aligned} v_1(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^5(s-1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x} v_0(t)dt\right\}\right\} \\ &\quad + \mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t} B_0(t)dt\right\}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^5(s-1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^5(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^5(s-1)}\right\} \\ &= 0 \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} v_{m+1}(x) &= -\mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^x e^{t-x} v_m(t)dt\right\}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^1 e^{x+3t} B_m(t)dt\right\}\right\} \\ &= 0 \quad \text{for } m \geq 1. \quad (\text{Since } v_1 = 0 \text{ implies } B_1 = 0 \text{ and consequently so on.}) \end{aligned} \tag{5.11}$$

Hence the solution is

$$v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots = e^{-x},$$

which is the exact solution.

Example 5.2. Consider the VFIDE:

$$v^{(iv)}(x)=1+e^x-xe^x+\int_0^x e^{x-2t}[v(t)+v^2(t)]dt-\int_0^1 e^{x-3t}v^3(t)dt \tag{5.12}$$

with the initial conditions: $v(0)=v'(0)=v''(0)=v'''(0)=1$, where $v(x)=e^x$ is the exact solution.

Here, $g(x)=1+e^x-xe^x$, $\Omega_1(x,t)=e^{x-2t}$, $\Omega_2(x,t)=-e^{x-3t}$, $L_1(v(t))=v(t)$, $L_2(v(t))=0$, $N_1(v(t))=v^2(t)$ and $N_2(v(t))=v^3(t)$.

Choose, $g_1(x)=1+e^x$ and $g_2(x)=-xe^x$.

Applying Laplace transform on (5.12), we get

$$\mathcal{L}\{v^{(iv)}(x)\}=\mathcal{L}\{1+e^x\}-\mathcal{L}\{xe^x\}+\mathcal{L}\left\{\int_0^x e^{x-2t}[v(t)+v^2(t)]dt\right\}-\mathcal{L}\left\{\int_0^1 e^{x-3t}v^3(t)dt\right\}$$

$$\Longrightarrow \quad \mathcal{L}\{v(x)\}=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^5}+\frac{1}{s^4(s-1)}-\frac{1}{s^4(s-1)^2}+\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}[v(t)+v^2(t)]dt\right\}-\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}v^3(t)dt\right\}. \tag{5.13}$$

Now applying MADM, we get

$$\mathcal{L}\left\{\sum_{m=0}^{\infty}v_m(x)\right\}=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^5}+\frac{1}{s^4(s-1)}-\frac{1}{s^4(s-1)^2}+\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}\left[\sum_{m=0}^{\infty}v_m(x)+A_m(t)\right]dt\right\}-\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}B_m(t)dt\right\}, \tag{5.14}$$

where

$$v(x)=\sum_{m=0}^{\infty}v_m(x),\quad A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_1\left(\sum_{i=0}^{k}\lambda^i v_i\right)\right]_{\lambda=0} \quad \text{and} \quad B_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[N_2\left(\sum_{i=0}^{k}\lambda^i v_i\right)\right]_{\lambda=0}. \tag{5.15}$$

Applying inverse laplace transform on (5.14), we get

$$\sum_{m=0}^{\infty}v_m(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^5}+\frac{1}{s^4(s-1)}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^4(s-1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}\left(\sum_{m=0}^{\infty}v_m(x)+\sum_{m=0}^{\infty}A_m(t)\right)dt\right\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}\sum_{m=0}^{\infty}B_m(t)dt\right\}\right\}. \tag{5.16}$$

On comparing both sides of eqn. (5.16), formula for the recursive relationship is

$$v_0(x)=\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^5}+\frac{1}{s^4(s-1)}\right\}=e^x+\frac{x^4}{24}, \tag{5.17}$$

$$v_1(x)=-\mathcal{L}^{-1}\left\{\frac{1}{s^4(s-1)^2}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}(v_0(t)+A_0(t))dt\right\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}B_0(t)dt\right\}\right\}, \tag{5.18}$$

and for $m\geq 1$,

$$v_{m+1}(x)=\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}(v_m(x)+A_m(t))dt\right\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}B_m(t)dt\right\}\right\}. \tag{5.19}$$

The numerical results are given in Table 2.

Table 2. Numerical results of Example 5.2

Value of x	Exact value	ADM	LDMADM	error (ADM)	error (LDMADM)
0	1.00000000000	1.002173330213613	1.000012814118250	2.17×10^{-3}	1.28×10^{-5}
0.1	1.10517091808	1.103461661744839	1.105183946973069	1.7×10^{-3}	1.30×10^{-5}
0.2	1.22140275816	1.211789414102073	1.221416004951750	9.6×10^{-3}	1.32×10^{-5}
0.3	1.34985880758	1.359321204686033	1.349872275474366	9.4×10^{-3}	1.34×10^{-5}
0.4	1.49182469764	1.493344121624684	1.491838389997030	1.5×10^{-3}	1.37×10^{-5}
0.5	1.64872127070	1.659280542839274	1.648735190965107	1.05×10^{-2}	1.39×10^{-5}
0.6	1.82211880039	1.832702316188225	1.822132952067463	1.06×10^{-2}	1.41×10^{-5}
0.7	2.01375270747	2.016346442231609	2.013767094086068	2.6×10^{-3}	1.43×10^{-5}
0.8	2.22554092849	2.228132416143231	2.225555553613046	2.6×10^{-3}	1.46×10^{-5}
0.9	2.45960311116	2.450181401869930	2.459617978455912	9.4×10^{-3}	1.48×10^{-5}
1	2.71828182846	2.738837429959827	2.718296941837674	2.06×10^{-2}	1.51×10^{-5}

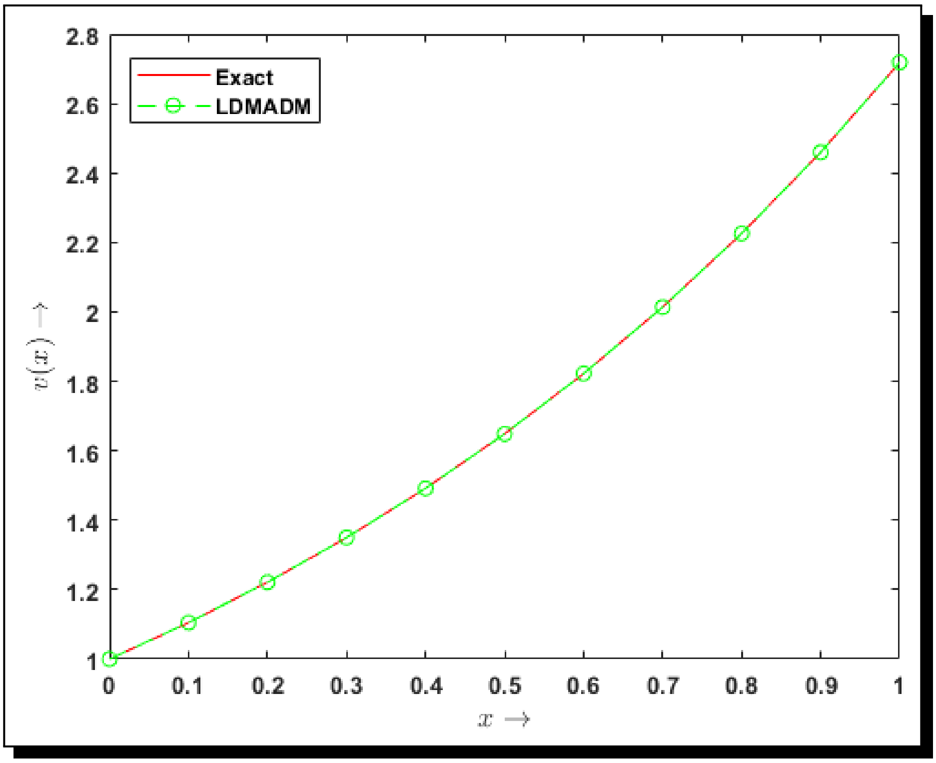

Figure 2. Numerical results of Example 5.2

But, if we choose $g_1(x) = e^x$ and $g_2(x) = 1 - xe^x$, then from eqn. (5.16) we get

$$v_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4(s-1)}\right\} = e^x, \tag{5.20}$$

$$\begin{aligned} v_1(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s^5} - \frac{1}{s^4(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}(v_0(t) + A_0(t))dt\right\}\right\} \\ &\quad - \mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}B_0(t)dt\right\}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^5} - \frac{1}{s^4(s-1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\left(\frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{s}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^4(s-1)}\right\} \\ &= 0, \end{aligned} \tag{5.21}$$

and for $m \geq 1$,

$$\begin{aligned} v_{m+1}(x) &= \mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^x e^{x-2t}(v_m(x) + A_m(t))dt\right\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 e^{x-3t}B_m(t)dt\right\}\right\} \\ &= 0. \quad \text{(Since } v_1 = 0 \text{ implies } A_1 = B_1 = 0 \text{ and consequently so on.)} \end{aligned} \tag{5.22}$$

Hence,

$$v(x) = v_0(x) + v_1(x) + v_2(x) + \ldots = e^x,$$

which is the exact solution.

6. Conclusion

In this study, we introduce a new modification to the MADM method based on the discretization property. We propose a Laplace Discrete Modified Adomian decomposition method (LDMADM) that can effectively solve nonlinear higher-order VFIDEs. The LDMADM method is shown to outperform the ADM method by providing approximate solutions with fewer computational steps, as demonstrated in Table 1, Table 2, Figure 1, and Figure 2. The results indicate that the LDMADM approach is both user-friendly and efficient. The existence of unique solutions guarantees that the solutions obtained are definitive and unambiguous.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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