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On Weak symmetries of Generalized Sasakian-Space-Forms

Research Article

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Abstract. The purpose of the paper is to study weakly symmetric and weakly Ricci-symmetric generalized Sasakian-space-forms. We consider the locally symmetric and recurrent type of weakly symmetric generalized Sasakian-space-forms. Also, locally Ricci-symmetric and Ricci-recurrent weakly Ricci-symmetric generalized Sasakian-space-forms are discussed.

Keywords. Generalized Sasakian-space-forms; Weakly symmetric; Weakly Ricci-symmetric; Specially weakly Ricci-symmetric

MSC. 53C15; 53C25

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1. Introduction

The notion of generalized Sasakian space forms was introduced and studied by P. Alegre et al. [1] with several examples. A generalized Sasakian-space-form is an almost contact metric manifold $M(\phi,\xi,\eta,g)$ whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

where f_1, f_2, f_3 are differentiable functions and X,Y,Z are vector fields on M. In such case we will write the manifold as $M(f_1,f_2,f_3)$. This kind of manifolds appears as a natural generalization of the Sasakian-space-forms: $f_1 = \frac{c+3}{4}$ and $f_1 = f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The ϕ-sectional curvature of generalized Sasakian-space-forms $M(f_1,f_2,f_3)$ is f_1+3f_2. Moreover, cosymplectic space-forms and Kenmotsu space-forms also consider as particular types of generalized Sasakian-space-forms. Generalized Sasakian-space-forms have been studied by many authors. For example see [2–4, 6, 9, 10, 12, 14].

The notion of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamassy and T.Q. Binh ([18] and [19]). These types of manifold were studied with different structures by several authors (see [7, 8, 11, 13, 15, 16]). In this connection, we would mention the works of Yadav and Suthar [20] on generalized Sasakian-space-forms.

The paper is organized as follows: Section 2 is devoted to preliminaries on generalized Sasakian-space-forms. In Section 3, we consider weakly symmetric generalized Sasakian-space-forms and study the characteristic properties of locally symmetric and recurrent spaces. Section 4 deals with the study on weakly Ricci-symmetric generalized Sasakian-space-forms. We study the characteristic properties of locally Ricci-symmetric and locally Ricci-recurrent spaces. Also, we show that special weakly Ricci-symmetric generalized Sasakian-space-forms cannot be locally Ricci-symmetric.

2. Preliminaries

In almost contact metric manifold we have [5]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0. \tag{2.3}$$

Again, for a $(2n+1)$-dimensional generalized Sasakian-space-form we have [1]

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \tag{2.4}$$

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{2.5}$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \tag{2.7}$$

where R and S are the curvature tensor and the Ricci tensor of the space-form, respectively.

3. Weakly Symmetric Generalized Sasakian-Space-Forms

In this section, we study the characterizations of locally symmetric and recurrent spaces.

Definition 3.1. Generalized Sasakian space form $M(f_1, f_2, f_3)$ $(n > 2)$ is called weakly symmetric if there exists 1-forms A, B, C, D and their curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,V) = {} & A(X)R(Y,Z,V) + B(Y)R(X,Z,V) + C(Z)R(Y,X,V) \\ & + D(V)R(Y,Z,X) + g(R(Y,Z,V),\, X)P,\end{aligned} \tag{3.1}$$

Definition 3.2. *A weakly symmetric generalized Sasakian space form* $M(f_1, f_2, f_3)$ $(n > 2)$ *is said to be locally symmetric, if*

$$\nabla R = 0.$$

Suppose a weakly symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ $(n > 2)$ is locally symmetric with $f_1 - f_3 \neq 0$. Then from (3.1) and Definition 3.2, we have

$$A(X)S(Z,V) + B(R(X,Z)V) + C(Z)S(X,V) + D(V)S(X,Z) + E(R(X,V)Z) = 0 \tag{3.2}$$

Replacing V by ξ in (3.2) and then using (2.5) and (2.7) we obtain

$$(f_1 - f_3)[(n-1)\{A(X)\eta(Z) + C(Z)\eta(X)\} + \{B(X)\eta(Z) - B(Z)\eta(X) + E(X)\eta(Z)\}$$
$$+ E(\xi)g(X,Z)] + D(\xi)S(X,Z) = 0. \tag{3.3}$$

Putting $X = Z = \xi$ in (3.3), we can easily get

$$(n-1)(f_1 - f_3)[A(\xi) + C(\xi) + D(\xi)] = 0$$

which implies that

$$A(\xi) + C(\xi) + D(\xi) = 0. \tag{3.4}$$

Next, plugging Z in (3.2) and then using (2.5) and (2.7) we have

$$(f_1 - f_3)[(n-1)\{A(X)\eta(V) + D(V)\eta(X)\} + \{B(X)\eta(V) + E(X)\eta(V) - E(V)\eta(X)$$
$$- B(\xi)g(X,V)\}] + C(\xi)S(X,V) = 0. \tag{3.5}$$

Setting $V = \xi$ in (3.5), we get

$$(f_1 - f_3)[(n-1)\{A(X) + D(\xi)\eta(X)\} + \{B(X) + E(X) - E(\xi)\eta(X)$$
$$- B(\xi)\eta(X)\} + (n-1)C(\xi)\eta(X)] = 0. \tag{3.6}$$

Similarly, if we set $X = \xi$ in (3.5) we obtain

$$(f_1 - f_3)[(n-1)\{A(\xi)\eta(V) + D(V)\} + \{E(\xi)\eta(V) - E(V)\} + (n-1)\{C(\xi)\eta(V)\}] = 0. \tag{3.7}$$

Replacing V by X in (3.7), we have

$$(f_1 - f_3)[(n-1)\{A(\xi)\eta(V) + D(V)\} + \{E(\xi)\eta(V) - E(V)\} + (n-1)\{C(\xi)\eta(V)\}] = 0. \tag{3.8}$$

Adding (3.6) and (3.8) and using (3.4), we have

$$(f_1 - f_3)[(n-1)\{A(X) - A(\xi)\eta(X)\} + \{B(X) - B(\xi)\eta(X)\} + \{D(X) - D(\xi)\eta(X)\}] = 0. \tag{3.9}$$

Next, putting $X = \xi$ in (3.3), we have

$$(f_1 - f_3)[(n-1)\{C(Z) - C(\xi)\eta(Z)\} + \{B(\xi)\eta(Z) - B(Z)\}] = 0. \tag{3.10}$$

Replacing Z by X in above equation and then adding with equation (3.9), we get

$$A(X) + C(X) + D(X) = 0.$$

Hence we are able to state the following;

Theorem 3.1. *If a weakly symmetric generalized Sasakian space form* $M(f_1,f_2,f_3)$ $(n>2)$ *with* $f_1-f_3\neq 0$ *is locally symmetric, then the sum of the associated* 1*-forms* A*,* C *and* D *is zero everywhere.*

Definition 3.3. *A weakly symmetric generalized Sasakian space form* $M(f_1,f_2,f_3)$ $(n>2)$ *is said to recurrent if*

$$\nabla R = A\otimes R\,.$$

On the other hand, let us consider a weakly symmetric generalized Sasakian space forms $M(f_1,f_2,f_3)$ $(n>2)$ with $f_1-f_3\neq 0$ is recurrent, then from (3.1) and Definition 3.3 we find,

$$B(Y)R(X,Z,V)+C(Z)R(Y,X,V)+D(V)R(Y,Z,X)+g(R(Y,Z,V),X)\rho=0. \tag{3.11}$$

Next, putting $X=Y=Z=V=\xi$ in (3.11) and then using (2.5), we obtain

$$C(\xi)+D(\xi)=0.$$

Further proceeding as in the proof of the previous theorem and using the fact that $C(\xi)+D(\xi)=0$, obviously, one can get $C(X)+D(X)=0$ for any vector field X on $M(f_1,f_2,f_3)$, so that $C+D=0$ everywhere on M. Hence we state the following result:

Theorem 3.2. *If a weakly symmetric generalized Sasakian space form* $M(f_1,f_2,f_3)$ $(n>2)$ *with* $f_1-f_3\neq 0$ *is recurrent, then the* 1*-forms* C *and* D *are in the opposite direction.*

4. Weakly Ricci-Symmetric Generalized Sasakian Space Forms

In this section, we investigate characterizations of locally Ricci-symmetric and Ricci-recurrent spaces.

Definition 4.1. *A generalized Sasakian space form* $M(f_1,f_2,f_3)$ $(n>2)$ *called weakly Ricci-symmetric if there exist* 1*-forms* α*,* β *and* γ *and their Ricci tensor* S *of type* $(0,2)$ *satisfies the conditions*

$$(\nabla_X S)(Y,Z)=\alpha(X)S(Y,Z)+\beta(Y)S(X,Z)+\gamma(Z)S(Y,X) \tag{4.1}$$

for all vector fields X*,* Y *and* Z *on* $M(f_1,f_2,f_3)$*.*

Definition 4.2. *A weakly Ricci-symmetric generalized Sasakian space form* $M(f_1,f_2,f_3)$ $(n>2)$ *is said to be locally Ricci-symmetric if*

$$\nabla S=0$$

Let us consider a weakly Ricci-symmetric generalized Sasakian space form $M(f_1,f_2,f_3)$ $(n>2)$ with $f_1-f_3\neq 0$ which is locally Ricci-symmetric. Then by virtue of (4.1) and Definition 4.2, we have

$$\alpha(X)S(Y,Z)+\beta(Y)S(X,Z)+\gamma(Z)S(Y,X)=0. \tag{4.2}$$

Setting $X = Y = Z = \xi$ in (4.2), we find

$$\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0. \tag{4.3}$$

Now taking $Y = Z = \xi$ in (4.2) and then using (2.7), we get

$$\alpha(X) - \alpha(\xi)\eta(X) = 0. \tag{4.4}$$

In a similar manner, we can obtain

$$\beta(X) - \beta(\xi)\eta(X) = 0 \tag{4.5}$$

and

$$\gamma(X) - \gamma(\xi)\eta(X) = 0. \tag{4.6}$$

Adding (4.4), (4.5) and (4.6) and using (4.3) we obtain

$$\alpha(X) + \beta(X) + \gamma(X) = 0, \tag{4.7}$$

for all X on $M(f_1, f_2, f_3)$. Thus we state following:

Theorem 4.1. *In weakly Ricci-symmetric generalized Sasakian-space-form $M(f_1, f_2, f_3)$ $(n > 2)$ with $f_1 - f_3 \neq 0$ is locally Ricci-symmetric, then the sum of associated 1-forms α, β and γ is zero everywhere.*

If in (4.1) the 1-form α is replaced by 2α and β and γ are equal to α then we have

$$(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(Y, X) \tag{4.8}$$

where α is a non-zero 1-form defined by $\alpha(X) = g(X, \rho)$. A manifold which satisfies (4.8) is called a specially weakly Ricci-symmetric manifold (see [17]).

Suppose that $M(f_1, f_2, f_3)$ $(n > 2)$ is a specially weakly Ricci-symmetric generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is locally Ricci-symmetric, then from (4.7), we have

$$2\alpha(X) + \alpha(X) + \alpha(X) = 0,$$

for any X on $M(f_1, f_2, f_3)$, that is $\alpha(X) = 0$. Which is contradicts the definition. Hence $M(f_1, f_2, f_3)$ can not be locally Ricci-symmetric. This gives us to state:

Theorem 4.2. *Let $M(f_1, f_2, f_3)$ $(n > 2)$ be a specially weakly Ricci-symmetric generalized Sasakian-space-form with $f_1 - f_3 \neq 0$. Then $M(f_1, f_2, f_3)$ cannot be locally Ricci-symmetric.*

Definition 4.3. *A weakly Ricci-symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ $(n > 2)$ is said to be Ricci-recurrent if it satisfies the condition*

$$\nabla S = \alpha \otimes S.$$

Suppose, weakly Ricci-symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ $(n > 2)$ with $f_1 - f_3 \neq 0$ is Ricci-recurrent, then from (4.1) and Definition 4.3, we have

$$\beta(Y)S(X,Z) + \gamma(Z)S(Y,X) = 0. \tag{4.9}$$

Putting $X = Y = Z = \xi$ in (4.9) and then using (2.7), we obtain

$$\beta(\xi) + \gamma(\xi) = 0. \tag{4.10}$$

Setting $X = Y = \xi$ in (4.9), we get

$$\gamma(Z) = -\beta(\xi)\eta(Z). \tag{4.11}$$

Similarly, we have

$$\beta(Z) = -\gamma(\xi)\eta(Z). \tag{4.12}$$

Adding the above equation with (4.11) and using (4.10), we get

$$\beta(Z) + \gamma(Z) = 0,$$

for any vector field Z on M. So that β and γ are in opposite direction. Hence we state

Theorem 4.3. *If a weakly Ricci-symmetric generalized Sasakian space form $M(f_1, f_2, f_3)$ $(n > 2)$ with $f_1 - f_3 \neq 0$ is locally Ricci-recurrent, then the 1-forms β and γ are in opposite direction.*

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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