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Research Article

A Study on the Applications of Negacyclic Matrices in the Construction of Hadamard Matrices

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Abstract. This article deals with the proper placement of negacyclic matrices as block matrices in a matrix so as to construct a Hadamard matrix. The article contains many new examples of suitable negacyclic matrices of such type with some new observations. The article also includes some negative results.

Keywords. Hadamard matrix, Negacyclic matrix, Circulant matrix, Orthogonal design

Mathematics Subject Classification (2020). 05B15, 05B20, 05B30

1. Introduction

A ± 1 matrix H is called a Hadamard matrix if $HH' = nI_n$. Correctly placing block matrices is one of the famous ways to construct Hadamard matrices. The first example was given by Sylvester [16] in 1867 and he showed that whenever H is a Hadamard matrix, so is the matrix $\begin{pmatrix} H & H \\ -H & H \end{pmatrix}$. This result is the fundamental result in the construction of Sylvester's family of Hadamard matrices (for more detail see, Horadam [10]). Later in 1893, J. Hadamard [9] published examples of Hadamard matrices of orders 12 and 20. In 1933, Paley[14] constructed a block matrix Q with the help of quadratic residues modulo p (p is a prime number) in the field $GF(p^\alpha)$ and using this block matrix Q he constructed Hadamard matrices of order $2(p^\alpha+1)$ and $(p^\alpha+1)$ for $p^\alpha \equiv 1 \mod (4)$ and $p^\alpha \equiv 3 \mod (4)$, respectively (for more detail see, Hadamard [9], and Seberry and Yamada [15]).

The first method of constructing Hadamard matrices for any order was proposed by Williamson [17] in 1944. He used the block matrix

$$H = \begin{pmatrix} A & B & C & D \\ B & -A & D & -C \\ C & -D & -A & B \\ D & C & -B & -A \end{pmatrix} \tag{1.1}$$

and tried to find suitable matrices A, B, C, and D so that H would become a Hadamard matrix (for details of the method see Hadamard [9], and Seberry and Yamada [15]). The process of this method of constructing such Hadamard matrices required computer search and was the entry of computational mathematics into the problem. However, this method does not work in the construction of Hadamard matrices for every order (Đoković [4]).

It is a well-known fact that if H is a Hadamard matrix of order $n\,(>2)$, then n must be some multiple of 4. But the converse of this statement is still an open problem in mathematics. The problem is known as the Hadamard matrix conjecture. The nomenclature of the conjecture is due to the proof of the result that the possible sequences of Hadamard matrices are of orders $1, 2$ or some multiple of 4 and it was first proposed by J. Hadamard [9]. However, he was not the first to postulate Hadamard matrices.

There are many methods available for constructing Hadamard matrices, yet the Hadamard matrix conjecture is one of the longest-standing conjectures in mathematics. The unknown orders of Hadamard matrices less than 2000 are 668, 716, 892, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, and 1964 (Đoković *et al*. [5], and Manjhi and Kumar [13]). The most recent new Hadamard matrix of order 428 was constructed by Kharaghani and Tayfeh-Rezaie [11] in 2005.

This article focuses on the construction of Hadamard matrices with the help of suitable negacyclic block matrices.

In 2004, Finlayson and Seberry [6] put forward a method of constructing a special kind of Hadamard matrix with the help of suitable four negacyclic matrices, although this method does not provide any new Hadamard matrix. However, this method is useful and incorporates negacyclic matrices into the supporting role of the Hadamard matrix construction.

In 2006, Ang *et al*. [1] put forward a method of constructing Hadamard matrices with the help of four negacyclic ± 1 matrices A, B, C, and D each of order n such that $AA' + BB' + CC' + DD' = 4nI_n$ so that the matrix

$$H = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}$$

becomes a Hadamard matrix, where R is a back diagonal matrix. Their work has focused on the inner product equivalence of selected negacyclic matrices. The above form of block matrix H was originally forwarded by Geothals and Seidel [8] regarding the construction of the skew Hadamard matrix of order 36 (for more details see, Seberry and Yamada [15]). Some more suitable negacyclic matrices for the construction of Hadamard matrices of the Goethal-Seidel type have been studied by Xia *et al*. [18]. Most recently, Xia *et al*. [19] have constructed weighing matrices $W(2n, w)$ with the help of 2-suitable negacyclic matrices in the following range:

(i) for odd $n (3 \le n \le 15)$,
(ii) $n = 6, 10, 14, 18$.

2. Preliminaries

This section contains basic definitions and relevant pieces of information for the understanding of this article.

Definition 2.1. A circulant matrix is a square matrix generated from its first row in which each subsequent row is derived from the right circular shift of the previous row.

Example 2.1. $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{pmatrix}$ is a circulant matrix generated by its first row (x_1, x_2, x_3), it is denoted by $\operatorname{circ}(x_1, x_2, x_3)$. That is

$$\operatorname{circ}(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{pmatrix}.$$

In general, we can write

$$\operatorname{circ}(x_1, x_2, \dots, x_n) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_n & x_1 & \dots & x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & \dots & x_1 \end{pmatrix}.$$

Some properties of circulant matrices are listed below:
(i) The sum of two circulant matrices is a circulant matrix.
(ii) The product of two circulant matrices is a circulant matrix.
(iii) Circulant matrices obey the commutative property with respect to matrix multiplication.
(iv) Circulant matrices are row-regular matrices.

Definition 2.2. A negacyclic matrix is a square matrix generated from its first row in which each successive row is obtained by a right cyclic shift of the previous row, with the sign changed at the first position.

Example 2.2. Following are some examples of negacyclic matrices:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 2 \\ -2 & -3 & 1 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_4 & x_1 & x_2 & x_3 \\ -x_3 & -x_4 & x_1 & x_2 \\ -x_2 & -x_3 & -x_4 & x_1 \end{pmatrix}.$$

The basic properties of negacyclic matrices are listed below:
(i) The sum of two negacyclic matrices is a negacyclic matrix.
(ii) The product of two negacyclic matrices is a negacyclic matrix.
(iii) Negacyclic matrices obey the commutative property with respect to matrix multiplication.
(iv) Conjugate transpose of negacyclic matrix is a negacyclic matrix.
(v) Inverse of a negacyclic matrix is a negacyclic matrix.

(For detail see, Davis [3], and Georgiou *et al*. [7]).

3. Main Results

In 2018, Manjhi and Kumar [12] considered the matrix

$$H=\begin{pmatrix} J_n & A & A & A \\ -A & J_n & A & -A \\ -A & -A & J_n & A \\ -A & A & -A & J_n \end{pmatrix} \tag{3.1}$$

and showed that H will become a Hadamard if $3A^2+nJ_n=4nI_n$. Further, they demonstrated the theory with the help of the matrix

$$A=\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Matrix given in (3.1) is a particular case of the Williamson Hadamard matrix. Herewith a similar result has been obtained in terms of negacyclic matrices.

Consider the negacyclic matrix P with the first row $(x_1,x_2,\dots,x_n)$ for odd n such that

$$x_i=\begin{cases} 1, & \text{if } i \text{ is odd,} \\ -1, & \text{if } i \text{ is even.} \end{cases}$$

Let $P=(p_{ij})$, then

$$p_{ij}=\begin{cases} 1, & \text{if } i \text{ and } j \text{ are of the same parity,} \\ -1, & \text{if } i \text{ and } j \text{ are of opposite parity.} \end{cases} \tag{3.2}$$

This means that P is a symmetric matrix with equal alternating rows, and hence the entries y_{ij} of matrix P^2 will be given by the product of ith and jth rows of the matrix P. This implies that

$$y_{ij}=\begin{cases} n, & \text{if } i \text{ and } j \text{ are of the same parity,} \\ -n, & \text{if } i \text{ and } j \text{ are of opposite parity.} \end{cases}$$

Thus, $P^2=nP$. Also, $P^2+3(2I_n-P)^2=4nP+12I_n-12P$ if and only if $P^2+3(2I_n-P)^2=12I_n$ if and only if $n=3$.

We summarize the above result as a theorem given below:

Theorem 3.1. *The block matrix*

$$H=\begin{pmatrix} P & (2I_n-P) & (2I_n-P) & (2I_n-P) \\ -(2I_n-P) & P & (2I_n-P) & -(2I_n-P) \\ -(2I_n-P) & -(2I_n-P) & P & (2I_n-P) \\ -(2I_n-P) & (2I_n-P) & -(2I_n-P) & P \end{pmatrix} \tag{3.3}$$

with negacyclic P and $P^2=nP$ is a Hadamard matrix if and only if $n=3$.

Remark 3.1. The block matrices $\begin{pmatrix} 1 & -1 \\ -1' & J_{(n-1)} \end{pmatrix}$ and $\begin{pmatrix} J_{(n-1)} & -1 \\ -1' & 1 \end{pmatrix}$ are solutions of the matrix equation $P^2=nP$, and these are not negacyclic matrices. These matrices also satisfy the condition for H to be a Hadamard matrix given in (3.3) for $n=3$.

Another particular case of the Williamson-Hadamard matrix is considered in the following theorem:

Theorem 3.2. *If A is a symmetric ± 1 matrix of order n with zero diagonal such that $A^2 = (2n-5)I_n - (n-4)J_n$, then the matrix*

$$H = \begin{pmatrix} 2I_n - J_n & 2I_n - J_n & I_n + A & I_n - A \\ -2I_n + J_n & 2I_n - J_n & -I_n + A & I_n + A \\ -I_n - A & I_n - A & 2I_n - J_n & -2I_n + J_n \\ -I_n + A & -I_n - A & 2I_n - J_n & 2I_n - J_n \end{pmatrix} \tag{3.4}$$

is a Hadamard matrix.

Proof. Since the identity matrix I_n and the all 1 matrix J_n are symmetric matrices each of order n with $I_nJ_n = J_nI_n$, therefore H will be a Hadamard matrix if

$$(2I_n - J_n)^2 + (2I_n - J_n)^2 + (I_n + A)^2 + (I_n - A)^2 = 4nI_n$$

that is, if $2(2I_n - J_n)^2 + (I_n + A)^2 + (I_n - A)^2 = 4nI_n$

that is, if $2(4I_n + nJ_n - 4J_n) + 2(I_n + A^2) = 4nI_n$

that is, if $A^2 = (n-4)I_n + (n-1)I_n - (n-4)J_n$

that is, if $A^2 = (2n-5)I_n - (n-4)J_n$.

This proves the theorem. □

Example 3.1. The matrices that satisfy the conditions of the above theorem are given below:

(i) for $n = 3$: $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$,

(ii) for $n = 5$: $\begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \end{pmatrix}$.

Remark 3.2. Both the above matrices (in (ii)) are symmetric and circulant with row sum zero. Apart from the above two matrices, further circulant and negacyclic matrices of order $n\,(3 \le n \le 22)$ do not satisfy the conditions of the theorem. Thus, no negacyclic matrix of order $n\,(3 \le n \le 22)$ can be found that satisfies the conditions of the above theorem.

Remark 3.3. If in the matrix H given in (3.4), J_n is replaced by the negacyclic matrix P (3.2) then we get H to be a Hadamard for the following negacyclic matrices A:

(i) for $n = 3$: $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$,

(ii) for $n = 5$: $\begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{pmatrix}$.

If all the matrices in the Hadamard matrix (1.1) are equal to the matrix A, then $4AA' = 4nI_n \Rightarrow AA' = nI_n \Rightarrow A$ is a Hadamard matrix of order n. Bolonin and Djokovic [2] conjectured that no negacyclic Hadamard matrix exists of order $n > 2$, the results are verified up to $n = 40$. So, to date, only the following matrices satisfy this condition

$$(1),\ (-1),\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},\ \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Therefore, if the conjecture of Bolonin and Djokovic [2] is correct, we cannot construct Hadamard matrices H (1.1) with negacyclic A,B,C,D and $A = B = C = D$ beyond the order 8.

An exhaustive search results of symmetric negacyclic blocks A,B,C and D with first entry 1 in the Williamson Hadamard matrix (1.1) have been obtained and are tabulated below:

(I) A,B are distinct; $B = C = D$.

Sl. No.	Value of n	First rows of A and B respectively, whenever exist
1	$2,4,5,6,7,8,9,11,12,13,14,15,16,17$	Does not exist
2	3	$(1,-1,1),(1,1,-1)$

(II) A,B,C are distinct; $C = D$.

Sl. No.	Value of n		First rows of A,B, and C respectively, whenever exist
1	$2,3,4,6,8,10,11,12$		Does not exist
2	5		$(1,1,1,-1,-1),(1,-1,-1,1,1),(1,1,-1,1,-1)$
3	7	(i)	$(1,1,1,1,-1,-1,-1)(1,-1,-1,-1,1,1,1)$, $(1,-1,1,1,-1,-1,1)$
		(ii)	$(1,1,1,-1,1,-1,-1),(1,-1,-1,1,-1,1,1)$, $(1,-1,-1,-1,1,1,1)$
		(iii)	$(1,1,-1,-1,1,1,-1),(1,-1,1,1,-1,-1,1)$, $(1,1,1,-1,1,-1,-1)$
4	9	(i)	$(1,1,1,1,-1,1,-1,-1,-1),(1,-1,-1,-1,1,-1,1,1,1)$, $(1,1,-1,-1,1,-1,1,1,-1)$
		(ii)	$(1,1,1,-1,1,-1,1,-1,-1),(1,-1,-1,1,-1,1,-1,1,1)$, $(1,1,1,-1,-1,1,1,-1,-1)$
		(iii)	$(1,1,-1,1,1,-1,-1,1,-1),(1,-1,1,-1,-1,1,1,-1,1)$, $(1,-1,-1,-1,-1,1,1,1,1)$

(III) A,B,C, and D all are distinct.

Sl. No.	Value of n		First rows of A,B,C, and D are respectively, whenever exist
1	$2,3,4,5,6,8,$		Does not exist
2	7	(i)	$(1,1,1,1,-1,-1,-1),(1,1,-1,1,-1,1,-1)$, $(1,1,-1,-1,1,1,-1),(1,-1,-1,1,-1,1,1)$
3	9	(i)	$(1,1,1,1,-1,1,-1,-1,-1),(1,1,1,-1,-1,1,1,-1,-1)$, $(1,1,-1,-1,1,-1,1,1,-1),(1,-1,1,-1,-1,1,1,-1,1)$
		(ii)	$(1,1,1,1,-1,1,-1,-1,-1),(1,1,-1,1,1,-1,-1,1,-1)$, $(1,1,-1,-1,-1,1,1,1,-1),(1,-1,-1,1,-1,1,-1,1,1)$
		(iii)	$(1,1,1,-1,1,-1,1,-1,-1),(1,1,-1,1,1,-1,-1,1,-1)$, $(1,1,-1,-1,1,-1,1,1,-1),(1,-1,-1,-1,-1,1,1,1,1)$
		(iv)	$(1,1,1,-1,-1,1,1,-1,-1),(1,-1,-1,1,-1,1,-1,1,1)$, $(1,-1,-1,-1,1,-1,1,1,1),(1,-1,-1,-1,-1,1,1,1,1)$

Remark 3.4. The symmetric negacyclic Williamson matrices do not exist in the case when A,C are distinct and $A = B$, $C = D$ for the orders $2,3,4,5,6,7,8,9,10,11,12,13,14,15,16$ and 17.

Remark 3.5. The obtained symmetric negacyclic Williamson matrices A and B in the case when A,B,C are distinct and $C = D$ given in Table (II) can be expressed as $A = I_n + X$ and $B = I_n - X$, X being a negacyclic matrix with first row $(0, p, \text{reverse}(-p))$ where p is a sequence of length $\frac{n-1}{2}$ with ± 1 entries and n being odd. Now $A + B = 2I_n$, therefore, $(A+B)^2 = 4I_n$. This implies $A^2 + B^2 + 2AB = 4In$ (because $AB = BA$ as negacyclic matrices commute). Since $A^2 + B^2 + 2C^2 = 4nI_n$, therefore $C^2 - AB = 2(n-1)I_n$. Thus, $C^2 - (I_n + X)(I_n - X) = 2(n-1)I_n$ which implies that

$$C^2 = (2n-1)I_n - X^2. \tag{3.5}$$

Remark 3.5 implies that finding out A,B,C amounts to finding just those X for which such C exist satisfying (3.5). As C is also a symmetric negacyclic matrix, and therefore it can be expressed as $(1, q, \text{reverse}(-q))$ where q is a sequence of length $\frac{n-1}{2}$ with ± 1 entries and n being odd. Thus, we need only specify the corresponding p's and q's. Proceeding in this manner, enabled us to go higher up the order, and accordingly, the following information regarding such matrices has been obtained for this case:

(i) *Existence*: for the matrices of order $13,15,19,21,25,27,31$ and 37.

(ii) *Non-existence*: for the matrices of order $11,17,23,29$ and 33.

As discussed above, the matrices A and B can be obtained from the sequence p, and the matrix C can be obtained from the sequence q. The sequences p and q of existing matrices for odd orders n greater than 11 are tabulated below:

Sl. No.	n	p	q
1	13	(1,1,-1,-1,1,-1)	(1,-1,1,1,1,1)
2	15	(1,-1,1,1,-1,-1,-1)	(1,1,1,-1,1,-1,-1)
3	19	(1,1,1,1,-1,1,-1,-1,-1)	(-1,-1,-1,1,1,-1,1,-1,-1)
4	21	(1,-1,-1,1,1,-1,-1,-1,-1,-1)	(-1,-1,1,-1,-1,-1,-1,1,-1,1)
5	25	(1,-1,1,1,1,1,-1,-1,-1,1,-1,-1)	(-1,-1,1,-1,1,-1,-1,-1,-1,-1,1,1)
6	27	(1,-1,-1,-1,1,1,1,1,-1,1,1,-1,-1)	(1,-1,1,1,-1,1,-1,1,1,1,1,1,-1)
7	31	(1,-1,1,1,-1,1,1,1,1,1,-1,1,-1,-1,-1)	(1,1,-1,-1,1,-1,1,-1,-1,-1,-1,1,1,-1,-1)
8	37	(1,1,-1,1,-1,1,1,1,-1,-1,1,1,-1,-1,-1,-1,-1,-1)	(1,1,-1,1,1,1,-1,-1,1,-1,1,1,1,-1,1,-1,-1,1)

Note 3.1. The above table includes one output for each existing case.

4. Conclusions

This article contains some new general forms of Hadamard block matrices (Vide (3.3) and (3.4)) and focuses on negacyclic block matrices. This is a new approach and adds dimension to the construction of Hadamard matrices as well as the study of negacyclic matrices.

A negacyclic matrix P (3.2) behaves like a matrix J of all 1's in the sense that it satisfies the equation $x^2 = x$. It is observed that replacing J with P gives corresponding outputs of a similar nature that have been obtained.

Many new symmetric negacyclic Williamson matrices have been generated through a computer search. It gives examples relevant to researchers in this field. In the search result it is observed that in the particular case when $A = C$ and $B = D$, no Williamson Hadamard matrices have been obtained up to the order $n = 17$, and it seems that no such symmetric negacyclic Williamson matrices can be found. It stimulates us to investigate and find out symmetric ± 1 negacyclic matrices A and B such that $A^2 + B^2 = 2nI_n$.

For the case when A, B, C are distinct and $C = D$ it seems that existence occurs for every odd order beyond 3 except for the orders of the form $11+6k$ where k is a non-negative integer(Verified for k up to 3). Đoković [4] showed that no Williamson matrices can be found for order 35, which means that the special category of matrices talked about cannot be found for $k = 4$. However, the result for order 33 does not exist and is not in the form of $11+6k$. Therefore, there are likely more non-existent classes for the type discussed.

Another interesting question addressed by the study is whether or not Williamson matrices of the type discussed above can be found for order $11+6k$ where k is some natural number.

If Williamson matrices of the type discussed above can not be found for the order $11+6k$ type, then Williamson matrices of the special class considered above do not exist for order 167. This implies that the Hadamard matrices of the order 668 which is the lowest order of unknown Hadamard matrix, can not be found in discussed special case.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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