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Research Article

An Efficient Tri-Parametric Jarratt Family for Nonlinear Models Solution

O. Ogbereyivwe*[1], K. Emumejaye[2] and S. S. Umar[3]

[1] *Department of Mathematics, Delta State University of Science and Technology, Ozoro, Delta State, Nigeria*
[2] *Department of Physics, Delta State University of Science and Technology, Ozoro, Delta State, Nigeria*
[3] *Department of Statistics, Federal Polytechnic, Auchi, Edo State, Nigeria*
***Corresponding author:** ogbereyivweo@dsust.edu.ng

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Abstract. This paper offers an efficient and competitive tri-parametric family of iterative schemes for deciding nonlinear and systems of nonlinear models solution. The family has quartic-convergence order and is based on the composition of the Jarratt's perturbed Newton method with a designed iterative function that involves rational approximation function of degree two in both its denominator and numerator. The new tri-parametric family is further extended to solving nonlinear models in n-dimensional form and its convergence investigation established to retain its quartic-convergence order. By varying the parameters in the family, enabled the rediscovery of many well established iterative schemes. The applicability and computational performance of some specified family examples, on some nonlinear models were also verified and results compared with some of known and established schemes that are also family members.

Keywords. Nonlinear model, Iterative scheme, Jarratt's scheme, Rational approximation function, Convergence order

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1. Introduction

Nonlinear models of the form $\Psi(x)=0$, $\Psi:\Omega\subseteq\Re\to\Re$ and nonlinear system models described as $\Psi(X)=\mathbf{0}$, $\Psi:\Theta\subseteq\Re^n\to\Re^n$, spring up in diverse applications in science, engineering, economics, medical and many other disciplines. Obtaining solution of the nonlinear models or nonlinear system models is a vital problem in numerical analysis that have several applications. In the absence of a general analytic approach for dealing with the solution of the nonlinear model or nonlinear system model, the numerical iterative schemes were designed and utilised.

The general numerical *Iterative Scheme* (IS) is $x_{k+1} = Q(x)$, where k is iterative counter and $Q(x)$ an iterative function. Its implementation require an initial guess x_0 (a neighbourhood of the true solution x_* of the nonlinear model), used in $Q(x)$ to generate sequence of iterations results $\{x_{k+1}\}_{k=0}^{n}$ that are approximations of x_*, such that $\lim_{k\to n}\{x_{k+1}\} = x_*$. This assertion also holds for nonlinear system in n-dimensional space. In this case, for $X_{k+1} = Q(X)$, $\lim_{k\to n}\{X_{k+1}\} = X_*$, where X_* is true solution of nonlinear system model.

The most famous and widely used IS is the *Newton's Iterative Scheme* (NIS) (Traub [19]). Its scalar version is:

$$x_{k+1} = x_k - \eta(x_k), \quad k = 0,1,2,\cdots, \tag{1}$$

where $\eta(x_k) = \frac{\Psi(x_k)}{\Psi'(x_k)}$. One important benefit of the NIS is that, it retains its convergence order (CO) two when modified to handle nonlinear problems in n-dimensional space. In the past, significant attention has been devoted to modification of the NIS with the aim of improving its CO, efficiency or redesigned to circumvent certain problem encountered during its computational implementation. Many of these modifications resulted to the development of new IS that are based on one or combination of techniques such as the composition, decomposition, homotopy, geometric, variational iteration, perturbation, quadrature formulas and weight function. Some excellent literature on techniques for developing IS that are modifications of the NIS can be seen in the works Abbasbandy [1], Chun *et al*. [4], Jarratt [6], Khirallah and Hafiz [8], Ogbereyivwe and Muka [11], Ogbereyivwe and Izevbizua [12], Ogbereyivwe and Ojo-Orobosa [13], Petković [14], Qureshi *et al*. [15], Sharma and Behl [16], Sivakumar and Jayaraman [18], and references in therein.

One early modification of the NIS is the *Jarratt Scheme* (JS) (Jarratt [6]) and was based on composition of perturbed NIS and an iterative function that involves a rational weight function of degree one. It's structure is:

$$x_{k+1} = x_k - \left[\frac{\Psi'(x_k) + 3\Psi'(y_k)}{6\Psi'(y_k) - 2\Psi'(x_k)}\right]\eta(x_k), \tag{2}$$

where $y_k = x_k - \frac{2}{3}\eta(x_k)$. The JS has CO four and also retains same CO when modified to handle problems in n-dimensional space. Behl *et al*. [2] developed family of the JS in (2) as:

$$x_{k+1} = x_k - \frac{[(p_1^2 - 22p_1p_2 - 27p_2^2)\Psi'(x_k) + 3(p_1^2 + 10p_1p_2 + 5p_2^2)\Psi'(y_k)]\Psi(x_k)}{2[p_1\Psi'(x_k) + 3p_2\Psi'(y_k)][3(p_1+p_2)\Psi'(y_k) - (p_1+5p_2)\Psi'(x_k)]}, \tag{3}$$

where $p_i \in \Re$ such that $p_1 \neq p_2$ and $p_1 \neq -3p_1$. In Kanwar *et al*. [7], eq. (3) was extended to solve nonlinear system model and presented as:

$$\left.\begin{aligned}
X_{k+1} &= X_k - \frac{1}{2}[(p_1I + 3p_2B(X_k))(3(p_1+p_2)B(X_k) - (p_1+5p_2)I)]^{-1} \\
&\qquad \times[(p_1^2 - 22p_1p_2 - 27p_2^2)I + 3(p_1^2 + 10p_1p_2 + 5p_2^2)B(X_k)]G(X_k), \\
B(X_k) &= \Gamma'(X_k)^{-1}\Gamma'(Y_k), \\
G(X_k) &= \Gamma'(X_k)^{-1}\Gamma'(X_k).
\end{aligned}\right\} \tag{4}$$

where $Y_k = X_k - \frac{2}{3}G(X_k)$ and I is identity matrix of same dimension as the nonlinear system.

Consequent upon the IS put forward in (3) and (4), a new family of IS that can generate the JS in [6], Sharma and Behl scheme in [16], schemes presented in Abbasbandy [1], Chun *et al*. [4], Khirallah and Hafiz [8] and their variants, in both scalar and vector form, is developed and offered in this manuscript. The schemes family was derived by replacing the weight function in the second step of the JS with a parameterised *Rational Approximation Function* (RAF) of degree 2. The parameters in the RAF, makes the family flexible in the construction of its concrete members.

The organisation of this manuscript follow the next described format: Section 2 presents the formation of the schemes family in scalar form and its convergence investigation. Section 3 deals with extension of the scheme to n-dimensional form and its convergence study. Section 4 put forward the developed scheme implementation on some real life nonlinear models. The conclusions drawn from the work are in Section 5.

2. The Iterative Scheme Family

Consider a parameterised weighted scheme structured as:

$$x_{k+1} = x_k - R_{2,2}[\Psi'(x_k), \Psi'(y_k)]\eta(x_k), \tag{5}$$

where the weight function $R_{2,2}[\Psi'(x_k), \Psi'(y_k)]$ is a real valued parameterised bi-variate *Scalar Rational Approximation Function* (SRAF) of degree two in both its denominator and numerator given as:

$$R_{2,2}(\Psi'(x_k), \Psi'(y_k)) = \frac{\alpha_1\Psi'(y_k)[\Psi'(y_k) - \Psi'(x_k)] + \Psi'(x_k)[\Psi'(y_k) + \alpha_2\Psi'(y_k) + \alpha_3\Psi'(x_k)]}{\alpha_4\Psi'(y_k)[\Psi'(y_k) - \Psi'(x_k)] + \Psi'(x_k)[\Psi'(y_k) + \alpha_5\Psi'(y_k) + \alpha_6\Psi'(x_k)]}, \tag{6}$$

$\alpha_i \in \Re$, $1 \le i \le 6$ are parameters not all equal zero. Our interest is that, given certain conditions imposed on the parameters α_i, the scheme in (5) will iteratively approximate the solution of nonlinear model with a CO of four. The scheme in (5) require the computational evaluation of function at a point x_k and derivative of function at two distinct points x_k and y_k. According to the conjecture by Kung and Traub in [9], which stated that an iterative scheme without memory that require sum of all distinct function assessment T in one iteration cycle, has a bound on its CO with 2^{T-1}. For a scheme that attain this bound, it is said to be optimal. Consequent on the Kung-Traub conjecture, we shall prove next that the scheme in (5) converge under some hypotheses and is also optimal.

2.1 Method Convergence Analysis

This subsection establishes the convergence of the IS in (5). The Taylor expansion technique is utilised in proving its convergence. The technique requires the Taylor expansion of the functions $\Psi(\cdot)$ and $\Psi'(\cdot)$ as contain in the scheme (5) and then obtain an equation in the form $\omega_{k+1} = \delta\omega_k^{\rho} + O(\omega_k^{\rho+1})$ from it, where $\omega_k = x_k - x_*$ is the kth iteration error of the scheme. When this is achieved, ω_{k+1}, δ and ρ are referred as Asymptotic error, Asymptotic constant and CO respectively (see Ogbereyivwe and Izevbizua [12], Ogbereyivwe and Ojo-Orobosa [13], Traub [19]). Further details on the use of this technique in analysing the convergence of schemes can be found in Abbasbandy [1], Behl *et al*. [2], Chun *et al*. [4], Jarratt [6], Kanwar *et al*. [7], Kung and Traub [9], Ogbereyivwe and Muka [11], Ogbereyivwe and Izevbizua [12], Ogbereyivwe and Ojo-Orobosa [13], Petković, [14], Qureshi *et al*. [15], Sharma and Behl [16], Sharma *et al*. [17], Sivakumar and Jayaraman [18], Traub [19] and some references in them.

Definition 2.1 ([12,19])**.** An iterative scheme efficiency is measured using the numerical value obtained by $\rho^{\frac{1}{T}}$, where T is the sum of all different functions computation in a complete iteration cycle.

In the case of the scheme in (5), its efficiency will be 1.5874 if proven to have CO four.

Definition 2.2 ([14])**.** The quantity obtained by the formula:

$$\rho_{coc} \approx \frac{\log Z_1}{\log Z_2}, \tag{7}$$

where $Z_1 = \left|\frac{\Psi(x_k)}{\Psi(x_{k-1})}\right|$, $Z_2 = \left|\frac{\Psi(x_{k-1})}{\Psi(x_{k-2})}\right|$ and x_{k-2}, x_{k-1}, x_k are last three consecutive iteration results of a scheme, is known as the computational order of convergence (ρ_{coc}) of a scheme.

The equations and definitions given above, are in scalar form and can be modified directly to give the equivalent vector form.

Theorem 2.1. *Suppose that $\Psi : \Omega \subseteq \Re \to \Re$ is defined and sufficiently differentiable on Ω, where Ω is an open interval and $\Psi'(\cdot) \neq 0$. For an initial guess x_0 in the neighbourhood of x_*, the scheme in* (5) *will generate sequence of approximations of x_* that converge to x_* provided the parameters satisfies the conditions: $\alpha_4 = \frac{28\alpha_1+3\alpha_2-9\alpha_3+3}{16}$, $\alpha_5 = \frac{-12\alpha_1+25\alpha_2+21\alpha_3+9}{16}$ and $\alpha_6 = \frac{12\alpha_1-9\alpha_2-5\alpha_3-9}{16}$. Consequently, its error equation satisfies:*

$$\omega_{k+1} = \left(\frac{c_2\left(2(3+2\alpha_1+3\alpha_2+9\alpha_3)c_2^2 - 9(1+\alpha_2+\alpha_3)c_3\right)}{9(1+\alpha_2+\alpha_3)}\right)\omega_k^4 + O(\omega_k^5), \tag{8}$$

where $c_i = \frac{1}{i!}\frac{\Psi^{(i)}(x^*)}{\Psi'(x^*)}$, $i = 1,2,\dots$.

Proof. Let $x = x_k$ in Taylor's expansion of $\Psi(x)$ and $\Psi'(x)$ around x_*, then

$$\Psi(x_i) = \Psi'(x_*)[\omega_k + c_2\omega_k^2 + c_3\omega_k^3 + c_4\omega_k^4 + c_5\omega_k^5 + c_6\omega_k^6 + O(\omega_k^7)], \tag{9}$$

and

$$\Psi'(x_i) = \Psi'(x_*)[1 + 2c_2\omega_k + 3c_3\omega_k^2 + 4c_4\omega_k^3 + 5c_5\omega_k^4 + 6c_6\omega_k^5 + 7c_7\omega_k^6 + O(\omega_k^7)]. \tag{10}$$

Using (9) and (10), the following equation is obtained.

$$\begin{aligned} y_k &= x_k - \frac{2}{3}\eta(x_k) \\ &= \frac{1}{3}\omega_k + \frac{2c_2}{3}\omega_k^2 + (c_3 - c_2^2)\omega_k^3 + \frac{2(3c_4 - 7c_2c_3 + 4c_2^2)}{3}\omega_k^4 \\ &\quad + \frac{4}{3}(4c_2^4 + 5c_2c_4 - 2c_5 - 10c_2^2c_3 + 3c_3^2)\omega_k^5 + O(\omega_k^6). \end{aligned} \tag{11}$$

Using (11) to expand $\Psi(y_k)$ and $\Psi'(y_k)$, gives

$$\begin{aligned} \Psi(y_i) &= \frac{1}{3}\omega_k + \frac{7}{9}c_2\omega_i^2 + \frac{1}{9}\left(8c_2^2 + \frac{37}{3}\right)\omega_k^3 + \frac{2}{9}(9c_4 - 16c_2c_3 + 10c_2^3)\omega_k^4 \\ &\quad + \frac{4}{9}(12c_2c_4 - 6c_5 + 12c_2^4 - 27c_2^2c_3 + 8c_3^2)\omega_k^5 + O(\omega_k^6) \end{aligned} \tag{12}$$

and

$$\Psi'(y_k) = 1 + \frac{2}{3}\omega_k + \frac{1}{3}(4c_2^2 - c_3)\omega_k^2 + \left(4c_2c_3 - \frac{8c_2^3}{3}\right)\omega_k^3$$

$$+\frac{4}{3}(4c_2^4+2c_3^2-8c_2c_3+3c_2c_4)\omega_k^4$$
$$+\frac{4}{3}((8c_2^5-20c_2^3c_3+10c_2^2c_4+9c_2c_3^2-3c_3c_4-4c_2c_5))\omega_k^5+O(\omega_k^6), \tag{13}$$

respectively.

Applying the equations in (9)-(13) into (5), the following error equation is obtained as

$$\omega_{k+1}=x_*+\left(1-\frac{1+\alpha_2+\alpha_3}{1+\alpha_5+\alpha_6}\right)\omega_k$$
$$+\frac{1}{3\Phi^2}\Big(3-\alpha_3-4\alpha_4-4\alpha_3\alpha_4+3\alpha_5-\alpha_3\alpha_5+7\alpha_6+3\alpha_3\alpha_6$$
$$+4\alpha_1(1+\alpha_5+\alpha_6)+\alpha_2(3-4\alpha_4+3\alpha_5+7\alpha_6)c_2\Big)\omega_k^2$$
$$+\frac{2}{9\Phi^3}\Big(-9+7\alpha_3+224\alpha_4+16\alpha_3\alpha_4-8\alpha_4^2-8\alpha_3\alpha_4^2+\cdots$$
$$+\alpha_2(9+8\alpha_4^2+18\alpha_5+\cdots-8\alpha_4(3+3\alpha_5+5\alpha_6))c_2^2+3(1+\alpha_5+\alpha_6)$$
$$\times(3-\alpha_3-4\alpha_4+\cdots+4\alpha_1(1+\alpha_5+\alpha_6)+\cdots+7\alpha_6)c_3\Big)\omega_k^3+O(\omega_k^4), \tag{14}$$

where $\Phi=1+\alpha_5+\alpha_6$.

To inflate the order of error equation in (14) to 4, the next set of equations must hold.

$$\left.\begin{aligned}&\alpha_2+\alpha_3-\alpha_5-\alpha_6=0,\\&4\alpha_1+7\alpha_2+3\alpha_3-4\alpha_4-4\alpha_5=-3,\\&28\alpha_1+3\alpha_2-9\alpha_3-16\alpha_4=-3.\end{aligned}\right\} \tag{15}$$

When the set of equations in (15) were solved, the following relations in terms of α_1, α_2 and α_3 were obtained,

$$\left.\begin{aligned}&\alpha_4=\frac{1}{16}(28\alpha_1+3\alpha_2-9\alpha_3+3),\\&\alpha_5=\frac{1}{16}(-12\alpha_1+25\alpha_2+21\alpha_3+9),\\&\alpha_4=\frac{1}{16}(12\alpha_1-9\alpha_2-5\alpha_3-9).\end{aligned}\right\} \tag{16}$$

The substitution of the equation in (16) into the equation in (14), reduced the equation in (14) to the next expression.

$$\omega_{k+1}=x_*+\left(\frac{c_2(2(3+2\alpha_1+3\alpha_2+9\alpha_3)c_2^2-9(1+\alpha_2+\alpha_3)c_3)}{9(1+\alpha_2+\alpha_3)}\right)\omega_k^4+O(\omega_k^5). \tag{17}$$

This brings the proof to a conclusion. □

Corollary 2.1. *Under the hypothesis in Theorem* 2.1*, the scheme in* (5) *is of CO three whenever* $\alpha_3=\frac{1}{3}(1+\alpha_2)$*, provided* $\alpha_2\neq\frac{\alpha_1}{3}-1$*. In this case, the scheme will have* $E.I=1.4422$.

Remark 2.1. The substitution of the relations in equation (16) into (6) and after simplification, (5) becomes:

$$x_{k+1} = x_k - 16\left[\frac{\alpha_3\Psi'(x_k)^2 + \Psi'(y_k)(\tau_1\Psi'(x_k) + \alpha_1\Psi'(y_k))}{\tau_2\Psi'(x_k)^2 + 2\tau_3\Psi'(x_k)\Psi'(y_k) + \tau_4\Psi'(y_k)^2}\right]\eta(x_k), \tag{18}$$

where $\tau_1 = 1-\alpha_1+\alpha_2$, $\tau_2 = 12\alpha_1 - 9\alpha_2 - 5\alpha_3 - 9$, $\tau_3 = -20\alpha_1 + 11\alpha_2 + 15\alpha_3 + 11$ and $\tau_4 = 28\alpha_1 + 3\alpha_2 - 9\alpha_3 + 3$.

Remark 2.2. For $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = -1$ the scheme in (18) fails.

2.2 The Scheme Family Specified Members

This subsection provides some typical concrete members obtained by assigning values to the parameters α_i. Some good and well known schemes that are of CO four are specific elements of the scheme in (18).

Case 1: For $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = \frac{3}{4}$ or $\alpha_1 = 0$ and $\alpha_3 = \frac{1+\alpha_2}{3}$, the famous Jarratt scheme (JS) [6] is re-obtained in this case as:

$$x_{k+1} = x_k + \left[\frac{\Psi'(x_k) + 3\Psi'(y_k)}{2\Psi'(x_k) - 6\Psi'(y_k)}\right]\eta(x_k). \tag{19}$$

Case 2: Take $\alpha_1 = \frac{3}{8}$, $\alpha_2 = -\frac{9}{8}$ and $\alpha_3 = \frac{9}{8}$ or $\alpha_1 = \frac{3}{64}$, $\alpha_2 = -\frac{65}{64}$ and $\alpha_3 = \frac{9}{64}$, the *Sharma and Behl method* (SBS) [16] is discovered as:

$$x_{k+1} = x_k - \left[-\frac{1}{2} + \frac{9\Psi'(x_k)}{8\Psi'(y_k)} + \frac{3\Psi'(y_k)}{8\Psi'(x_k)}\right]\eta(x_k). \tag{20}$$

Case 3: When $\alpha_1 = \frac{9}{128}$, $\alpha_2 = -\frac{143}{128}$ and $\alpha_3 = \frac{129}{8}$, a CO four scheme ($KACS$) put forward in Khirallah and Hafiz [8], Abbasbandy [1] and Chun *et al*. [4], is rediscovered as:

$$x_{k+1} = x_k - \left[\frac{23}{8} - 3\frac{\Psi'(y_k)}{\Psi'(x_k)} + \frac{9}{8}\left(\frac{\Psi'(y_k)}{\Psi'(x_k)}\right)^2\right]\eta(x_k). \tag{21}$$

Case 4: The substitution of $\alpha_1 = \alpha_3 = 0$ produces a new CO four scheme (OS_4^1) obtained as:

$$x_{k+1} = x_k + \left[\frac{16\Psi'(x_k)\Psi'(y_k)}{9\Psi'(x_k)^2 - 22\Psi'(x_k)\Psi'(y_k) - 3\Psi'(y_k)^2}\right]\eta(x_k). \tag{22}$$

Case 5: Consider $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = -\frac{1}{3}$, a new CO four scheme (OS_4^2) is derived as:

$$x_{k+1} = x_k - \left[\frac{8\Psi'(x_k)(\Psi'(x_k) - 3\Psi'(y_k))}{11\Psi'(x_k)^2 - 18\Psi'(x_k)\Psi'(y_k) - 9\Psi'(y_k)^2}\right]\eta(x_k). \tag{23}$$

Case 6: Let $\alpha_1 = 0$ and $\alpha_2 = -1$, a new CO four scheme (OS_4^3) is next:

$$x_{k+1} = x_k + \left[\frac{16\Psi'(x_k)^2}{5\Psi'(x_k)^2 - 30\Psi'(x_k)\Psi'(y_k) + 9\Psi'(y_k)^2}\right]\eta(x_k). \tag{24}$$

Case 7: Put $\alpha_1 = \alpha_3 = 0$ and $\alpha_2 = -1$, a new CO three scheme (OS_3^1) is derived as:

$$x_{k+1} = x_k - \left[\frac{4\Psi'(x_k)}{\Psi'(x_k) + 3\Psi'(y_k)}\right]\eta(x_k). \tag{25}$$

3. Method Extension to n-Dimensional Case

Suppose $\Psi(X) = (\Psi_1(X), \Psi_2(X), \cdots, \Psi_n(X))^T$ such that $\Psi(X) = \mathbf{0}$, describes the nth -dimensional nonlinear model, where $\Psi : \Theta \subset \Re^n \to \Re^n$ and $X = (X_1, X_2, X_3, \cdots X_n)^T$ is a vector with dimension as $\mathbf{0}$, then the corresponding nth dimensional form of the method in (5) is given as:

$$X_{k+1} = X_k - \mathbf{R}_{2,2}(\Psi'(X_k), \Psi'(Y_k))G(X_k), \tag{26}$$

where

$$\mathbf{R}_{2,2}(\Psi'(X_k), \Psi'(Y_k)) = M^{-1}N, \tag{27}$$

is a bi-variate vector rational approximation function (BVRAF) of degree 2, and

$$\left.\begin{aligned} M &= [A_4\Psi'(X_k)\Psi'(Y_k)(B(X_k)-I) + \Psi'(X_k)^2(B(X_k) + A_5B(X_k) + A_6I)], \\ N &= [A_1\Psi'(X_k)\Psi'(Y_k)(B(X_k)-I) + \Psi'(X_k)^2(B(X_k) + A_2B(X_k) + A_3I)], \end{aligned}\right\} \tag{28}$$

$Y_k = X_k - \frac{2}{3}G(X_k)$, $B(X_k) = \Psi'(X_k)^{-1}\Psi(Y_k)$, $\Psi'(\cdot)$ is first order Frechet derivative of vector functions and A_i $(0 \le i \le 6)$ are scalar parameters.

Next, the convergence requirements of the method in (26) is explored by using the Taylor's expansion on vector functions.

Theorem 3.1. *Suppose that $\Psi : \Theta \subseteq \Re^n \to \Re^n$ is sufficiently differentiable in the convex set Ω that contains the solution X_* of $\Psi(X) = \mathbf{0}$ such that $\Psi'(\cdot) \neq \mathbf{0}$ in Θ. For an initial guess X_0 in the neighbourhood X_*, the scheme in* (26) *will generate sequence of approximations of X_* that converge to X_* with CO four provided the parameters A_i satisfies the conditions: $A_4 = \frac{1}{16}(28A_1+3A_2-9A_3+3)$, $A_5 = \frac{1}{16}(-12A_1+25A_2+21A_3+9)$ and $A_6 = \frac{1}{16}(12A_1-9A_2-5A_3-9)$.*

Proof. Consider the expansion of $\Psi(X)$ about X_k using the Taylor's expansion, then

$$\begin{aligned}\Psi(X) &= \Psi(X_k) + \Psi'(X_k)(X - X_k) + \frac{1}{2!}\Psi''(X_k)(X - X_k)^2 \\ &\quad + \frac{1}{3!}\Psi'''(X_k)(X - X_k)^3 + \frac{1}{4!}\Psi^{(iv)}(X_k)(X - X_k)^4 + O(\|X - X_k\|^5).\end{aligned} \tag{29}$$

Suppose $E_K = X_k - X_*$ is error at kth iteration point and set $X = X_*$ in (29), then

$$\Psi(X_k) = \sum_{n=1}^{4}\left[(-1)^{n+1}\frac{1}{n!}\Psi^{(n)}(X_k)(E_k)^n\right] + O(\|E_k\|^5), \tag{30}$$

where $\Psi^{(n)}(X_k)$ is nth Frechet derivatives of $\Psi(X_k)$.

The expression in (30) is then pre-multiplied by $\Psi(X_k)^{-1}$ to have

$$G(X_k) = E_k + \sum_{n=2}^{4}\left[(-1)^{n+1}\frac{1}{n!}\Psi(X_k)^{-1}\Psi^{(n)}(X_k)(E_k)^n\right] + O(\|E_k\|^5). \tag{31}$$

Using (31) the equation $Y_k = X_k - \frac{2}{3}G(X_k)$ can be rewritten and expressed as

$$Y_k - X_k = \frac{2}{3}\left(-E_k + \sum_{n=2}^{4}\left[(-1)^n\frac{1}{n!}\Psi(X_k)^{-1}\Psi^{(n)}(X_k)(E_k)^n\right] + O(\|E_k\|^5)\right). \tag{32}$$

Consequently,

$$\begin{aligned}(Y_k - X_k)^2 &= \frac{4}{9}E_k^2 - \frac{4}{9}\Psi(X_k)^{-1}\Psi''(X_k)E_k^3 \\ &\quad + \frac{1}{27}\Psi(X_k)^{-1}[4\Psi'''(X_k) + 3\Psi''(X_k)\Psi(X_k)^{-1}\Psi''(X_k)]E_k^3 + O(\|E_k\|^5),\end{aligned} \tag{33}$$

$$(Y_k - X_k)^3 = -\frac{8}{27}E_k^3 + \frac{4}{9}\Psi(X_k)^{-1}\Psi''(X_k)E_k^4 + O(\|E_k\|^5) \tag{34}$$

and

$$(Y_k - X_k)^4 = -\frac{16}{81}E_k^4 + O(\|E_k\|^5). \tag{35}$$

Now, the Taylor's expansion of $\Psi(Y_k)$ about X_k is:

$$\Psi'(Y_k) = \sum_{n=1}^{4}\left[\frac{1}{n!}\Psi^{(n)}(X_k)(Y_k - X_k)^{n-1}\right] + O(\|Y_k - X_k\|^5). \tag{36}$$

Using (36) the next two expansions are obtained,

$$\begin{aligned} N = &(1+A_2+A_3)\Psi'(X_k)^2 - \frac{2}{3}(1+A_1+A_2)[\Psi'(X_k)\Psi''(X_k)]E_k \\ &+ \frac{1}{9}(1+A_2)[\Psi'(X_k)(2\Psi'''(X_k)+3\Psi''(X_k)\Psi'(X_k)^{-1}\Psi''(X_k)) \\ &+ A_1(2\Psi'(X_k)\Psi'''(X_k)+7\Psi''(X_k)^2)]E_k^2 \\ &+ \frac{1}{81}[-((1+A_2)\Psi'(X_k)(4\Psi^{(4)}(X_k)+27\Psi''(X_k)\Psi'''(X_k)\Psi'(X_k)^{-1}\Psi'''(X_k))) \\ &- A_1(4\Psi'(X_k)\Psi^{(4)}(X_k)+36\Psi''(X_k)\Psi'(X_k)^{-1}\Psi''(X_k)+20\Psi''(X_k)\Psi'''(X_k))]E_k^3 \\ &+ \frac{1}{972}[A_1(8\Psi'(X_k)\Psi^{(5)}(X_k)+\cdots+9\Psi'(X_k)^{-1}(8\Psi'''(X_k)^2+\cdots \\ &+ 6\Psi''(X_k)^2\Psi'(X_k)^{-1}\Psi'''(X_k)))]E_k^4 + O(\|E_k^5\|) \end{aligned} \tag{37}$$

and

$$\begin{aligned} M = &(1+A_5+A_6)\Psi'(X_k)^2 - \frac{2}{3}(1+A_4+A_5)[\Psi'(X_k)\Psi''(X_k)]E_k \\ &+ \frac{1}{9}(1+A_5)[\Psi'(X_k)(2\Psi'''(X_k)+3\Psi''(X_k)\Psi'(X_k)^{-1}\Psi''(X_k)) \\ &+ A_4(2\Psi'(X_k)\Psi'''(X_k)+7\Psi''(X_k)^2)]E_k^2 \\ &+ \frac{1}{81}[-((1+A_5)\Psi'(X_k)(4\Psi^{(4)}(X_k)+27\Psi''(X_k)\Psi'''(X_k)\Psi'(X_k)^{-1}\Psi'''(X_k))) \\ &- A_4(4\Psi'(X_k)\Psi^{(4)}(X_k)+36\Psi''(X_k)\Psi'(X_k)^{-1}\Psi''(X_k)+20\Psi''(X_k)\Psi'''(X_k))]E_k^3 \\ &+ \frac{1}{972}[A_1(8\Psi'(X_k)\Psi^{(5)}(X_k)+\cdots+9\Psi'(X_k)^{-1}(8\Psi'''(X_k)^2+\cdots \\ &+ 6\Psi''(X_k)^2\Psi'(X_k)^{-1}\Psi'''(X_k)))]E_k^4 + O(\|E_k^5\|). \end{aligned} \tag{38}$$

Now, (26) can be rewritten as:

$$M\Psi'(X_k)E_{k+1} = M\Psi'(X_k)E_k - N\Psi(X_k). \tag{39}$$

Using the expansions in (37) and (38) to expand (39), we obtained the following:

$$\begin{aligned} M\Psi'(X_k)E_{k+1} = &(A_6+A_5-A_3-A_2)\Psi'(X_k)^3E_k \\ &+ \frac{1}{6}(3+4A_1+7A_2+3A_3-4A_4-4A_5)\Psi'(X_k)^2\Psi''(X_k)E_k^2 \\ &+ \frac{1}{18}(\Psi'(X_k)((3+4A_1+7A_2+3A_3-4A_4-4A_5)\Psi'(X_k)\Psi'''(X_k)) \\ &\quad + 2\Psi''(X_k)^2(3+6A_2-7A_4-3A_5+10A_1))E_k^3 \end{aligned}$$

$$+\frac{1}{648}(4A_1(8\Psi'(X_k)^2\Psi^{(4)}(X_k)+\cdots+24\Psi''(X_k)\Psi'''(X_k)(6+\cdots-9A_5)))E_k^4$$
$$+O(\|E_k^5\|). \tag{40}$$

The sequence of approximations of the scheme in (26), will converge to the solution X_* of $\Psi(X_k)=\mathbf{0}$ if the coefficients of E_k^i $(1\le i\le 3)$ are annihilated. This can be achieved when the next sets of equations holds,

$$\left.\begin{aligned}&A_6+A_5-A_3-A_2=0,\\&4A_1+7A_2+3A_3-4A_4-4A_5+3=0,\\&28A_1+3A_2-9A_3-16A_4+3=0.\end{aligned}\right\} \tag{41}$$

The solutions to the equations in (41) are obtained as:

$$\left.\begin{aligned}&A_4=\frac{1}{16}(28A_1+3A_2-9A_3+3),\\&A_5=\frac{1}{16}(21A_3+25A_2-12A_1+9),\\&A_6=\frac{1}{16}(12A_1-9A_2-5A_3-9).\end{aligned}\right\} \tag{42}$$

The substitution of the results in (42) into (40) resulted to the next equation.

$$M\Psi'(X_k)E_{k+1}=\frac{1}{216}[\Lambda\Psi'(X_k)\Psi^{(iv)}(X_k)\Psi'(X_k)+6\Psi''(X_k)\Psi'(X_k)(-3\eta\Psi'''(X_k)$$
$$+\mu\Psi''(X_k)\Psi'(X_k)^{-1}\Psi''(X_k))]E_k^4+O(\|E_k^5\|), \tag{43}$$

where $\Lambda=1+A_2+A_3$ and $\mu=3+2A_1+3A_2+9A_3$.
The expression in (43) shows that the scheme in (26) has order 4 convergence.
Consequent upon the proof of Theorem 3.1 and after some simplifications, the scheme in (26) is finally presented as a free tri-parametric scheme given as:

$$X_{k+1}=X_k-16[\varpi_1I+2\varpi_2B(X_k)+\varpi_3B(X_k)^2]^{-1}[A_3I+B(X_k)(\varpi_4I+A_1B(X_k))], \tag{44}$$

where $\varpi_1=12A_1-9A_2-5A_3-9$, $\varpi_2=-20A_1+11A_2+15A_3+11$, $\varpi_3=28A_1+3A_2-9A_3+3$ and $\varpi_4=1-A_1+A_2$. □

3.1 Some Concrete Forms

Some special forms of the scheme in (44) are presented next.

Form 1: By assigning $A_2=A_3=0$ and $A_2=\frac{3}{4}$ or $a_3=\frac{1+A_2}{3}$ and $A_1=0$, the nth-dimensional Jarratt scheme (nJS) is recovered as:

$$X_{k+1}=X_k+[2I-6B(X_k)]^{-1}[I+3B(X_k)]G(X_k). \tag{45}$$

Form 2: When $A_1=\frac{3}{8}$, $A_2=-\frac{9}{8}$ and $A_2=\frac{9}{8}$ the method by Sharma *et al*. [17] (nSS) is obtained as:

$$X_{k+1}=X_k-\left[-\frac{1}{2}I+\frac{9}{8}H(X_k)+\frac{3}{8}B(X_k)\right]G(X_k), \tag{46}$$

where $H(X_k)=\Psi'(Y_k)^{-1}\Psi'(X_k)$.

Form 3: Taking $A_1 = \frac{9}{128}$, $A_2 = -\frac{143}{128}$ and $A_2 = \frac{23}{128}$, behold the equivalent n-dimensional form of the $KACS$ (21) denoted as $nKACS$ and presented as:

$$X_{k+1} = X_k - \left[\frac{23}{8}I - 3H(X_k) + \frac{9}{8}H(X_k)^2\right]G(X_k). \tag{47}$$

Form 4: For $A_1 = A_3 = 0$, a new scheme of CO four (nOS_4^1) is derived as:

$$X_{k+1} = X_k + 16[9I - 22B(X_k) - 3B(X_k)^2]^{-1}B(X_k)G(X_k). \tag{48}$$

Form 5: The substitution of $A_1 = A_2 = -0$ and $A_3 = -\frac{1}{3}$ in (44), reduced the family of schemes to a new CO four scheme (nOS_4^2) presented as:

$$X_{k+1} = X_k - 8[11I - 18B(X_k) - 9B(X_k)^2]^{-1}[I - 3B(X_k)]G(X_k). \tag{49}$$

Form 6: For $A_1 = 0$, $A_2 = -1$, a scheme with CO four (nOS_4^3) is constructed as:

$$X_{k+1} = X_k + 16[5I - 30B(X_k) + 9B(X_k)^2]^{-1}G(X_k). \tag{50}$$

Form 7: Consider $A_1 = A_3 = 0$ and $A_2 = -1$, the new scheme (nOS_3^4) with CO is discovered as:

$$X_{k+1} = X_k - 4[I - 3B(X_k)]^{-1}G(X_k). \tag{51}$$

4. Numerical Applications

The numerical applications of some constructed typical members of the developed schemes (OS_4^1, OS_4^2, OS_4^3, and OS_3^4 from (18) and $nOS_4^1, nOS_4^2, nOS_4^3$, and nOS_4^3 from (44)) are presented in this section. The practical examples used for the developed schemes implementation includes some nonlinear and system of nonlinear models from mathematical sciences and engineering fields. The obtained numerical results from the developed schemes implementation, where compared with the results by some robust existing schemes ($JS_4, SBS_4, nJS_4, nSBS_4$ taken from [6] and [16]) that are also members of the developed family of schemes. The mpmath-PYTHON program was designed and executed for all the schemes. In all programs, $|\Psi(x)| \le 10^{-100}$ was adopted for program termination criterion and 1000sf used as approximation precision. To compute ρ_{coc}, Definition 2.2 was utilised.

4.1 Numerical Applications on Scalar NL Models

The presented schemes were applied to solve some scalar nonlinear models. Numerical results obtained were presented in the format $E.Fe-H$ which represents $E.F \times 10^H$, ($E, F, H \in \Re$) in Table 1.

Application 4.1 (Environmental Engineering Model [3]). The nonlinear model in environmental engineering for determining downstream oxygen level Q mg/L when sewage is discharged in a river is given as

$$\Psi_1(x) = 10 - 20(\exp(-.15x) - \exp(-.5x)) = 0, \tag{52}$$

where x is the downstream distance in kilometres. A problem may arise to determine x when the oxygen level Q is 5 mg/L. Here, the actual solution is $x_* = 0.9762298\ldots$.

Application 4.2 (Chemical Engineering Model [3]). Water vapor (H_2O) heated at high temperatures splits apart hydrogen (H) and oxygen (O) into fractions. To determine the H_2O

mole fraction x that dissociates is modeled as:

$$\Psi_2(x) = K - \left(\frac{2\rho}{2+\rho}\right)^{\frac{1}{2}} \left(\frac{x}{1-x}\right) = 0, \tag{53}$$

where K and ρ are constant of reaction equilibrium and mixture total pressure. Suppose $\rho = 3$ and $K = 0.05$, the H_2O mole fraction $x_* = 0.0282494411\ldots$ that dissociates can be determined using the developed schemes.

Application 4.3 (Anti-Symmetric Buckling (ASB) [15]). The ASB equation is:

$$\Psi_3(x) = \exp(x) + x - 20, \quad x_* = 2.842\ldots. \tag{54}$$

Application 4.4 (Vertical Stress [18]). One basic stresses experienced in an underground structures that is finite is the vertical stress and is described as:

$$\Psi_4(x) = -\left(\frac{1}{4}\right) + \left(\frac{\cos x \sin x + x}{\pi}\right) = 0. \tag{55}$$

The solution of $\Psi_4(x) = 0$ at $x = 0$ and $x = 0.9$ is $0.4160\ldots$.

Application 4.5 (Van der Waals Equation (VDWE) [15]). The VDWE is given as:

$$\Psi_5(x) = -5.289 + 9.067 * x - 5.181 * x^2 + .986 * x^3 = 0 \tag{56}$$

and its solution is $x_* = 1.9298\ldots$.

Application 4.6 (Falling Parachutist Velocity [3]). The model describing falling parachutist is given as:

$$\Psi_6(x) = v - \frac{gx}{c}\left(1 - \exp\left(-\frac{c}{x}\right)t\right) = 0, \tag{57}$$

where $g = 9.8$ N is gravitational force, $c = 14$ kg/s is drag coefficient, $v = 35$ m/s is velocity and $t = 7$ s is time. The mass x kg can be calculated. The solution is $x_* = 63.6496\ldots$.

Application 4.7 (Jumper Mass (JS) [15]). The JS equation is:

$$\Psi_7(x) = \sin x - x + 1 = 0 \tag{58}$$

and its solution is $x_* = 1.9345\ldots$.

4.2 Numerical Application on System of Nonlinear Models

Some real life models in vector form where solved using the developed schemes and their computation performance compared in this subsection.

Application 4.8 (Chemical Equilibrium (CE) [10]). The CE system model that described the combination of carbon oxide (x_1), oxygen (x_2), Hydrogen (x_3), Nitrogen (x_4) and x_5 a factor depicting the moles number of a product formed for each mole of consumed propane is given as:

$$\Psi_8(X) = \begin{bmatrix} x_1 + x_1x_2 - 3x_5 \\ x_1 + 2x_1x_2 + x_2x_3^2 + \rho_8x_2 - \rho x_5 + \rho_{10}x_2^2 + \rho_7x_1x_3 + \rho_9x_2x_4 \\ 2x_2x_3^2 + 2\rho_5x_3^2 - 8x_5 + \rho_6x_3 + \rho_7x_2x_3 \\ \rho_9x_2x_4 + 2x_4^2 - 4\rho_5 \\ x_1(x_2+1) + \rho_{10}x_2^2 + x_2x_3^2 + \rho_8x_2 + \rho_5x_3^2 + x_4^2 - 1 + \rho_6x_3 + \rho_7x_2x_3 + \rho_9x_2x_4 \end{bmatrix}$$

$$= \mathbf{0}.$$

Table 1. Numerical experiments results on nonlinear models

Applications	IM	x_0	n	$\lvert\Psi(x_{i+1})\rvert$	ρ_{coc}	x_0	N	$\lvert\Psi(x_{i+1})\rvert$	ρ_{coc}
	JS		4	1.8e-293	4.0		4	3.3e-173	4.0
	SBS		4	1.3e-243	4.0		4	2.8e-136	4.0
	$KACS$		4	7.3e-207	4.0		4	1.6e-112	4.0
$\Psi_1(x)$	OS_4^1	1.2	4	2.0e-334	4.0	0.2	4	1.2e-200	4.0
	OS_4^2		4	1.5e-296	4.0		4	2.5e-182	4.0
	OS_4^3		4	1.2e-250	4.0		4	4.2e-143	4.0
	OS_3^4		5	6.6e-256	3.0		5	1.3e-144	3.0
	JS		4	9.2e-202	4.0		4	2.3e-159	4.0
	SBS		5	1.7e-356	4.0		4	7.6e-156	4.0
	$KACS$		5	1.0e-246	4.0		5	5.6e-333	4.0
$\Psi_2(x)$	OS_4^1	0.5	4	1.3e-113	4.0	-0.5	4	2.0e-143	4.0
	OS_4^2		5	1.9e-242	4.0		4	4.8e-127	4.0
	OS_4^3		4	1.8e-101	4.0		4	1.6e-169	4.0
	OS_3^4		6	9.5e-265	3.0		5	2.5e-126	3.0
	JS		4	1.2e-138	4.0		4	4.3e-329	4.0
	SBS		5	7.0e-243	4.0		4	3.4e-273	4.0
	$KACS$		6	2.7e-211	4.0		4	7.7e-241	4.0
$\Psi_3(x)$	OS_4^1	2.0	4	1.9e-207	4.0	3.0	3	1.5e-109	4.0
	OS_4^2		4	8.9e-107	4.0		4	4.8e-305	4.0
	OS_4^3		5	1.4e-214	4.0		4	1.4e-281	4.0
	OS_3^4		6	1.8e-264	3.0		5	1.4e-275	3.0
	JS		4	1.4e-202	4.0		6	4.0e-127	4.0
	SBS		4	1.7e-196	4.0			failed	
	$KACS$		4	1.7e-188	4.0		6	1.4e-106	4.0
$\Psi_4(x)$	OS_4^1	0.0	4	1.4e-204	4.0	0.9	9	9.7e-310	4.0
	OS_4^2		4	1.2e-209	4.0		7	8.0e-111	4.0
	OS_4^3		4	7.4e-198	4.0		>30	2.4e-139	4.0
	OS_3^4		5	7.7e-243	3.0		12	3.3e-213	3.0
	JS		8	4.0e-244	4.0		4	6.0e-138	4.0
	SBS		15	3.1e-223	4.0		4	1.0e-110	
	$KACS$		>15	6.8e-248	4.0		5	3.6e-356	4.0
$\Psi_5(x)$	OS_4^1	-1.0	>15	1.3e-228	4.0	2	4	9.2e-154	4.0
	OS_4^2		8	1.6e-384	4.0		4	2.0e-203	4.0
	OS_4^3		>15	1.3e-188	4.0		4	3.5e-116	4.0
	OS_3^4			failed			5	4.3e-121	3.0
	JS		3	9.4e-110	4.0		4	1.3e-280	4.0
	SBS		4	2.6e-367	4.0		4	2.2e-195	4.0
	$KACS$		4	3.0e-311	4.0		4	7.5e-152	4.0
$\Psi_6(x)$	OS_4^1	60	3	4.1e-113	4.0	80	4	1.9e-279	4.0
	OS_4^2		4	1.4e-389	4.0		4	2.2e-223	4.0
	OS_4^3		4	1.4e-376	4.0		4	5.5e-203	4.0
	OS_3^4		4	2.2e-120	3.0		5	3.9e-203	3.0
	JS		5	7.5e-311	4.0		4	3.0e-148	4.0
	SBS		5	2.2e-135	4.0		4	3.7e-138	4.0
	$KACS$			Failed			4	7.8e-127	4.0
$\Psi_7(x)$	OS_4^1	1.0	4	1.3e-116	4.0	3.0	4	4.5e-152	4.0
	OS_4^2		4	4.8e-114	4.0		4	1.7e-165	4.0
	OS_4^3		7	2.4e-149	4.0		4	2.4e-140	4.0
	OS_3^4		6	5.0e-226	3.0		5	5.0e-166	3.0

Application 4.9 (Neurophysiology [5]). The model is of the form:

$$\Psi_9(X) = \begin{bmatrix} x_4^2 + x_2^2 - 1 \\ x_4^3 x_6 + x_3^3 x_5 - c_1 \\ x_2^3 x_6 + x_1^3 x_5 - c_2 \\ x_4^2 x_6 x_2 + x_3^2 x_5 x_1 - c_3 \\ x_2^2 x_6 x_4 + x_1^2 x_5 x_3 - c_4 \end{bmatrix} = \mathbf{0}.$$

In this case, the constants c_i are assumed to be 0 for all $i = 1,2,3,4$. The solution of $\Psi_9(X) = 0$ was obtained using $X_0^1 = (1.8, 2.6, 1.5, 2.3, 3.8, 3.1)$ and $X_0^2 = (2,3,2,2,4,3)$ in all the methods compared.

Table 2. Numerical experiments results on system of nonlinear models

Applications	IS	X_0	n	$\lVert\Psi(X_{i+1})\rVert$	ρ_{coc}	X_0	n	$\lVert\Psi(X_{i+1})\rVert$	ρ_{coc}
	nJS		5	3.3e-19	4.0		5	1.1e-17	4.0
	$nSBS$		4	2.9e-19	4.0		11	1.2e-28	4.0
	$nKACS$		10	2.7e-48	4.0		9	3.0e-30	4.0
$\Psi_8(X)$	nOS_4^1	$X_0^{(1)}$	10	4.3e-34	4.0	$X_0^{(2)}$	11	6.1e-40	4.0
	nOS_4^2		10	3.5e-33	4.0		11	3.8e-34	4.0
	nOS_4^3		10	1.2e-37	4.0		10	1.4e-26	4.0
	nOS_3^4		10	2.3e-27	4.0		11	3.9e-33	4.0
	nJS		5	3.5e-35	4.0		5	4.6e-43	4.0
	$nSBS$		4	6.3e-47	4.0		4	1.0e-44	4.0
	$nKACS$		8	2.2e-31	4.0		8	9.0e-30	4.0
$\Psi_9(X)$	nOS_4^1	$X_0^{(1)}$	8	2.5e-25	4.0	$X_0^{(2)}$	8	9.3e-24	4.0
	nOS_4^2		8	6.6e-24	4.0		9	2.5e-50	4.0
	nOS_4^3		8	9.5e-27	4.0		8	3.8e-25	4.0
	nOS_3^4		9	2.3e-44	4.0		9	2.1e-47	4.0

4.3 Some Notes on Results

From Tables 1 and 2, the following can be observed:

(i) All constructed schemes from the developed family of schemes, solved the nonlinear and systems of nonlinear models.

(ii) The newly presented special forms of the developed family of schemes are tough competitors to the compared existing JS, SBS and KACS.

(iii) The numerical CO (ρ_{coc}) of all the special forms of the developed family of schemes, are in agreement with the theoretical CO of the families.

5. Conclusion

A new tri-parameter family of the JS [6] was developed by replacing the weight function in the second step of the JS with a SRAF. Under some hypothesis, the family of schemes was proven to have CO four and require the assessment of three distinct functions in an iteration cycle. Consequently, all its members, which includes the famous JS [6], SBS [17], KACS [1,4,8] are optimal as conjectured by Kung-Traub in [9]. The extension of the method from its scalar form to vector form by appropriate replacement of the SRAF with VRAF was also studied. Consequently, its analysis in norm space, have shown that it retained the CO established in the case of scalar space.

Nomenclature

IS	:	Iteration Scheme
CO	:	Convergence Order
NIS	:	Newton Iterative Scheme
RAF	:	Rational Approximation Function
BVRAF	:	Bi-Variate Rational Approximation Function

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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