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Research Article

Soft *n*-Normed Linear Spaces: Generalizations and Extensions from Soft Normed Spaces

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Abstract. This article presents the notion of soft 2-normed linear space (NDLS) and extends it to soft n-NDLS, providing a versatile framework beyond traditional soft NDLS and n-normed spaces (NDS). Established results in soft NDLS are adapted to soft n-NDLS, bolstered by practical examples.

Keywords. Soft sets, Soft linear spaces, Soft NDLS, Soft n-NDLS

Mathematics Subject Classification (2020). 46B20, 46A99, 46B99, 03E72

1. Introduction

The management of uncertainty constitutes a pivotal challenge across numerous academic domains, ranging from engineering to economics. Addressing this challenge, Molodtsov's seminal work on soft sets [10] revolutionized the field by introducing a formalism for handling imprecise or incomplete information. This concept has since found extensive applications in decision-making, pattern recognition, and data analysis.

Building on this foundation, subsequent researchers, such as Das *et al.* [6], have contributed significantly to the principle of soft NDLS. Their investigations provided a mathematical framework for modeling uncertainty within normed vector spaces. This extension not only enriched the theoretical foundation but also opened up new avenues for practical applications, particularly in fields like optimization and approximation theory.

Gunawan and Mashadi's [8] derivation of an $(n-1)$-norm from the n-norm represents a notable stride in the study of normed spaces. Their work illuminated the intricate relationships between different levels of norms, unveiling a profound structure within these mathematical structures. This revelation holds far-reaching implications, not only for the theoretical comprehension of normed spaces but also for their practical efficacy across diverse domains.

In the present study, we draw inspiration from these seminal contributions and aim to broaden the horizons of NDLS. We introduce the conceptions of soft 2-NDLS and soft n-NDLS, unifying and extending theories from soft NDLS (Das *et al*. [6]), n-NDS (Gunawan and Mashadi [8]), and insights from Narayanan and Vijayabalaji [11].

1.1 Historical Review

The management of uncertainty has been a perennial challenge in various fields, driving the need for formal frameworks to grapple with imprecise or incomplete information. A pivotal milestone in this endeavor was reached with the introduction of soft sets by Molodtsov [1] in his seminal work. This breakthrough, which happened in the latter half of the 20th century, marked a watershed moment in the field of uncertainty modeling.

Molodtsov's notion of soft sets provided a versatile mathematical tool to represent and manipulate uncertain information. It laid the groundwork for a systematic approach for handling vague, ambiguous, or contradictory data. This development found immediate applications in decision-making, where uncertainties abound, and in pattern recognition, where precise distinctions can be elusive.

Building on the foundation laid by Molodtsov, subsequent researchers further extended and refined the theory. Notable among these contributions is the work of Das *et al*. [6], who delved into the realm of soft (NDLS). Their investigations, spanning the late 20th and early 21st centuries, provided a rigorous mathematical framework for modeling uncertainty within normed vector spaces.

The incorporation of soft norm into the theory of normed spaces represented a significant advancement. This extension not only enriched the theoretical underpinnings but also opened up new avenues for practical applications. In particular, the newfound ability to quantify uncertainty within normed spaces had profound implications for fields like optimization and approximation theory.

In parallel, Gunawan and Mashadi [8] made a noteworthy contribution in the study of normed spaces by deriving an $(n-1)$-norm from the n-norm. This development, emerging in the early 21st century, shed light on the intricate relationships between different levels of norms. It revealed a deep-seated structure within these mathematical spaces, offering a fresh perspective on the nature of NDS.

This revelation had far-reaching implications, not only for the theoretical understanding of normed spaces but also for their practical utility across diverse domains. It provided researchers with a powerful tool to navigate the complexities of spaces equipped with norms, offering insights that could be leveraged in a wide range of applications, from mathematical analysis to engineering and beyond.

In summary, the historical evolution of uncertainty management reflects a trajectory of continuous innovation and refinement. From Molodtsov's [10] pioneering work on soft sets to the extensions into soft NDLS by Das *et al*. [6] and the insights gained from Gunawan and Mashadi's [8] study of normed spaces, each contribution has left an indelible mark on the field. These developments collectively form the foundation upon which the present study builds, seeking to further extend and unify these theories to address contemporary challenges in uncertainty modeling.

1.2 Need of the Present Study

While the existing body of work on soft NDLS and n-NDLS has been instrumental in handling uncertainty, there exists a compelling case for extending this theory. The current paradigm, though insightful, primarily addresses specific instances and may not fully capture the intricacies of uncertainty in more complex, multidimensional settings.

Moreover, Gunawan and Mashadi's [8] derivation of an $(n-1)$-norm from the n-norm represents a crucial advancement, prompting the exploration of higher-order norms and their potential implications. Investigating these extensions may uncover deeper insights into the underlying structures of normed spaces, offering a more nuanced understanding of their mathematical properties.

Practical applications of soft n-normed linear spaces remain an area ripe for exploration. By establishing concrete use cases across various domains, from optimization to decision-making under uncertainty, we aim to not only validate the theoretical underpinnings of this extended framework but also showcase its utility in real-world scenarios.

In light of these attentions, this paper seeks to bridge these gaps by introducing the concept of soft n-NDLS. Through unification and extension of existing theories, we endeavor to provide a versatile mathematical framework capable of addressing the diverse challenges posed by uncertainty. By means of rigorous analysis and practical demonstrations, we aim to underscore the relevance and significance of this extended theory in both theoretical and applied contexts.

2. Preliminaries

In this part, we undertake a comprehensive analysis of the essential notations and definitions outlined by Molodtsov [10], Das *et al*. [6], and Yazar *et al*. [14]. This critical examination sets the stage for subsequent discussions and extensions, providing the necessary theoretical groundwork for our exploration.

Definition 2.1 ([10]). Let $\mathbb{U}$ be a universe and $\mathbb{E}$ be a collection of parameters. We define a soft set over $\mathbb{U}$ as a pair $(\mathcal{F},\mathbb{A})$, where F is defined as $\mathcal{F}:\mathbb{A}\to\mathfrak{P}(\mathbb{U})$, with $\mathfrak{P}(U)$ signifying the power set of $\mathbb{U}$, and $\mathbb{A}\subset E$.

Definition 2.2 ([6]). A soft set $(\mathcal{F},\mathbb{E})$ is called an absolute soft set $(\check{\mathbb{U}})$ if $\forall\ \delta\in\mathbb{A}$, $\mathcal{F}(\delta)=\mathbb{U}$.

Definition 2.3 ([6]). A soft set $(\mathcal{F},\mathbb{E})$ is called a null soft set (Φ) if $\forall\ \delta\in\mathbb{A}$, $\mathcal{F}(\delta)=\Phi$.

Definition 2.4 ([6])**.** Let $\mathcal{R}$ be the collection of real numbers and $\mathfrak{B}(\mathcal{R})$ the set of all non-empty bounded subsets of $\mathcal{R}$ and $\mathbb{A}$ taken as a collection of parameters. Then $\mathcal{F}:\mathbb{A}\to\mathfrak{B}(\mathcal{R})$ is named as a soft real set and is represented by $(\mathcal{F},\mathbb{A})$. If $(\mathcal{F},\mathbb{A})$ is a singleton soft set, then $(\mathcal{F},\mathbb{A})$ with the equivalent soft element, it is called a soft real number.

We use $\widetilde{p},\widetilde{q},\widetilde{r},\dots$ for soft real numbers. For example, $\widetilde{0},\widetilde{1}$ are soft real numbers where $\widetilde{0}(\delta)=0$, $\widetilde{1}(\delta)=1$, $\forall\ \delta\in\mathbb{A}$.

Definition 2.5 ([6])**.** For two soft real numbers $\widetilde{p},\widetilde{q}$, the following hold:

(i) $\widetilde{p}\le\widetilde{q}$ if $\widetilde{p}(\delta)\le\widetilde{q}(\delta)$, $\forall\ \delta\in\mathbb{A}$,

(ii) $\widetilde{p}\ge\widetilde{q}$ if $\widetilde{p}(\delta)\ge\widetilde{q}(\delta)$, $\forall\ \delta\in\mathbb{A}$,

(iii) $\widetilde{p}\,\widetilde{<}\,\widetilde{q}$ if $\widetilde{p}(\delta)<\widetilde{q}(\delta)$, $\forall\ \delta\in\mathbb{A}$,

(iv) $\widetilde{p}\,\widetilde{>}\,\widetilde{q}$ if $\widetilde{p}(\delta)>\widetilde{q}(\delta)$, $\forall\ \delta\in\mathbb{A}$.

Definition 2.6 ([6])**.** A soft real number $\widetilde{p}$ is non-negative if $\widetilde{p}(\delta)\ge 0$, for all $\delta\in\mathbb{A}$. We symbolize the set of all non-negative soft real numbers by $\mathfrak{R}(\mathbb{A})^*$.

Definition 2.7 ([6])**.** A soft set (F,A), $F:A\to\mathcal{P}(V)$ will be indicated by F only. A soft set G over V called a *soft linear space* (SLS) of V over K if $G(\lambda)$ is a vector subspace of V, $\forall\ \lambda\in A$.

Definition 2.8 ([6])**.** Let $\mathbb{G}$ be a SLS of $\mathbb{V}$ over $\mathbb{K}$. Then a soft element of $\mathbb{G}$ is a *soft vector* (SV) of $\mathbb{G}$ and it is denoted by $\widetilde{p}_\delta$ if there is exactly one $\delta\in\mathbb{A}$ such that $\mathcal{F}(\delta)=\{p\}$ for some $p\in\mathbb{U}$ and $\mathcal{F}(\delta')=\phi$, $\forall\ \delta'\in\mathbb{A}\setminus\{\delta\}$. A soft element of the soft set $(\mathbb{K},\mathbb{A})$ is called a soft scalar and $\mathbb{K}$ being the scalar field.

Definition 2.9 ([6])**.** A SV $\widetilde{p}_\delta$ in a SLS $\mathbb{G}$ is the null SV if $\widetilde{p}_\delta=\widetilde{p}(\delta)=\theta$, $\forall\ \delta\in\mathbb{A}$, θ being the zero element of $\mathbb{V}$ and indicated by Θ. A SV is non-null if it is not a null SV.

We use $\widetilde{p}_\delta,\widetilde{q}_\delta,\widetilde{r}_\delta,\dots$ for SVs of a SLS and $\widetilde{k},\widetilde{l},\widetilde{m},\dots$ for soft real numbers.

The collection of all SVs over $\check{\mathbb{U}}$ will be represented by $\mathcal{SV}(\check{\mathbb{U}})$ and it is called SLS.

Definition 2.10 ([6])**.** Let $\widetilde{p}_\delta,\widetilde{q}_\delta$ be SVs of $\mathbb{G}$ and $\widetilde{k}$ be a soft scalar. Then

(i) $\widetilde{p}_\delta+\widetilde{q}_\delta=\widetilde{p}(\delta)+\widetilde{q}(\delta)=(\widetilde{p}+\widetilde{q})(\delta)$,

(ii) $(\widetilde{k}\widetilde{p})_\delta=(\widetilde{k}\widetilde{p})(\delta)=\widetilde{k}(\delta)\widetilde{p}(\delta)$, $\forall\ \delta\in\mathbb{A}$. Obviously, $\widetilde{p}_\delta+\widetilde{q}_\delta$, $\widetilde{k}\widetilde{p}_\delta$ are SVs of $\mathbb{G}$.

Definition 2.11 ([11])**.** Let $\|\cdot,\cdot\|$ be a real-valued function on $\mathbb{U}\times\mathbb{U}$ satisfying the succeeding conditions:

(c-2-N1) $\|p,q\|=0$ if and only if p and q are linearly dependent,

(c-2-N2) $\|p,q\|=\|q,p\|$,

(c-2-N3) $\|kp,q\|=|k|\,\|p,q\|$, where k is real,

(c-2-N4) $\|p,q+r\|\le\|p,q\|+\|p,r\|$,

$\|\cdot,\cdot\|$ is called a 2-norm on $\mathbb{U}$ and the pair $(\mathbb{U},\|\cdot,\cdot\|)$ is called a 2-NDLS.

Definition 2.12 ([11])**.** A real-valued function $\|\cdot,\dots,\cdot\|$ on $\underbrace{\mathbb{U}\times\cdots\times\mathbb{U}}_{n}$ satisfying the succeeding four properties,

(c-n-N1) $\|p_1,p_2,\dots,p_n\| = 0$if and only if $p_1,p_2,\dots,p_n$ are linearly dependent,

(c-n-N2) $\|p_1,p_2,\dots,p_n\|$is invariant under any permutation,

(c-n-N3) $\|p_1,p_2,\dots,kp_n\| = |k|\|p_1,p_2,\dots,p_n\|$, for any $k\in R$ (real),

(c-n-N4) $\|p_1,p_2,\dots,p_{n-1},q+r\| \le \|p_1,p_2,\dots,p_{n-1},q\| + \|p_1,p_2,\dots,p_{n-1},r\|$,

is called a n-norm on $\mathbb{U}$ and the pair $(\mathbb{U},\|\cdot,\dots,\cdot\|)$ is called a n-NDLS.

Definition 2.13 ([14])**.** Let $\check{\mathbb{U}}$ be the absolute SLS, i.e., $\check{\mathbb{U}}(\delta)=\mathbb{U}$, $\forall\ \delta\in\mathbb{A}$. Then a mapping $\|\cdot\| : \mathcal{SV}(\check{\mathbb{U}}) \to \mathfrak{R}(\mathbb{A})^*$ is a soft norm on the SLS $\check{\mathbb{U}}$ if $\|\cdot\|$ satisfies the succeeding conditions:

For all $\widetilde{p}_\delta,\widetilde{q}_\delta \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}})$,

(s-N1) $\|\widetilde{p}_\delta\| \ge \overline{0}$,

(s-N2) $\|\widetilde{p}_\delta\| = \widetilde{0}$ if and only if $\widetilde{p}_\delta = \theta$,

(s-N3) $\|\widetilde{k}\cdot\widetilde{p}_\delta\| = |\widetilde{k}|\cdot\|\widetilde{p}_\delta\|$ for every soft scalar $\widetilde{k}$,

(s-N4) $\|\widetilde{p}_\delta+\widetilde{q}_\delta\| \le \|\widetilde{p}_\delta\| + \|\widetilde{q}_\delta\|$.

The SLS $\mathcal{SV}(\check{\mathbb{U}})$ with a soft norm $\|\cdot\|$ on $\check{\mathbb{U}}$ is said to be a soft NDLS and is symbolized by $(\check{\mathbb{U}},\|\cdot\|,\mathbb{A})$ or $(\check{\mathbb{U}},\|\cdot\|)$.

3. Soft 2-NDLS and soft n-NDLS

In this section, we embark on the starter of the concept of soft 2-NDLS and soft n-NDLS, offering a broader perspective that extends beyond the scope of Definition 2.13, as outlined below.

Definition 3.1. Let $\check{\mathbb{U}}$ be the absolute SLS, i.e., $\check{\mathbb{U}}(\delta)=\mathbb{U}$, $\forall\ \delta\in\mathbb{A}$, of dimension greater than 1, then $\|\cdot,\cdot\| : \mathcal{SV}(\check{\mathbb{U}})\times\mathcal{SV}(\check{\mathbb{U}}) \to \mathfrak{R}(\mathbb{A})^*$ is said to be a soft 2-norm on the SLS on $\check{\mathbb{U}}\times\check{\mathbb{U}}$ satisfying succeeding conditions:

For all $\widetilde{p}_\delta,\widetilde{q}_\delta,\widetilde{r}_\delta \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}})$,

(s-2-N1) $\|\widetilde{p}_\delta,\widetilde{q}_\delta\|$ if and only if $\widetilde{p}_\delta$ and $\widetilde{q}_\delta$ are linearly dependent,

(s-2-N2) $\|\widetilde{p}_\delta,\widetilde{q}_\delta\| = \|\widetilde{q}_\delta,\widetilde{p}_\delta\|$,

(s-2-N3) $\|\widetilde{k}\widetilde{p}_\delta,\widetilde{q}_\delta\| = |\widetilde{k}|\|\widetilde{p}_\delta,\widetilde{q}_\delta\|$, for every soft scalar $\widetilde{k}$,

(s-2-N4) $\|\widetilde{p}_\delta,\widetilde{q}_\delta+\widetilde{r}_\delta\| \le \|\widetilde{p}_\delta,\widetilde{q}_\delta\| + \|\widetilde{p}_\delta,\widetilde{r}_\delta\|$.

The pair $(\check{\mathbb{U}},\|\cdot,\cdot\|)$ is called a soft 2-NDLS.

Definition 3.2. The function $\|\cdot,\dots,\cdot\| : \mathcal{SV}(\check{\mathbb{U}})\times\cdots\times\mathcal{SV}(\check{\mathbb{U}}) \to \mathfrak{R}(\mathbb{A})^*$ is said to be a soft n-norm on the SLS on $\underbrace{\check{\mathbb{U}}\times\cdots\times\check{\mathbb{U}}}_{n}$ satisfying the succeeding four properties:

For all $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}$, $\widetilde{q}_{\delta_n}$ and $\widetilde{r}_{\delta_n} \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}})$,

(s-n-N1) $\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\| = \theta$ if and only if $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}$ are linearly dependent,

(s-n-N2) $\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|$ is invariant under any permutation, (s-n-N3) $\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{k}\widetilde{p}_{\delta_n}\| = |\widetilde{k}|\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|$, for every soft scalar $\widetilde{k}$,

(s-n-N4) $\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}+\widetilde{r}_{\delta_n}\| \le \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}\| + \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{r}_{\delta_n}\|$.
The pair $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ is called a soft n-NDLS.

Proposition 3.3. *Every parameterized family of n-norms $\{\|\cdot,\dots,\cdot\|_\delta : \delta \in \mathbb{A}\}$ on a vector space $\mathbb{U}$ can be deliberated as a soft n-norm on the soft vector space $\mathcal{SV}(\check{\mathbb{U}})$.*

Proof. Let $\check{\mathbb{U}}$ be the absolute SLS over a field $\mathbb{K}$, $\mathbb{A}$ be a non-empty collection of parameters. Let $\{\|\cdot,\dots,\cdot\|_\delta : \delta \in \mathbb{A}\}$ be a parameterized set of n-norms. Let $\widetilde{p}_\delta \,\widetilde{\in}\, \check{\mathbb{U}}$ then

$$\widetilde{p}(\delta) \in \mathbb{U}, \quad \forall\ \delta \in \mathbb{A}.$$

Let us define a mapping $\|\cdot,\dots,\cdot\| : \mathcal{SV}(\check{\mathbb{U}}) \times \cdots \times \mathcal{SV}(\check{\mathbb{U}}) \to \mathfrak{R}(\mathbb{A})^*$ by

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|(\delta) = \|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_n}(\delta)\|_\delta, \quad \forall\ \delta \in \mathbb{A},\ \forall\ \widetilde{p}_{\delta_i} \,\widetilde{\in}\, \mathcal{SV}(\check{\mathbb{U}}).$$

Then $\|\cdot,\dots,\cdot\|$ is a soft n-norm on $\underbrace{\check{\mathbb{U}}\times\cdots\times\check{\mathbb{U}}}_{n}$.

We now verify the conditions (s-n-N1), (s-n-N2), (s-n-N3) and (s-n-N4) for soft n-norm.

(s-n-N1): For all $\widetilde{p}_\delta \,\widetilde{\in}\, \mathcal{SV}(\check{\mathbb{U}})$ and for every $\delta \in \mathbb{A}$,

$$\begin{aligned}
& \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|(\delta) = \theta \\
\iff\ & \|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_n}(\delta)\|_\delta = \theta \\
\iff\ & \widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_n}(\delta) = \theta \\
\iff\ & \widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n} = \Theta \\
\iff\ & \widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n} \text{ are linearly dependent}
\end{aligned}$$

$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\| = \theta$ if and only if $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}$ are linearly dependent.

(s-n-N2): For all, $\delta \in \mathbb{A}$,

$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|(\delta) = \|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_n}(\delta)\|_\delta$ is invariant under any permutation.

(s-n-N3):

$$\begin{aligned}
\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{k}\widetilde{p}_{\delta_n}\|(\delta) &= \|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{k}(\delta)\widetilde{p}_{\delta_n}(\delta)\|_\delta \\
&= |\widetilde{k}(\delta)|\|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_n}(\delta)\|_\delta \\
&= |\widetilde{k}(\delta)|\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|(\delta) \\
&= |\widetilde{k}|\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|(\delta).
\end{aligned}$$

So,

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{k}\widetilde{p}_{\delta_n}\| = |\widetilde{k}|\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|, \quad \forall\ \widetilde{p}_{\delta_i} \,\widetilde{\in}\, \mathcal{SV}(\check{\mathbb{U}})$$

and for all soft scalar $\widetilde{k}$.

(s-n-N4): For all $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n},\widetilde{q}_{\delta_n}$ and $\widetilde{r}_{\delta_n} \,\widetilde{\in}\, \mathcal{SV}(\check{\mathbb{U}})$

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}+\widetilde{r}_{\delta_n}\| \le \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}\| + \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{r}_{\delta_n}\|,$$

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}+\widetilde{r}_{\delta_n}\|(\delta) = \|\widetilde{p}_{\delta_1}(\delta),\widetilde{p}_{\delta_2}(\delta),\dots,\widetilde{p}_{\delta_{n-1}}(\delta),(\widetilde{q}_{\delta_n}+\widetilde{r}_{\delta_n})(\delta)\|_\delta$$

$$
\begin{aligned}
&= \|\widetilde{p}_{\delta_1}(\delta), \widetilde{p}_{\delta_2}(\delta), \dots, \widetilde{p}_{\delta_{n-1}}(\delta), (\widetilde{q}_{\delta_n}(\delta) + \widetilde{r}_{\delta_n}(\delta))\|_\delta \\
&\le \|\widetilde{p}_{\delta_1}(\delta), \widetilde{p}_{\delta_2}(\delta), \dots, \widetilde{p}_{\delta_{n-1}}(\delta), \widetilde{q}_{\delta_n}(\delta)\|_\delta \\
&\quad + \|\widetilde{p}_{\delta_1}(\delta), \widetilde{p}_{\delta_2}(\delta), \dots, \widetilde{p}_{\delta_{n-1}}(\delta), \widetilde{r}_{\delta_n}(\delta)\|_\delta \\
&\le \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{q}_{\delta_n}\|(\lambda) + \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{r}_{\delta_n}\|(\lambda) \\
&\le (\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{q}_{\delta_n}\| + \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{r}_{\delta_n}\|)(\lambda) \\
&\le \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{q}_{\delta_n}\| + \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{r}_{\delta_n}\|.
\end{aligned}
$$

Hence, (s-n-N4) is satisfied.

Therefore, $\|\cdot,\dots,\cdot\|$ is a soft n-norm on $\underbrace{\check{\mathbb{U}} \times \dots \times \check{\mathbb{U}}}_{n}$ and subsequently $(\check{\mathbb{U}}, \|\cdot,\dots,\cdot\|)$ is a soft n-NDLS. □

Proposition 3.4. *Every n-norm $\|\cdot,\dots,\cdot\|_{\mathbb{U}}$ on $\mathbb{U}$ can be extended to a soft n-norm on the SLS $\mathcal{SV}(\check{\mathbb{U}})$.*

Proof. Let $\check{\mathbb{U}}$ be the absolute SLS and $\mathbb{A}$ be a non-empty collection of parameters.

We define a mapping $\|\cdot,\dots,\cdot\| : \mathcal{SV}(\check{\mathbb{U}}) \times \dots \times \mathcal{SV}(\check{\mathbb{U}}) \to \mathfrak{R}(\mathbb{A})^*$ by

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}\|(\delta) = \|\widetilde{p}_{\delta_1}(\delta), \widetilde{p}_{\delta_2}(\delta), \dots, \widetilde{p}_{\delta_n}(\delta)\|_{\mathbb{U}}, \quad \forall\ \delta \in \mathbb{A},\ \forall\ \widetilde{p}_{\delta_i} \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}}).$$

By using the same technique as in Proposition 3.3, it can be easily shown that $\|\cdot,\dots,\cdot\|$ is a soft n-norm on $\underbrace{\check{\mathbb{U}} \times \dots \times \check{\mathbb{U}}}_{n}$.

This soft n-norm is generated using the n-norm $\|\cdot,\dots,\cdot\|_{\mathbb{U}}$ and is called the soft n-norm generated by $\|\cdot,\dots,\cdot\|_{\mathbb{U}}$. □

Example 3.5. Let $(\check{\mathbb{U}}, \|\cdot,\dots,\cdot\|)$ be a soft n-NDS. In this case, for every $\widetilde{p}_{\delta_i} \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}})$, $\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}\| = \sum_{i=1}^{n} |\delta_i| + \|p_1, p_2, \dots, p_n\|$ is a soft n-norm.

For all $\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}, \widetilde{q}_{\delta_n}$ and $\widetilde{r}_{\delta_n} \widetilde{\in} \mathcal{SV}(\check{\mathbb{U}})$ and for every soft scalar $\widetilde{k}$:

(s-n-N1):

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}\| = \sum_{i=1}^{n} |\delta_i| + \|p_1, p_2, \dots, p_n\|,$$

$$
\begin{aligned}
& \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}\| = \theta \\
\iff & \sum_{i=1}^{n} |\delta_i| + \|p_1, p_2, \dots, p_n\| = \theta \\
\iff & \delta_i = 0, \quad \|p_1, p_2, \dots, p_n\| = 0 \\
\iff & \delta_i = 0, \quad p_1, p_2, \dots, p_n \text{ are linearly dependent}
\end{aligned}
$$

$\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}$ are linearly dependent.

(s-n-N2): Clearly,

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_n}\| = \sum_{i=1}^{n} |\delta_i| + \|p_1, p_2, \dots, p_n\|$$

is invariant under any permutation.

(s-n-N3):

$$\begin{aligned}\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{k}\widetilde{p}_{\delta_n}\| &= \sum_{i=1}^{n}|k\delta_i| + \|p_1,p_2,\dots,kp_n\| \\ &= |k|\left(\sum_{i=1}^{n}|\delta_i| + \|p_1,p_2,\dots,p_n\|\right) \\ &= |k|\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_n}\|.\end{aligned}$$

(s-n-N4):

$$\begin{aligned}&\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}+\widetilde{r}_{\delta_n}\| \\ &\quad= \sum_{i=1}^{n}|\delta_i+\delta'_i| + \|p_1,p_2,\dots,p_{n-1},q+r\| \\ &\quad\le \sum_{i=1}^{n}|\delta_i| + \sum_{i=1}^{n}|\delta'_i| + \|p_1,p_2,\dots,p_{n-1},q\| + \|p_1,p_2,\dots,p_{n-1},r\| \\ &\quad= \left(\sum_{i=1}^{n}|\delta_i| + \|p_1,p_2,\dots,p_{n-1},q\|\right) + \left(\sum_{i=1}^{n}|\delta'_i| + \|p_1,p_2,\dots,p_{n-1},r\|\right) \\ &\quad\le \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{q}_{\delta_n}\| + \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{r}_{\delta_n}\|.\end{aligned}$$

4. Convergence and Completeness of Soft *n*-NDLS

Converge and completeness of soft n-NDLS are discussed below.

Definition 4.1. A sequence of SV $\{\widetilde{p}^n_{\delta_n}\}$ in soft n-NDLS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ is said to converge to the SV $\widetilde{p}^0_{\delta_0}\,\widetilde{\in}\,\check{\mathbb{U}}$, whenever $\lim_{n\to\infty}\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}^0_{\delta_0}\| = \widetilde{0}$, for every $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}}\,\widetilde{\in}\,\check{\mathbb{U}}$ and it is denoted by $\widetilde{p}^n_{\delta_n}\to\widetilde{p}^0_{\delta_0}$ as $n\to\infty$.

Definition 4.2. A sequence of SVs $\{\widetilde{p}^n_{\delta_n}\}$ in soft n-NDS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ is said to be a Cauchy sequence on soft n-NDLS if $\lim_{m,n\to\infty}\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}^m_{\delta_m}\| = \widetilde{0}$ for every $\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}}\,\widetilde{\in}\,\check{\mathbb{U}}$.

Definition 4.3. A soft n-NDLS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ is called complete soft n-norm if every Cauchy sequence is convergent on soft n-NDLS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$.

Theorem 4.4. *Every convergent sequence in a soft n-NDLS is Cauchy and also every Cauchy sequence is bounded.*

Proof. Let $\{\widetilde{p}^n_{\delta_n}\}$ be a convergent sequence of SVs with limit $\widetilde{p}_\delta$ in soft n-NDLS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$. Then equivalent to each $\widetilde{\epsilon}>\widetilde{0}$, there exists $m\in N$ such that $\widetilde{p}_\delta\,\widetilde{\in}\,\widetilde{B}\left(\widetilde{p}_\delta,\frac{\widetilde{\epsilon}}{2}\right)$, i.e.,

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_\delta\| \le \frac{\widetilde{\epsilon}}{2}, \quad \forall\; n\ge m,$$

then, for $i,j\ge m$,

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|$$

$$\leq \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_\delta\|+\|\widetilde{p}_\delta-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|$$
$$\leq \frac{\widetilde{\epsilon}}{2}+\frac{\widetilde{\epsilon}}{2}=\widetilde{\epsilon}.$$

Hence $\{\widetilde{p}^n_{\delta_n}\}$ is a Cauchy sequence.

Let $\{\widetilde{p}^n_{\delta_n}\}$ be a Cauchy sequence of SVs in soft n-NDLS $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$. Then there exists $s\in N$ such that

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|<\widetilde{1},\quad \forall\, m,n\geq s.$$

Take $\widetilde{M}$ with

$$\widetilde{M}(\lambda)=\max_{1\leq m,n\leq s}\{\|\widetilde{p}^n_{\delta_n}-\widetilde{p}^m_{\delta_m}\|(\lambda)\},\quad \forall\, \lambda\,\widetilde{\in}\,\mathbb{A},$$

then, for $1\leq n\leq s$ and $m\geq s$,

$$\begin{aligned}&\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|\\ &\quad\leq \|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{s-1}},\widetilde{p}^s_{\delta_s}\|\\ &\qquad+\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{s-1}},\widetilde{p}^s_{\delta_s}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|\\ &\quad<\widetilde{M}+\widetilde{1}.\end{aligned}$$

thus,

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{m-1}},\widetilde{p}^m_{\delta_m}\|<\widetilde{M}+\widetilde{1},\quad \forall\, m,n\,\widetilde{\in}\,N$$

and consequently the sequence is bounded. □

Theorem 4.5. *Let $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ be a soft n-NDLS. Then*

(i) *if $\widetilde{p}^n_{\delta_n}\to\widetilde{p}_\delta$ and $\widetilde{q}^n_{\delta_n}\to\widetilde{q}_\delta$ then $\widetilde{p}^n_{\delta_n}+\widetilde{q}^n_{\delta_n}\to\widetilde{p}_\delta+\widetilde{q}_\delta$,*

(ii) *if $\widetilde{p}^n_{\delta_n}\to\widetilde{p}_\delta$ and $\widetilde{\lambda}_{\delta_n}\to\widetilde{\lambda}_\delta$ then $\widetilde{\lambda}_{\delta_n}\cdot\widetilde{p}^n_{\delta_n}\to\widetilde{\lambda}_\delta\widetilde{p}_\delta$, where $\{\widetilde{\lambda}_{\delta_n}\}$ is a sequence of soft scalars,*

(iii) *if $\{\widetilde{p}^n_{\delta_n}\}$ and $\{\widetilde{q}^n_{\delta_n}\}$ are Cauchy sequences in $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$ and $\{\widetilde{\lambda}_{\delta_n}\}$ is a Cauchy sequence of soft scalars, then $\{\widetilde{p}^n_{\delta_n}+\widetilde{q}^n_{\delta_n}\}$ and $\{\widetilde{\lambda}_{\delta_n}\cdot\widetilde{p}^n_{\delta_n}\}$ are also Cauchy sequences in $(\check{\mathbb{U}},\|\cdot,\dots,\cdot\|)$.*

Proof. (i) Since $\widetilde{p}^n_{\delta_n}\to\widetilde{p}_\delta$ and $\widetilde{q}^n_{\delta_n}\to\widetilde{q}_\delta$, for $\widetilde{\epsilon}>\widetilde{0}$, there exists positive integers N_1,N_2 such that

$$\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_\delta\|<\frac{\widetilde{\epsilon}}{2},\quad \forall\, n\geq N_1$$

and

$$\|\widetilde{q}_{\delta_1},\widetilde{q}_{\delta_2},\dots,\widetilde{q}_{\delta_{n-1}},\widetilde{q}^n_{\delta_n}-\widetilde{q}_\delta\|<\frac{\widetilde{\epsilon}}{2},\quad \forall\, n\geq N_2.$$

Let $N=\max\{N_1,N_2\}$, then both the above relations hold for $n\geq N$, then

$$\begin{aligned}&\|(\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}+\widetilde{q}_{\delta_1},\widetilde{q}_{\delta_2},\dots,\widetilde{q}_{\delta_{n-1}},\widetilde{q}^n_{\delta_n})-(\widetilde{p}_\delta+\widetilde{q}_\delta)\|\\ &\quad=\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}+\widetilde{q}_{\delta_1},\widetilde{q}_{\delta_2},\dots,\widetilde{q}_{\delta_{n-1}},\widetilde{q}^n_{\delta_n}-\widetilde{p}_\delta-\widetilde{q}_\delta\|\\ &\quad\leq\|\widetilde{p}_{\delta_1},\widetilde{p}_{\delta_2},\dots,\widetilde{p}_{\delta_{n-1}},\widetilde{p}^n_{\delta_n}-\widetilde{p}_\delta\|+\|\widetilde{q}_{\delta_1},\widetilde{q}_{\delta_2},\dots,\widetilde{q}_{\delta_{n-1}},\widetilde{q}^n_{\delta_n}-\widetilde{q}_\delta\|\\ &\quad<\frac{\widetilde{\epsilon}}{2}+\frac{\widetilde{\epsilon}}{2}=\widetilde{\epsilon},\quad \forall\, n\geq N\end{aligned}$$

$$\Longrightarrow\quad \widetilde{p}^n_{\delta_n}+\widetilde{q}^n_{\delta_n}\to\widetilde{p}_\delta+\widetilde{q}_\delta.$$

(ii) Since $\widetilde{p}^n_{\delta_n} \to \widetilde{p}_\delta$, for $\widetilde{\epsilon} > \widetilde{0}$, there is a positive integers N such that

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta\| < \widetilde{\epsilon}, \quad \forall\ n \geq N.$$

Now,

$$\begin{aligned}\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}\| &= \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta + \widetilde{p}_\delta\| \\ &\leq \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta\| + \|\widetilde{p}_\delta\| \\ &< \widetilde{\epsilon} + \|\widetilde{p}_\delta\|, \quad \forall\ n \geq N\end{aligned}$$

$$\implies \quad \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}\| < \widetilde{\epsilon} + \|\widetilde{p}_\delta\|, \quad \forall\ n \geq N.$$

Thus, the sequence $\{\widetilde{p}^n_{\delta_n}\}$ is bounded.
Now,

$$\begin{aligned}&\|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_\delta \widetilde{p}_\delta\| \\ &\quad = \|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_\delta (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) + \widetilde{\lambda}_\delta (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_\delta \widetilde{p}_\delta\| \\ &\quad = \|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n})(\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_\delta) + \widetilde{\lambda}_\delta (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta)\| \\ &\quad \leq \|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n})(\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_\delta)\| + \|\widetilde{\lambda}_\delta (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta)\| \\ &\quad = |\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_\delta| \cdot \|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n})\| + |\widetilde{\lambda}_\delta| \cdot \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta\|\end{aligned}$$

implies

$$\begin{aligned}&\|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_\delta \widetilde{p}_\delta\| \\ &\quad \leq |\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_\delta| \cdot \|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n})\| + |\widetilde{\lambda}_\delta| \cdot \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta\|.\end{aligned}$$

Since $\widetilde{p}^n_{\delta_n} \to \widetilde{p}_\delta$ and $\widetilde{\lambda}_{\delta_n} \to \widetilde{\lambda}_\delta$, then, we have $|\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_\delta| \to \widetilde{0}$ and $\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_\delta\| \to \widetilde{0}$ as $n \to \infty$.
Therefore, $\|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_\delta \cdot \widetilde{p}_\delta\| \to \widetilde{0}$ as $n \to \infty$.
Hence, $\widetilde{\lambda}_{\delta_n} \cdot \widetilde{p}^n_{\delta_n} \to \widetilde{\lambda}_\delta \widetilde{p}_\delta$.

(iii) Let $\{\widetilde{p}^n_{\delta_n}\}$ and $\{\widetilde{q}^n_{\delta_n}\}$ be Cauchy sequences in $(\check{\mathbb{U}}, \|\cdot, \ldots, \cdot\|)$, then for $\widetilde{\epsilon} > \widetilde{0}$, there exists positive integers N_1, N_2 such that

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}\| < \frac{\widetilde{\epsilon}}{2}, \quad \forall\ m, n \geq N_1$$

and

$$\|\widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{n-1}}, \widetilde{q}^n_{\delta_n} - \widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{m-1}}, \widetilde{q}^m_{\delta_m}\| < \frac{\widetilde{\epsilon}}{2}, \quad \forall\ m, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$, then both the above relations hold for $m, n \geq N$.
Now,

$$\begin{aligned}&\|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} + \widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{n-1}}, \widetilde{q}^n_{\delta_n}) - (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m} + \widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{n-1}}, \widetilde{q}^m_{\delta_m})\| \\ &= \|(\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{m-1}} \widetilde{p}^m_{\delta_m}) + (\widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{n-1}}, \widetilde{q}^n_{\delta_n} - \widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{m-1}}, \widetilde{q}^m_{\delta_m})\| \\ &\leq \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \ldots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}\| + \|\widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{n-1}}, \widetilde{q}^n_{\delta_n} - \widetilde{q}_{\delta_1}, \widetilde{q}_{\delta_2}, \ldots, \widetilde{q}_{\delta_{m-1}}, \widetilde{q}^m_{\delta_m}\| \\ &< \frac{\widetilde{\epsilon}}{2} + \frac{\widetilde{\epsilon}}{2} = \widetilde{\epsilon}, \quad \forall\ m, n \geq N\end{aligned}$$

implies

$$\{\widetilde{p}^n_{\delta_n} + \widetilde{q}^n_{\delta_n}\}$$

is a Cauchy sequence in $(\check{\mathbb{U}}, \|\cdot,\dots,\cdot\|)$.

Since $\{\widetilde{p}^n_{\delta_n}\}$ is a Cauchy sequences in $(\check{\mathbb{U}}, \|\cdot,\dots,\cdot\|)$, for $\widetilde{\epsilon} > \widetilde{0}$, there exists positive integers N such that

$$\|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}\| < \widetilde{\epsilon}, \quad \forall\, m, n \geq N.$$

Taking in particular $n = m+1$,

$$\|\widetilde{p}^{m+1}_{\delta_{m+1}}\| < \widetilde{\epsilon}, \quad \forall\, m, n \geq N,$$

so $\{\|\widetilde{p}^n_{\delta_n}\|\}$ is bounded.

Now, $\|\widetilde{\lambda}_{\delta_n}\|$ is bounded too.

Then,

$$\begin{aligned}
&\|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_{\delta_m} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m})\| \\
&\quad = \|\widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n}) - \widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}) \\
&\qquad + \widetilde{\lambda}_{\delta_n} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}) - \widetilde{\lambda}_{\delta_m} \cdot (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m})\| \\
&\quad = \|\widetilde{\lambda}_{\delta_n} (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}) \\
&\qquad + (\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m})(\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_{\delta_m})\| \\
&\quad \leq |\widetilde{\lambda}_{\delta_n}| \cdot \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{n-1}}, \widetilde{p}^n_{\delta_n} - \widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{m-1}}, \widetilde{p}^m_{\delta_m}\| \\
&\qquad + |\widetilde{\lambda}_{\delta_n} - \widetilde{\lambda}_{\delta_m}| \cdot \|\widetilde{p}_{\delta_1}, \widetilde{p}_{\delta_2}, \dots, \widetilde{p}_{\delta_{nm1}}, \widetilde{p}^m_{\delta_m}\| \\
&\quad \to \widetilde{0} \text{ as } n \to \infty.
\end{aligned}$$

Therefore $\{\widetilde{\lambda}_{\delta_n} \widetilde{p}^n_{\delta_n}\}$ is also a Cauchy sequence in $(\check{\mathbb{U}}, \|\cdot,\dots,\cdot\|)$. □

5. Conclusion

We have expanded the theoretical landscape of normed linear spaces by introducing the innovative concepts of soft 2-NDLS and soft n-NDLS. This extension transcends the boundaries set by the conventional notion of a norm, allowing for a more encompassing framework to tackle uncertainty within normed spaces.

Furthermore, we have addressed the critical topics of convergence and completeness within the context of soft n-NDLS. These discussions lay the foundation for a deeper understanding of the behavior and properties of spaces equipped with a soft norm, offering valuable insights for both theoretical explorations and practical applications.

By unifying and extending existing theories, we have endeavored to provide a versatile mathematical framework capable of addressing the diverse challenges posed by uncertainty. The integration of soft norms with normed spaces opens up new avenues for research in various domains, from optimization to decision-making in complex, uncertain environments.

In conclusion, the introduction of soft 2-NDLS and soft n-NDLS, equipped with the notion of a soft norm, marks a significant advancement in the field of uncertainty modeling. This framework not only enriches the theoretical underpinnings of normed spaces but also offers practical tools for navigating uncertainty in a wide range of applications. We anticipate that this work will serve as a catalyst for further exploration and innovation in this burgeoning area of study.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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