Communications in Mathematics and Applications
Vol. 15, No. 1, pp. 95–110, 2024
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v15i1.2420

http://www.rgnpublications.com

Research Article

Super Restrained Domination in the Join of Some Graphs

Monrille L. Lorono and Stephanie O. Espinola*

Department of Mathematics and Statistics, University of Southeastern Philippines, Davao City, Philippines
***Corresponding author:** saomega@usep.edu.ph

Received: September 20, 2023 **Accepted:** January 1, 2024

Abstract. Let $G=(V(G),E(G))$ be a simple graph. A set $S\subseteq V(G)$ is a restrained dominating set S if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G)\setminus S$. It is a super restrained dominating set if for every vertex $u\in V(G)\setminus S$, there exists $v\in S$ such that $N_G(v)\cap(V(G)\setminus S)=\{u\}$. The minimum cardinality of a super restrained dominating set in G, denoted by $\gamma_{spr}(G)$, is called the super restrained domination number of G. In this paper, the researchers obtained the super restrained domination number of the following graphs: $F_n\cong K_1+P_n$, $W_n\cong K_1+C_n$, $S_n\cong K_1+\overline{K}_n$, $D_n^{(m)}\cong K_1+mK_{n-1}$ and $K_{m,n}\cong\overline{K}_m+\overline{K}_n$.

Keywords. Domination, Restrained domination, Super domination, Super restrained domination, Join

Mathematics Subject Classification (2020). 05C38, 05C69, 05C76

1. Introduction

All graphs considered in this paper are all connected, finite, and simple. Let graph $G=(V,E)$, be connected, finite, and simple. The graph G has a vertex set $V=V(G)$ and an edge set $E=E(G)$. Further, let the order of the graph G be m, that is $|V|=|V(G)|=m$ and the size of the graph G be n, that is, $|E|=|E(G)|=n$.

A subset S of $V(G)$ is a dominating set of G if for every $v\in(V(G)\setminus S)$, there exists $x\in S$ such that $xv\in E(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G (Enriquez [3]).

One variant of domination is the restrained domination in graphs and it was introduced by Telle and Proskurowski [5] as a vertex partitioning problem. A set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in S denoted as $V(G)\backslash S$ is adjacent to a vertex in S and to a vertex in $V(G)\backslash S$ (Domke *et al.* [2]). Alternately, a subset S of $V(G)$ is a restrained dominating set if $S = V(G)$ or $\langle V(G)\backslash S\rangle$ has no isolated. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of graph G (Monsanto and Rara [4]). A set $D \subset V(G)$ is called a super dominating set if for every vertex $u \in V(G)\backslash D$, there exist $v \in D$ such that $N_G(v) \cap (V(G)\backslash D) = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating set in G denoted by $\gamma_{sp}(G)$. A restrained dominating set S is a super restrained dominating set in a graph G if for every vertex $u \in V(G)\backslash S$, there exists $v \in S$ such that $N_G(v) \cap (V(G)\backslash S) = \{u\}$. The minimum cardinality of a super restrained dominating set in G, denoted by $\gamma_{spr}(G)$, is called the super restrained domination number of G (Enriquez [3]). For general concepts and graph theoretic terminologies, may refer to the book of Chartrand and Zhang [1].

2. Results

This section presents the super restrained domination of $F_n \cong K_1 + P_n$, $W_n \cong K_1 + C_n$, $S_n \cong K_1 + \overline{K}_n$, $D_n^{(m)} \cong mK_{n-1} + K_1$, and $K_{m,n} \cong \overline{K}_m + \overline{K}_n$.

Remark 2.1 ([3])**.** The super retrained dominating set is a super dominating set and a restrained dominating set.

Theorem 2.2 ([3])**.** *Let $G = K_n$. Then $\gamma_{spr}(G) = n$.*

Theorem 2.3. *For any graphs G of order $n \geq 2$, $\gamma_{sp}(G) \leq \gamma_{spr}(G)$.*

Proof. Let G be a graph of order $n \geq 2$. Let S be a super restrained dominating set in G with minimum cardinality. Then by Remark 2.1, S is a super dominating set. Thus, $\gamma_{sp}(G) \leq |S| \leq \gamma_{spr}(G)$. Therefore, $\gamma_{sp}(G) \leq \gamma_{spr}(G)$. □

Theorem 2.4. *Let H be a graph of order $n \geq 2$ and $K_1 = \langle v\rangle$ be a trivial graph. Then $S \subseteq V(K_1+H)$ is a super restrained dominating set of $K_1 + H$ if and only if*

$$S = S_v \cup \{v\},$$

where S_v is a super restrained dominating set in H.

Proof. Let H be a graph of order $n \geq 2$ and $K_1 = \langle v\rangle$ be a trivial graph. Let $S \subseteq V(K_1 + H)$ be a super restrained dominating set of $K_1 + H$. Suppose to the contrary that $v \notin S$. Then $S \subseteq V(H)$. Since S is a restrained dominating set, $S \neq V(H)$, otherwise $\langle V(H)\backslash S\rangle$ is a trivial graph K_1. Thus, $|V(H)\backslash S| \geq 1$. Let $x \in V(H)\backslash S$. Then for every $u \in S \subseteq V(H)$,

$$\begin{aligned} N_{K_1+H}(u) \cap (V(K_1 + H)\backslash S) &= (N_H(u) \cup \{v\}) \cap ((V(H)\backslash S) \cup \{v\}) \\ &\neq \{x\}, \quad \text{since } v \neq x. \end{aligned}$$

This is a contradiction to the assumption since S is a super restrained dominating set. Hence, $v \in S$.

Next, let $S_v = S \cap V(H)$. Then $S = S_v \cup \{v\}$. If $S_v = V(H)$, then S_v is a super restrained dominating set in H. Suppose that $S_v \neq V(H)$. Since $S = S_v \cup \{v\}$ is a super restrained dominating set of $K_1 + H$, $\langle V(K_1+H)\setminus S\rangle = \langle V(H)\setminus S\rangle = \langle V(H)\setminus S_v\rangle$ has no isolated vertices. Suppose that S_v is not a dominating set in H. Then, there exists a vertex $w \in V(H)\setminus S_v$ that is not dominated by any vertex in S_v. Since $\langle V(H)\setminus S_v\rangle$ has no isolated vertices, w has a neighbor, say $y \in V(H)\setminus S_v$. This imply that for every $u \in S_v$,

$$\begin{aligned} N_{K_1+H}(u) \cap (V(K_1+H)\setminus S_v) &= (N_H(u) \cup \{v\}) \cap (V(H)\setminus S_v) \\ &\neq \{w\}, \quad \text{since } w \notin N_H(u). \end{aligned}$$

Also,

$$\begin{aligned} N_{K_1+H}(v) \cap (V(K_1+H)\setminus S) &= V(H) \cap (V(K_1+H)\setminus S) \\ &= V(H)\setminus S \\ &\neq \{w\}, \quad \text{since } w, y \in N_H(u) \cap (V(H)\setminus S) \text{ and } w \neq y. \end{aligned}$$

Thus, there exists $w \in V(K_1+H)\setminus S$ such that for all $d \in S = S_v \cup \{v\}$,

$$N_{K_1+H}(d) \cap (V(K_1+H)\setminus S) \neq \{w\}.$$

This is a contradiction since S is a super restrained dominating set of K_1+H. Thus, S_v is a restrained dominating set in H. Next, suppose that S_v is not a super dominating set in H. Then there exists a vertex $a \in V(H)\setminus S_v$ such that for every $b \in S_v$,

$$N_H(b) \cap (V(H)\setminus S_v) \neq \{a\}.$$

This implies that for every $b \in S_v$,

$$\begin{aligned} N_{K_1+H}(b) \cap (V(K_1+H)\setminus S) &= (N_H(b) \cup \{v\}) \cap (V(H)\setminus S_v) \\ &= (N_H(b) \cap V(H)\setminus S_v) \cup (\{v\} \cap (V(H)\setminus S_v)) \\ &= (N_H(b) \cap V(H)\setminus S_v) \cup \emptyset \\ &= N_H(b) \cap V(H)\setminus S_v \\ &\neq \{a\}. \end{aligned}$$

For $v \in S$,

$$\begin{aligned} &N_{K_1+H}(v) \cap (V(K_1+H)\setminus S) \\ &\quad = V(H) \cap (V(H)\setminus S_v) \\ &\quad = V(H)\setminus S_v \\ &\quad \neq \{a\}, \quad \text{since } \langle V(H)\setminus S_v\rangle \text{ has no isolated vertices.} \end{aligned}$$

Hence, there exists $a \in V(H)\setminus S_v = V(K_1+H)\setminus S$ such that for every $u \in S$,

$$N_{K_1+H}(u) \cap (V(K_1+H)\setminus S) \neq \{a\}.$$

This is a contradiction since S is a super restrained dominating set of K_1+H. Thus, S_v is a super dominating set in H. Therefore, S_v is a super restrained dominating set in H.

Suppose that $S = S_v \cup \{v\}$ where S_v is a super restrained dominating set in H. Since $v \in S$, it follows that S is a dominating set in $K_1 + H$. If $S_v = V(H)$, then $S = V(K_1 + H)$ is a super restrained dominating set in $K_1 + H$. If $S_v \neq V(H)$, then $\langle V(K_1 + H)\backslash S\rangle = \langle V(H)\backslash S_v\rangle$ has no isolated vertices since S_v is a restrained dominating set in H. Hence, S is a restrained dominating set in $K_1 + H$. Moreover, since S_v is a super restrained dominating set in H, for every $c \in V(H)\backslash S_v$, there exists a vertex $t \in S_v$ such that

$$N_H(t) \cap (V(H)\backslash S_v) = \{c\}.$$

Thus, for every $c \in V(H)\backslash S_v = V(K_1 + H)\backslash S$, there exists a vertex $t \in S_v \subseteq S$ such that

$$\begin{aligned} N_{K_1+H}(t) \cap (V(K_1 + H)\backslash S) &= (N_H(t) \cup \{v\}) \cap (V(H)\backslash S_v) \\ &= (N_H(t) \cap (V(H)\backslash S_v)) \cup (\{v\} \cap (V(H)\backslash S_v)) \\ &= \{c\} \cup \emptyset, \quad \text{since } v \notin V(H) \\ &= \{c\}. \end{aligned}$$

Hence, S is a super dominating set of $K_1 + H$. Consequently, $S = S_v \cup \{v\}$ is super restrained dominating set in $K_1 + H$. □

Corollary 2.5. *Let $F_n \cong K_1 + P_n$ be a fan graph of order $n + 1$ with $n \geq 2$. Then*

$$\gamma_{spr}(F_n) = 1 + \gamma_{spr}(P_n).$$

Corollary 2.6. *Let $W_n \cong K_1 + C_n$ be a wheel graph of order $n + 1$ with $n \geq 3$. Then*

$$\gamma_{spr}(W_n) = 1 + \gamma_{spr}(C_n).$$

Corollary 2.7. *Let $S_n \cong K_1 + \overline{K}_n$ be a star graph of order $n + 1$ with $n \geq 2$. Then*

$$\gamma_{spr}(S_n) = 1 + \gamma_{spr}(\overline{K}_n) = 1 + n.$$

Corollary 2.8. *Let $D_n^{(m)} \cong K_1 + mK_{n-1}$ be a windmill graph of order $m(n-1)+1$ with $m \geq 2$ and $n \geq 2$. Then*

$$\gamma_{spr}(D_n^{(m)}) = 1 + \gamma_{spr}(mK_{n-1}) = 1 + m(n-1).$$

Proof. By Theorem 2.2, $\gamma_{spr}(K_{n-1}) = n - 1$. Thus, by Theorem 2.4,

$$\gamma_{spr}(D_n^{(m)}) = 1 + m(n-1).$$ □

Theorem 2.9. *Let P_n be a path graph of order n such that $n \geq 4$, then*

$$\gamma_{spr}(P_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0 \pmod 4; \\ 2\lfloor \frac{n-1}{4} \rfloor + 1, & \text{if } n \equiv 1 \pmod 4; \\ 2\lfloor \frac{n-2}{4} \rfloor + 2, & \text{if } n \equiv 2 \pmod 4; \\ 2\lfloor \frac{n-3}{4} \rfloor + 3, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Consider the graph K_1+P_n in Figure 1, where $\{v_i : 1\le i\le n\}$ is the vertex set of P_n.

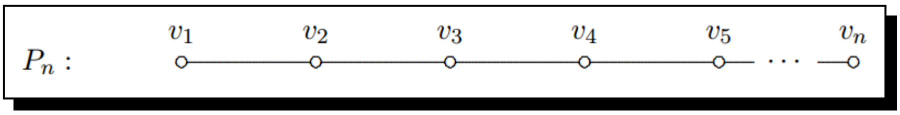

Figure 1. Path graph P_n

Consider the following cases:

Case 1: $n\equiv 0 (\text{mod } 4)$.

Let $n\equiv 0(\text{mod } 4)$. Then $n=4k$ for some positive integer k. This implies that $k=\frac{n}{4}$. Then we take the ceiling function of $\lceil k\rceil=\lceil\frac{n}{4}\rceil$. Let $S=S_1\cup S_2$, where $S_1=\{v_{4i}|i=1,2,\cdots,\lceil\frac{n}{4}\rceil\}$ and $S_2=\{v_{4i+1}|i=0,1,2,\cdots,\lceil\frac{n}{4}\rceil-1\}$. Then $V(P_n)\backslash S=\{v_{4i-1}|i=1,2,\cdots,\lceil\frac{n}{4}\rceil\}\cup\{v_{4i+2}|i=0,1,2,\cdots,\lceil\frac{n}{4}\rceil-1\}$. Observe that for each $1\le i\le\lceil\frac{n}{4}\rceil$, $v_{4i-1}v_{4i}\in E(P_n)$ and for each $0\le i\le\lceil\frac{n}{4}\rceil-1$, $v_{4i+2}v_{4i+1}\in E(P_n)$. Hence, for every $u\in V(P_n)\backslash S$, there exists $w\in S$ such that $uw\in E(P_n)$. Thus, S is a dominating set of P_n. Note that the subgraph of P_n induced by $V(P_n)\backslash S$ is shown Figure 3. Observe that $\langle V(P_n)\backslash S\rangle$ has no isolated vertices. Thus, S is a restrained dominating set of P_n.

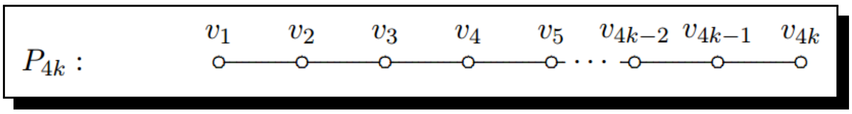

Figure 2. Path graph P_{4k}

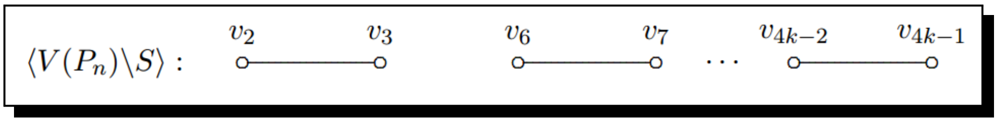

Figure 3. The subgraph of P_n induced by $V(P_n)\backslash S$

Now, we need to show that S is a super dominating set. Note that

$$V(P_n)\backslash S=\left\{v_{4i-1}|i=1,2,\cdots,\left\lceil\frac{n}{4}\right\rceil\right\}\cup\left\{v_{4i+2}|i=0,1,2,\cdots,\left\lceil\frac{n}{4}\right\rceil-1\right\}$$

and for each $v_{4i-1}\in V(P_n)\backslash S$, where $1\le i\le\lceil\frac{n}{4}\rceil$, there exists $v_{4i}\in S$ such that $N(v_{4i})\cap(V(P_n)\backslash S)=\{v_{4i-1}\}$. Also, for each $v_{4i+2}\in V(P_n)\backslash S$, where $0\le i\le\lceil\frac{n}{4}\rceil-1$, there exist $v_{4i+1}\in S$ such that $N(v_{4i+1})\cap(V(P_n)\backslash S)=\{v_{4i+2}\}$. Thus, S is a super dominating set of P_n. Consequently, S is a super restrained dominating set of P_n. Thus, for $n\equiv 0(\text{mod } 4)$,

$$\gamma_{spr}(P_n)\le|S|=\left\lceil\frac{n}{4}\right\rceil+\left\lceil\frac{n}{4}\right\rceil=2\left\lceil\frac{n}{4}\right\rceil=\left\lceil\frac{n}{2}\right\rceil.$$

Case 2: $n\equiv 1(\text{mod } 4)$

Let $n\equiv 1(\text{mod } 4)$. Then $n=4k+1$ for some positive integer k. This implies that $k=\frac{n-1}{4}$. Then we take the floor function of $\lfloor k\rfloor=\lfloor\frac{n-1}{4}\rfloor$. Let $T=T_1\cup T_2$, where $T_1=\{v_{4i}|i=1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\}$ and $T_2=\{v_{4i+1}|i=0,1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\}$. Then $V(P_n)\backslash T=\{v_{4i-1}|i=1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\}\cup\{v_{4i+2}|i=0,1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor-1\}$. Thus, for each $v_{4i-1}\in V(P_n)\backslash T$, there exists $v_{4i}\in T$ such that $v_{4i-1}v_{4i}\in E(P_n)$ for each $1\le i\le\lfloor\frac{n-1}{4}\rfloor$. Also, for each $v_{4i+2}\in V(P_n)\backslash T$, there exists $v_{4i+1}\in T$ such that $v_{4i+2}v_{4i+1}\in E(P_n)$ for each $0\le i\le\lfloor\frac{n-1}{4}\rfloor$. Thus, T is a dominating set of P_n. Note that the subgraph of P_n induced by $V(P_n)\backslash T$ is shown in Figure 5 has no isolated vertices. Thus, T is a restrained dominating set.

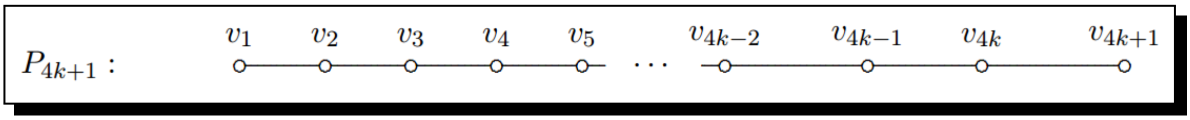

Figure 4. Path graph P_{4k+1}

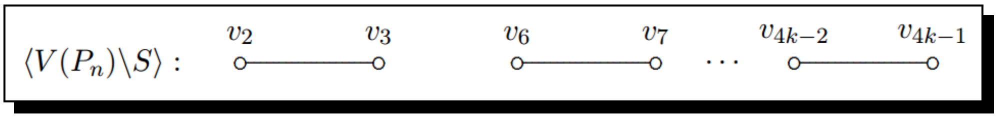

Figure 5. The subgraph of P_n induced by $V(P_n)\backslash T$

Now, we need to show that T is a super dominating set of P_n. Note that

$$V(P_n)\backslash T = \left\{v_{4i-1} | i = 1,2,\cdots,\left\lfloor\frac{n-1}{4}\right\rfloor\right\} \cup \left\{v_{4i+2} | i = 0,1,2,\cdots,\left\lfloor\frac{n-1}{4}\right\rfloor - 1\right\}$$

and for each $v_{4i-1} \in V(P_n)\backslash T$, where $1 \le i \le \lfloor\frac{n-1}{4}\rfloor$, there exists $v_{4i} \in T$ such that $N(v_{4i}) \cap (V(P_n)\backslash T) = \{v_{4i-1}\}$. Also, for each $v_{4i+2} \in V(P_n)\backslash T$, where $0 \le i \le \lfloor\frac{n-1}{4}\rfloor - 1$, there exist $v_{4i+1} \in T$ such that $N(v_{4i+1}) \cap (V(P_n)\backslash T) = \{v_{4i+2}\}$. Thus, T is a super dominating set of P_n. Consequently, T is a super restrained dominating set of P_n. Hence, for $n \equiv 1(\text{mod } 4)$,

$$\gamma_{spr}(P_n) \le |T| = \left\lfloor\frac{n-1}{4}\right\rfloor + \left\lfloor\frac{n-1}{4}\right\rfloor + 1 = 2\left\lfloor\frac{n-1}{4}\right\rfloor + 1.$$

Case 3: $n \equiv 2(\text{mod } 4)$

Let $n \equiv 2(\text{mod } 4)$. Then $n = 4k+2$ for some positive integer k. This implies that $k = \frac{n-2}{4}$. Then we take the floor of $\lfloor k\rfloor = \lfloor\frac{n-2}{4}\rfloor$. Let $X = X_1 \cup X_2 \cup \{v_n\}$, where $X_1 = \{v_{4i} | i = 1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$ and $X_2 = \{v_{4i+1} | i = 0,1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$. Then $V(P_n)\backslash X = \{v_{4i-1} | i = 1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\} \cup \{v_{4i+2} | i = 0,1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$. Thus, for each $v_{4i-1} \in V(P_n)\backslash X$, there exists $v_{4i} \in X$ such that $v_{4i-1}v_{4i} \in E(P_n)$ for each $1 \le i \le \lfloor\frac{n-2}{4}\rfloor$. Also, for each $v_{4i+2} \in V(P_n)\backslash X$, there exists $v_{4i+1} \in X$ such that $v_{4i+2}v_{4i+1} \in E(P_n)$ for each $0 \le i \le \lfloor\frac{n-2}{4}\rfloor$. Thus, X is a dominating set of P_n. Note that the subgraph of P_n induced by $V(P_n)\backslash X$ as shown in Figure 7 has no isolated vertices. Thus, X is a restrained dominating set.

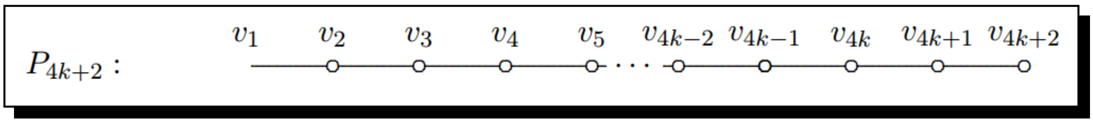

Figure 6. Path graph P_{4k+2}

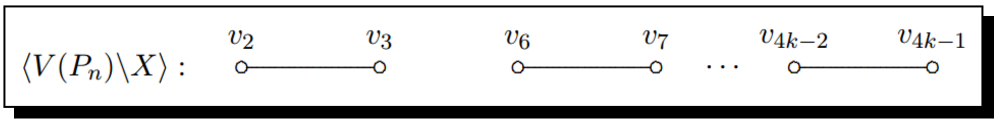

Figure 7. The subgraph of P_n induced by $V(P_n)\backslash X$

Now, we need to show that X is a super dominating set of P_n. Note that

$$V(P_n)\backslash X = \left\{v_{4i-1} | i = 1,2,\cdots,\left\lfloor\frac{n-2}{4}\right\rfloor\right\} \cup \left\{v_{4i+2} | i = 0,1,2,\cdots,\left\lfloor\frac{n-2}{4}\right\rfloor\right\}$$

and for each $v_{4i-1} \in V(P_n)\backslash X$, where $1 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exists $v_{4i} \in X$ such that $N(v_{4i}) \cap (V(P_n)\backslash X) = \{v_{4i-1}\}$. Also, for each $v_{4i+2} \in V(P_n)\backslash X$, where $0 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exist $v_{4i+1} \in X$

such that $N(v_{4i+1})\cap(V(P_n)\setminus X)=\{v_{4i+2}\}$. Thus, X is a super dominating set of P_n. Consequently, X is a super restrained dominating set of P_n. Hence, for $n\equiv 2(\text{mod } 4)$,

$$\gamma_{spr}(P_n)\le |X| = \left\lfloor\frac{n-2}{4}\right\rfloor + \left\lfloor\frac{n-2}{4}\right\rfloor + 2 = 2\left\lfloor\frac{n-2}{4}\right\rfloor + 2.$$

Case 4: $n\equiv 3(\text{mod } 4)$

Let $n\equiv 3(\text{mod } 4)$. Then $n=4k+3$ for some positive integer k. This implies that $k=\frac{n-3}{4}$. Then we take the floor of $\lfloor k\rfloor = \lfloor\frac{n-3}{4}\rfloor$. Let $Y=Y_1\cup Y_2\cup\{v_{n-1},v_n\}$, where $Y_1=\{v_{4i}|i=1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\}$, $Y_2=\{v_{4i+1}|i=0,1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\}$. Thus, for each $v_{4i-1}\in V(P_n)\setminus Y$, there exists $v_{4i}\in Y$ such that $v_{4i-1}v_{4i}\in E(P_n)$ for each $1\le i\le\lfloor\frac{n-3}{4}\rfloor$. Also, for each $v_{4i+2}\in V(P_n)\setminus X$, there exists $v_{4i+1}\in Y$ such that $v_{4i+2}v_{4i+1}\in E(P_n)$ for each $0\le i\le\lfloor\frac{n-3}{4}\rfloor$. Thus, Y is a dominating set of P_n. Note that the subgraph of P_n induced by $V(P_n)\setminus Y$ is shown in Figure 9. Observe that $\langle V(P_n)\setminus Y\rangle$ has no isolated vertices. Thus, Y is a restrained dominating set.

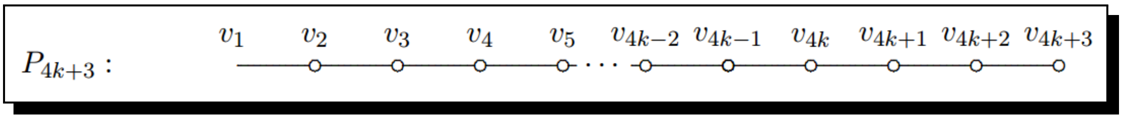

Figure 8. Path graph P_{4k+3}

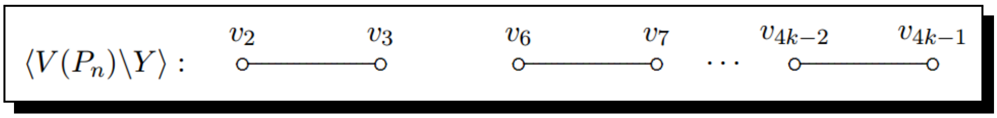

Figure 9. The subgraph of P_n induced by $V(P_n)\setminus Y$

Now, we need to show that Y is a super dominating set of P_n. Note that

$$V(P_n)\setminus Y = \left\{v_{4i-1}|i=1,2,\cdots,\left\lfloor\frac{n-3}{4}\right\rfloor\right\}\cup\left\{v_{4i+2}|i=0,1,2,\cdots,\left\lfloor\frac{n-3}{4}\right\rfloor\right\}$$

and for each $v_{4i-1}\in V(P_n)\setminus Y$, where $1\le i\le\lfloor\frac{n-3}{4}\rfloor$, there exists $v_{4i}\in Y$ such that $N(v_{4i})\cap(V(P_n)\setminus Y)=\{v_{4i-1}\}$. Also, for each $v_{4i+2}\in V(P_n)\setminus Y$, where $0\le i\le\lfloor\frac{n-3}{4}\rfloor$, there exist $v_{4i+1}\in Y$ such that $N(v_{4i+1})\cap(V(P_n)\setminus Y)=\{v_{4i+2}\}$. Thus, Y is a super dominating set of P_n. Consequently, Y is a super restrained dominating set of P_n. Hence, for $n\equiv 3(\text{mod } 4)$,

$$\gamma_{spr}(P_n)\le |Y| = \left\lfloor\frac{n-3}{4}\right\rfloor + \left\lfloor\frac{n-3}{4}\right\rfloor + 3 = 2\left\lfloor\frac{n-2}{4}\right\rfloor + 3.$$

Therefore,

$$\gamma_{spr}(P_n)\le\begin{cases}\lceil\frac{n}{2}\rceil, & \text{if } n\equiv 0(\text{mod } 4);\\ 2\lfloor\frac{n-1}{4}\rfloor+1, & \text{if } n\equiv 1(\text{mod } 4);\\ 2\lfloor\frac{n-2}{4}\rfloor+2, & \text{if } n\equiv 2(\text{mod } 4);\\ 2\lfloor\frac{n-3}{4}\rfloor+3, & \text{if } n\equiv 3(\text{mod } 4).\end{cases}$$

□

Corollary 2.10. *Let $F_n \cong K_1 + P_n$ be a fan graph of order $n+1$ with $n \geq 2$. Then*

$$\gamma_{spr}(F_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 0 (\text{mod } 4); \\ 2\lfloor \frac{n-1}{4} \rfloor + 2, & \text{if } n \equiv 1 (\text{mod } 4); \\ 2\lfloor \frac{n-2}{4} \rfloor + 3, & \text{if } n \equiv 2 (\text{mod } 4); \\ 2\lfloor \frac{n-3}{4} \rfloor + 4, & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

Proof. Follows from Theorem 2.4 and Theorem 2.9. □

Theorem 2.11. *Let graph C_n be a cycle graph of order n such that $n \geq 3$. Then*

$$\gamma_{spr}(C_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0 (\text{mod } 4); \\ 2\lfloor \frac{n-1}{4} \rfloor + 1, & \text{if } n \equiv 1 (\text{mod } 4); \\ 2\lfloor \frac{n-2}{4} \rfloor + 2, & \text{if } n \equiv 2 (\text{mod } 4); \\ 2\lfloor \frac{n-3}{4} \rfloor + 3, & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

Proof. Consider the graph C_n in Figure 10 where $\{v_i : 1 \leq i \leq n\}$ is the vertex set of C_n.

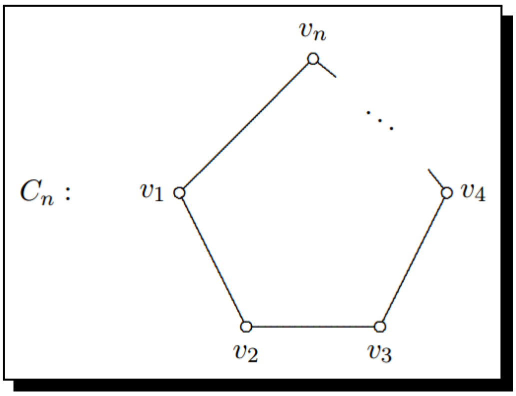

Figure 10. Cycle graph C_n

We consider the following four cases:

Case 1: $n \equiv 0 (\text{mod } 4)$

Let $n \equiv 0 (\text{mod } 4)$. Then $n = 4k$ for some positive integer k. This implies that $k = \frac{n}{4}$. Then we take the ceiling function of $\lceil k \rceil = \lceil \frac{n}{4} \rceil$. Let $D = D_1 \cup D_2$, where $D_1 = \{v_{4i} | i = 1, 2, \cdots, \lceil \frac{n}{4} \rceil\}$ and $D_2 = \{v_{4i+1} | i = 0, 1, 2, \cdots, \lceil \frac{n}{4} \rceil - 1\}$. Thus, for each $v_{4i-1} \in V(P_n) \backslash D$, where $1 \leq i \leq \lceil \frac{n}{4} \rceil$, there exists $v_{4i} \in D$ such that $v_{4i-1}v_{4i} \in E(C_n)$. Also, for each $v_{4i+2} \in V(C_n) \backslash D$, where $0 \leq i \leq \lceil \frac{n}{4} \rceil - 1$, there exists $v_{4i+1} \in D$ such that $v_{4i+1}v_{4i+2} \in E(C_n)$. Thus, D is a dominating set. Note that the subgraph of C_n induced by $V(C_n) \backslash D$ is shown in Figure 12. Observe that $\langle V(C_n) \backslash D \rangle$ has no isolated vertices. Thus D is a restrained dominating set of C_n.

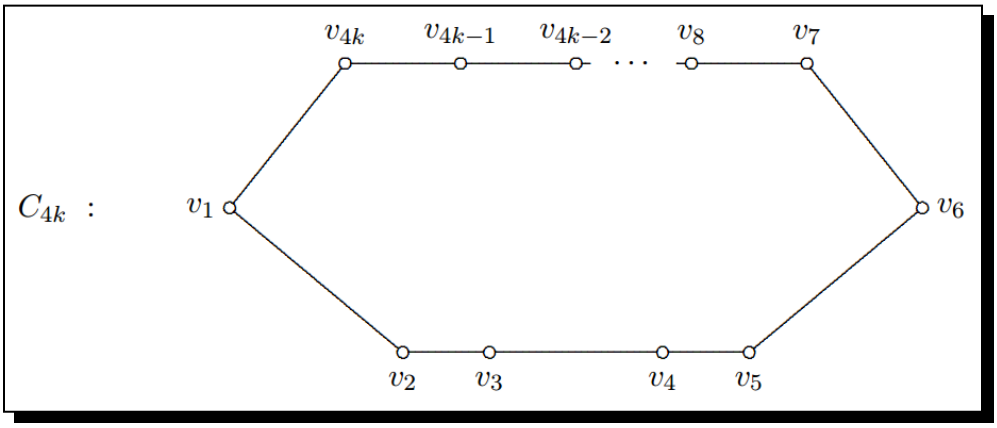

Figure 11. Cycle graph C_{4k}

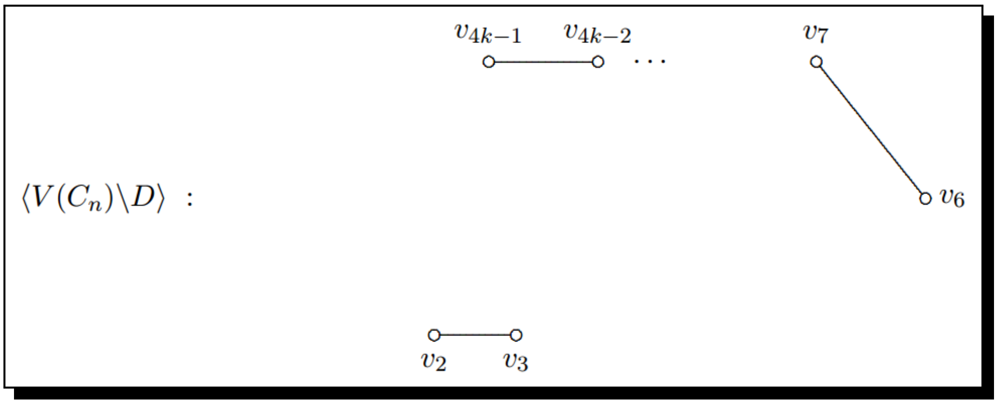

Figure 12. The subgraph of C_n induced by $V(C_n)\backslash D$

Now, we need to show that D is a super dominating set of C_n. Note that

$$V(C_n)\backslash D = \left\{v_{4i-1} | i = 1,2,\cdots,\left\lceil\frac{n}{4}\right\rceil\right\} \cup \left\{v_{4i+2} | i = 0,1,2,\cdots,\left\lceil\frac{n}{4}\right\rceil - 1\right\}$$

and for each $v_{4i-1} \in V(C_n)\backslash D$, where $1 \le i \le \lceil\frac{n}{4}\rceil$, there exists a vertex $v_{4i} \in D$ such that $N(v_{4i}) \cap (V(C_n)\backslash D) = \{v_{4i-1}\}$. Also, for each $v_{4i+2} \in V(C_n)\backslash D$, where $0 \le i \le \lceil\frac{n}{4}\rceil - 1$ there exists a vertex $v_{4i+1} \in D$ such that $N(v_{4i+1}) \cap (V(C_n)\backslash D) = \{v_{4i+2}\}$. Thus, D is a super dominating set of C_n. Consequently, D is a super restrained dominating set of C_n. Hence, for $n \equiv 0 (\text{mod } 4)$,

$$\gamma_{spr}(C_n) \le |D| = \left\lceil\frac{n}{4}\right\rceil + \left\lceil\frac{n}{4}\right\rceil = 2\left\lceil\frac{n}{4}\right\rceil = \left\lceil\frac{n}{2}\right\rceil.$$

Case 2: $n \equiv 1 (\text{mod } 4)$

Let $n \equiv 1 (\text{mod } 4)$. Then $n = 4k+1$ for some positive integer k. This implies that $k = \frac{n-1}{4}$. Then we take the floor function of $\lfloor k \rfloor = \lfloor\frac{n-1}{4}\rfloor$. Let $S = S_1 \cup S_2$, where $S_1 = \{v_{4i} | i = 1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\}$, $S_2 = \{v_{4i+1} | i = 0,1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\}$. Then $V(C_n)\backslash S = \{v_{4i-1} | i = 1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor\} \cup \{v_{4i+2} | i = 0,1,2,\cdots,\lfloor\frac{n-1}{4}\rfloor - 1\}$. Thus, for each $v_{4i-1} \in V(C_n)\backslash S$, where $1 \le i \le \lfloor\frac{n-1}{4}\rfloor$, there exists $v_{4i} \in S$ such that $v_{4i-1}v_{4i} \in E(C_n)$. Also, for each $v_{4i+2} \in V(C_n)\backslash S$, where $0 \le i \le \lfloor\frac{n-1}{4}\rfloor$, there exists $v_{4i+1} \in S$ such that $v_{4i+1}v_{4i+2} \in E(C_n)$. Thus, S is a dominating set of C_n. Note that the subgraph of C_n induced by $V(C_n)\backslash S$ is shown in Figure 14. Observe that $\langle V(C_n)\backslash S\rangle$ has no isolated vertices. Thus, S is a restrained dominating set of C_n.

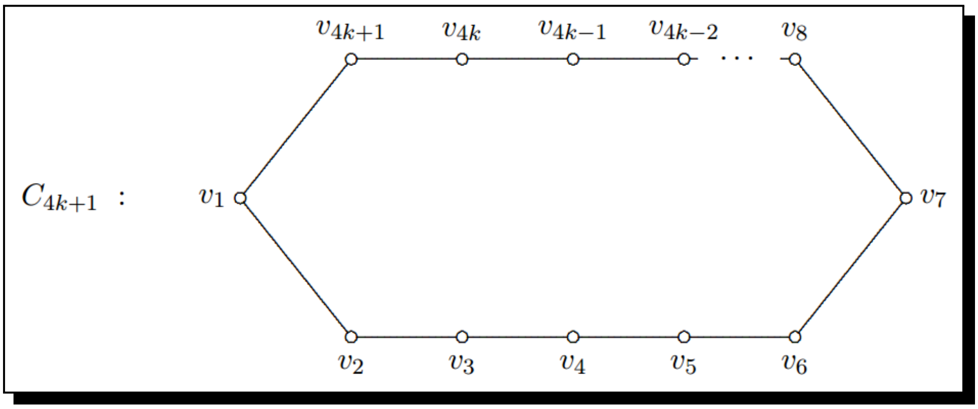

Figure 13. Join graph C_{4k+1}

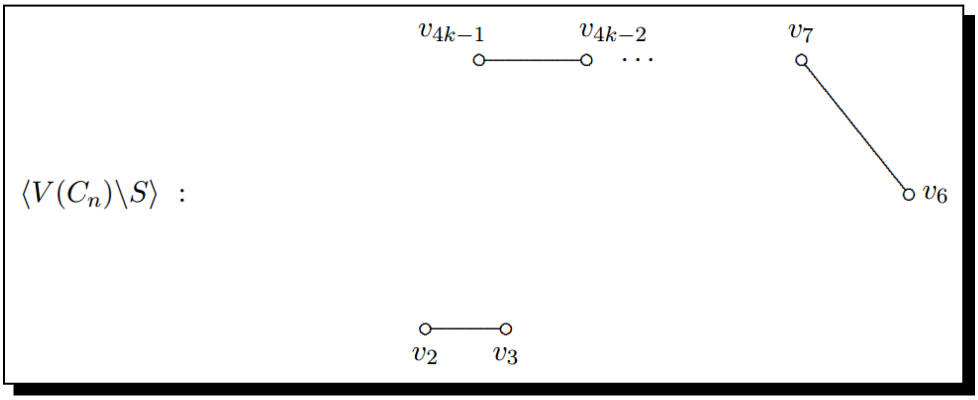

Figure 14. The subgraph of C_n induced by $V(C_n)\backslash S$

Now, we need to show that S is a super dominating set. Note that

$$V(C_n)\backslash S = \left\{v_{4i-1}|i=1,2,\cdots,\left\lfloor\frac{n-1}{4}\right\rfloor\right\} \cup \left\{v_{4i+2}|i=0,1,2,\cdots,\left\lfloor\frac{n-1}{4}\right\rfloor-1\right\}$$

and for each $v_{4i-1} \in V(C_n)\backslash S$, where $1 \le i \le \lfloor\frac{n-1}{4}\rfloor$, there exists a vertex $v_{4i} \in S$ such that $N(v_4 i) \cap (V(C_n)\backslash S) = \{v_{4i-1}\}$. Also, for each vertex $v_{4i+2} \in V(C_n)\backslash S$, where $0 \le i \le \lfloor\frac{n-1}{4}\rfloor - 1$, there exists a vertex $v_{4i+1} \in S$ such that $N(v_{4i+1}) \cap (V(C_n)\backslash S) = \{v_{4i+2}\}$. Thus S is a super dominating set of C_n. Consequently, S is a super restrained dominating set of C_n. Hence, for $n \equiv 1 (\text{mod } 4)$,

$$\gamma_{spr}(C_n) \le |S| = \left\lfloor\frac{n-1}{4}\right\rfloor + \left\lfloor\frac{n-1}{4}\right\rfloor + 1 = 2\left\lfloor\frac{n-1}{4}\right\rfloor + 1.$$

Case 3: $n \equiv 2 (\text{mod } 4)$

Let $n \equiv 2(\text{mod } 4)$. Then $n = 4k+2$ for some positive integer k. This implies that $k = \frac{n-2}{4}$. Then we take the floor function of $\lfloor k\rfloor = \lfloor\frac{n-2}{4}\rfloor$. Let $T = T_1 \cup T_2 \cup \{v_n\}$, where $T_1 = \{v_{4i}|i = 1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$, $T_2 = \{v_{4i+1}|i = 0,1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$. Then $V(C_n)\backslash T = \{v_{4i-1}|i = 1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\} \cup \{v_{4i+2}|i = 0,1,2,\cdots,\lfloor\frac{n-2}{4}\rfloor\}$. Thus, for each $v_{4i-1} \in V(C_n)\backslash T$, where $1 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exists $v_{4i} \in T$ such that $v_{4i-1}v_{4i} \in E(C_n)$. Also, for each $v_{4i+2} \in V(C_n)\backslash T$, where $0 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exists $v_{4i+1} \in T$ such that $v_{4i+1}v_{4i+2} \in E(C_n)$. Thus, T is a dominating set of C_n. Note that the subgraph of C_n induced by $V(C_n)\backslash T$ is shown in Figure 16. Observe that $\langle V(C_n)\backslash T\rangle$ has no isolated vertices. Thus, T is a restrained dominating set of C_n.

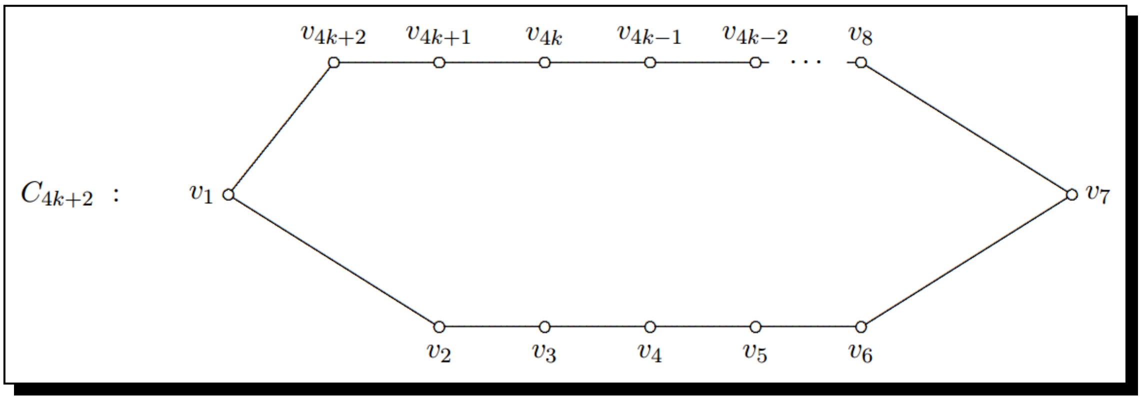

Figure 15. Cycle graph C_{4k+2}

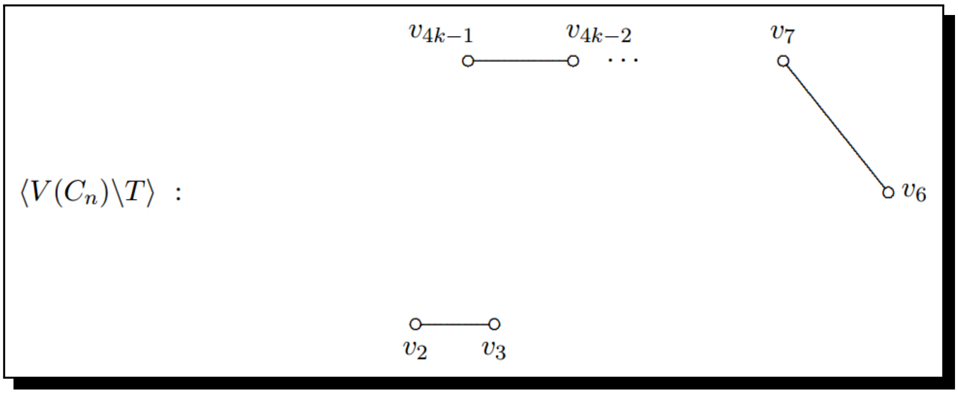

Figure 16. The subgraph of C_n induced by $V(C_{4k+2})\backslash T$

Now, we need to show that T is a super dominating set of C_n. Note that

$$V(C_n)\backslash T = \left\{v_{4i-1}|i=1,2,\cdots,\left\lfloor\frac{n-2}{4}\right\rfloor\right\} \cup \left\{v_{4i+2}|i=0,1,2,\cdots,\left\lfloor\frac{n-2}{4}\right\rfloor\right\}$$

and for each $v_{4i-1} \in V(C_n)\backslash T$, where $1 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exists a vertex $v_{4i} \in T$ such that $N(v_4 i) \cap (V(C_n)\backslash T) = \{v_{4i-1}\}$. Also, for each vertex $v_{4i+2} \in V(C_n)\backslash T$, where $0 \le i \le \lfloor\frac{n-2}{4}\rfloor$, there exists a vertex $v_{4i+1} \in T$ such that $N(v_{4i+1}) \cap (V(C_n)\backslash T) = \{v_{4i+2}\}$. Thus T is a super dominating set of C_n. Consequently, T is a super restrained dominating set of C_n. Hence, for $n \equiv 2 (\text{mod } 4)$,

$$\gamma_{spr}(C_n) \le |T| = \left\lfloor\frac{n-2}{4}\right\rfloor + \left\lfloor\frac{n-2}{4}\right\rfloor + 2 = 2\left\lfloor\frac{n-2}{4}\right\rfloor + 2.$$

Case 4: $n \equiv 3 (\text{mod } 4)$

Let $n \equiv 3 (\text{mod } 4)$. Then $n = 4k+3$ for some positive integer k. This implies that $k = \frac{n-3}{4}$. Then we take the floor function of $\lfloor k \rfloor = \lfloor\frac{n-3}{4}\rfloor$. Let $X = X_1 \cup X_2 \cup \{v_{n-1}, v_n\}$, where $X_1 = \{v_{4i}|i = 1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\}$, $X_2 = \{v_{4i+1}|i = 0,1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\}$. Then, $V(C_n)\backslash X = \{v_{4i-1}|i = 1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\} \cup \{v_{4i+2}|i = 0,1,2,\cdots,\lfloor\frac{n-3}{4}\rfloor\}$. Thus, for each $v_{4i-1} \in V(C_n)\backslash X$, where $1 \le i \le \lfloor\frac{n-3}{4}\rfloor$, there exists $v_{4i} \in X$ such that $v_{4i-1}v_{4i} \in E(C_n)$. Also, for each $v_{4i+2} \in V(C_n)\backslash X$, where $0 \le i \le \lfloor\frac{n-3}{4}\rfloor$, there exists $v_{4i+1} \in X$ such that $v_{4i+1}v_{4i+2} \in E(C_n)$. Thus, X is a dominating set of C_n. Note that the subgraph of C_n induced by $V(C_n)\backslash X$ is shown in Figure 18. Observe that $\langle V(C_n)\backslash X\rangle$ has no isolated vertices. Thus, X is a restrained dominating set.

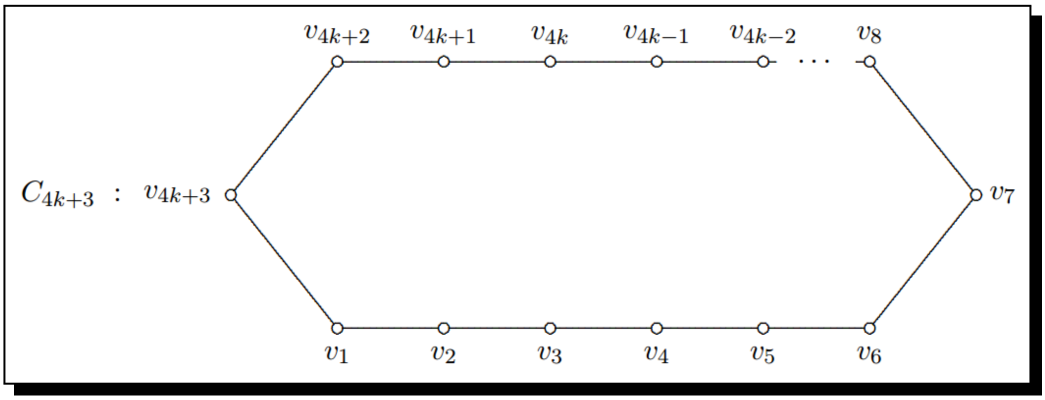

Figure 17. Cycle graph C_{4k+3}

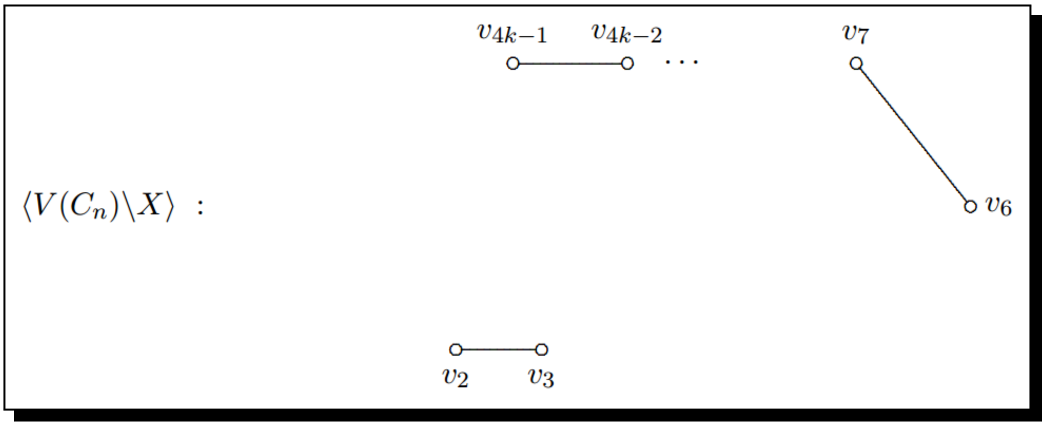

Figure 18. The subgraph of C_n induced by $V(C_n)\backslash X$

Now, we need to show that X is a super dominating set. Note that

$$V(C_n)\backslash X = \left\{v_{4i-1} | i = 1, 2, \cdots, \left\lfloor \frac{n-3}{4} \right\rfloor \right\} \cup \left\{v_{4i+2} | i = 0, 1, 2, \cdots, \left\lfloor \frac{n-3}{4} \right\rfloor \right\}$$

and for each $v_{4i-1} \in V(C_n)\backslash X$, where $1 \le i \le \lfloor \frac{n-3}{4} \rfloor$, there exists a vertex $v_{4i} \in X$ such that $N(v_4 i) \cap (V(C_n)\backslash X) = \{v_{4i-1}\}$. Also, for each vertex $v_{4i+2} \in V(C_n)\backslash X$, where $0 \le i \le \lfloor \frac{n-3}{4} \rfloor$, there exists a vertex $v_{4i+1} \in X$ such that $N(v_{4i+1}) \cap (V(C_n)\backslash X)$
$= \{v_{4i+2}\}$. Thus X is a super dominating set of C_n. Consequently, X is a super restrained dominating set of C_n. Hence, for $n \equiv 3 \pmod 4$,

$$\gamma_{spr}(C_n) \le |X| = \left\lfloor \frac{n-3}{4} \right\rfloor + \left\lfloor \frac{n-3}{4} \right\rfloor + 3 = 2\left\lfloor \frac{n-2}{4} \right\rfloor + 3.$$

Therefore,

$$\gamma_{spr}(C_n) \le \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 0 \pmod 4; \\ 2\lfloor \frac{n-1}{4} \rfloor + 1, & \text{if } n \equiv 1 \pmod 4; \\ 2\lfloor \frac{n-2}{4} \rfloor + 2, & \text{if } n \equiv 2 \pmod 4; \\ 2\lfloor \frac{n-3}{4} \rfloor + 3, & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

□

Corollary 2.12. *Let $W_n \cong K_1 + C_n$ be a wheel graph of order $n+1$ with $n \geq 3$. Then*

$$\gamma_{spr}(W_n) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 0 \pmod 4; \\ 2\lfloor \frac{n-1}{4} \rfloor + 2, & \text{if } n \equiv 1 \pmod 4; \\ 2\lfloor \frac{n-2}{4} \rfloor + 3, & \text{if } n \equiv 2 \pmod 4; \\ 2\lfloor \frac{n-3}{4} \rfloor + 4, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Follows from Theorem 2.4 and Theorem 2.11. □

Theorem 2.13. *Let $K_{m,n} \cong \overline{K}_m + \overline{K}_n$ be a complete bipartite graph such that $m \geq 2$ and $n \geq 2$. Then $D \subseteq V(K_{m,n})$ is a super restrained dominating set of $K_{m,n}$ if and only if $D = V(K_{m,n})$ or $D = V(K_{m,n}) \backslash \{u_i, v_j\}$, where $u_i \in V(\overline{K}_m)$ and $v_j \in V(\overline{K}_n)$.*

Proof. Let $m,n \geq 2$, $V(\overline{K}_m) = \{u_i | 1 \leq i \leq m\}$, $V(\overline{K}_n) = \{v_i | 1 \leq i \leq n\}$ and let $K_{m,n}$ be a complete bipartite graph as shown in Figure 19. Let $D \subseteq V(K_{m,n})$.

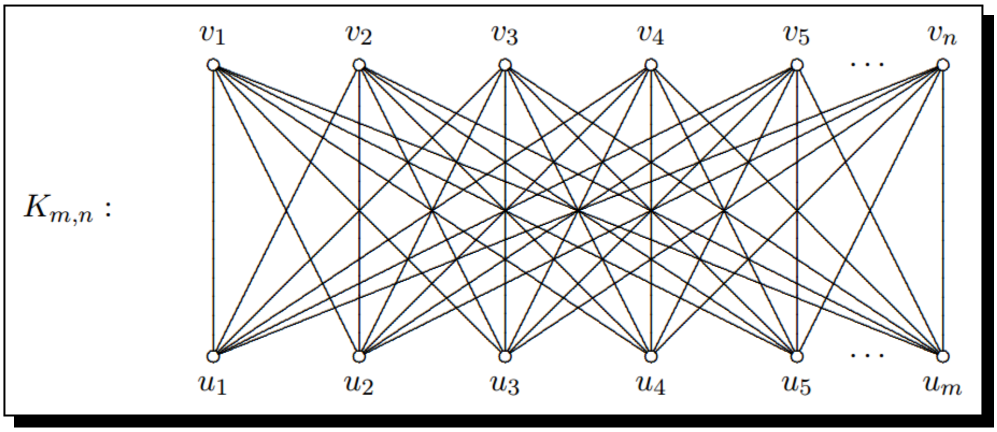

Figure 19. Complete bipartite graph $K_{m,n} \cong \overline{K}_m + \overline{K}_n$

Suppose that D is a super restrained dominating set of $K_{m,n}$. If $D = V(K_{m,n})$, then we are done. Suppose that $D \neq V(K_{m,n})$. Since D is a restrained dominating set, it follows that $|V(K_{m,n}) \backslash D| \geq 2$, since the subgraph of $K_{m,n}$ induced by $(V(K_{m,n}) \backslash D)$ has no isolated vertices.

Case 1: Suppose that $|V(K_{m,n}) \backslash D| = 2$ and consider the following subcases:

Subcase 1.1: $|V(\overline{K}_m) \backslash D| = 2$ and $|V(\overline{K}_n) \backslash D| = 0$.
Then $V(\overline{K}_m) \backslash D = \{u_i, u_k\}$ where $i,k \in \{1,2,3,\cdots,m\}$ with $i \neq k$. Let $c \in D$ and consider the following subcases:

Subcase 1.1.1: $c \in V(\overline{K}_n)$.
Then $N_{K_{m,n}}(c) \cap (V(K_{m,n}) \backslash D) = V(\overline{K}_m) \cap \{u_i, u_k\} = \{u_i, u_k\}$.

Subcase 1.1.2: $c \in V(\overline{K}_m)$.
Then $N_{K_{m,n}}(c) \cap (V(K_{m,n}) \backslash D) = V(\overline{K}_n) \cap \{u_i, u_k\} = \varnothing$.

Thus, there exist $u_i \in V(K_{m,n})\backslash D$ such that for all $c \in D$, $N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) \neq \{u_i\}$. Hence, if $u_i, u_k \in V(\overline{K}_m)$, D is not a super restrained dominating set of $K_{m,n}$.

Subcase 1.2: $|V(\overline{K}_n)\backslash D| = 2$ and $|V(\overline{K}_m)\backslash D| = 0$.
Then $V(\overline{K}_n)\backslash D = \{v_i, v_k\}$ where $i,k \in \{1,2,3,\cdots,m\}$ with $i \neq k$. Let $c \in D$ and consider the following subcases:

Subcase 1.2.1: $c \in V(\overline{K}_m)$.
Then $N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_n) \cap \{v_i, v_k\} = \{v_i, v_k\}$.

Subcase 1.2.2: $c \in V(\overline{K}_n)$.
Then $N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_m) \cap \{v_i, v_k\} = \emptyset$.
Thus, there exist $v_i \in V(K_{m,n})\backslash D$ such that for all $c \in D$, $N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) \neq \{v_i\}$. Hence, if $v_i, v_k \in V(\overline{K}_n)$, D is not a super restrained dominating set of $K_{m,n}$.

Subcase 1.3: $|V(\overline{K}_n)\backslash D| = 1$ and $|V(\overline{K}_m)\backslash D| = 1$.
Let $v_j \in V(\overline{K}_n)\backslash D$ and $u_i \in V(\overline{K}_m)\backslash D$, where $j \in \{1,2,3,\cdots,n\}$ and $i \in \{1,2,3,\cdots,m\}$. Since $m \geq 2$, then there exist $u_k \in V(\overline{K}_n)$ with $i \neq k$ such that

$$N_{K_{m,n}}(u_k) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_n) \cap \{v_j, u_i\} = \{v_j\}$$

and since $n \geq 2$, then there exist $v_t \in V(\overline{K}_n)$ with $t \neq j$ such that

$$N_{K_{m,n}}(v_t) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_m) \cap \{v_j, u_i\} = \{u_i\}.$$

Hence, $D = V(K_{m,n})\backslash\{v_j, u_i\}$ is a super restrained dominating set of $K_{m,n}$. Moreover, the subgraph of $K_{m,n}$ induced by $V(K_{m,n})\backslash D = \{v_j, u_i\}$ is isomorphic to path P_2. Thus, D is a super restrained dominating set of $K_{m,n}$.

Case 2: Suppose that $|(V(K_{m,n})\backslash D| > 2$, that is, $|(V(K_{m,n})\backslash D| \geq 3$.
Since $|V(K_{m,n})\backslash D| \geq 3$, it follows that either $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| \geq 2$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| \geq 1$ or $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| \geq 1$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| \geq 2$ or $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| \geq 3$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| = 0$ or $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| \geq 3$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| = 0$. Suppose that $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| \geq 2$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| \geq 1$. Let $p \in (V(K_{m,n})\backslash D) \cap V(\overline{K}_m)$ and let $c \in D$. If $c \in D \cap V(\overline{K}_m)$, then

$$N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_n) \cap ((V(K_{m,n})\backslash D) \neq \{p\}, \quad \text{since } p \notin V(\overline{K}_n).$$

If $c \in D \cap V(\overline{K}_n)$, then

$$N_{K_{m,n}}(c) \cap (V(K_{m,n})\backslash D) = V(\overline{K}_m) \cap ((V(K_{m,n})\backslash D) \neq \{p\}.$$

since $|V(K_{m,n})\backslash D \cap V(\overline{K}_m)| \geq 2$. Thus, there exists $p \in (V(K_{m,n})\backslash D$ such that for all $c \in D$,

$$N(c) \cap (V(K_{m,n})\backslash D) \neq \{p\}.$$

Thus, D is not a super restrained dominating set of $K_{m,n}$. Similarly, if $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_m)| \geq 1$ and $|(V(K_{m,n})\backslash D) \cap V(\overline{K}_n)| \geq 2$, then D is not a super restrained dominating set of $K_{m,n}$.

Suppose that $|(V(K_{m,n})\backslash D)\cap V(\overline{K}_m)| \geq 3$ and $|(V(K_{m,n})\backslash D)\cap V(\overline{K}_n)| = 0$. let $q \in (V(K_{m,n})\backslash D)\cap V(\overline{K}_m)$ and let $d \in D$. If $d \in D\cap V(\overline{K}_m)$, then

$$\begin{aligned} N_{K_{m,n}}(d)\cap(V(K_{m,n})\backslash D) &= V(\overline{K}_n)\cap((V(K_m)\backslash D) \\ &= \emptyset, \quad \text{since } |(V(K_{m,n})\backslash D)\cap V(\overline{K}_n)| = 0 \\ &\neq \{q\}. \end{aligned}$$

If $d \in D\cap V(\overline{K}_n)$, then

$$\begin{aligned} N_{K_{m,n}}(d)\cap(V(K_{m,n})\backslash D) &= V(\overline{K}_m)\cap((V(K_m)\backslash D) \\ &\neq \{q\}, \quad \text{since } |(V(K_{m,n})\backslash D)\cap V(\overline{K}_m)| \geq 3. \end{aligned}$$

Similarly, if $|(V(K_{m,n})\backslash D)\cap V(\overline{K}_n)| \geq 3$ and $|(V(K_{m,n})\backslash D)\cap V(\overline{K}_m)| = 0$. Let $r \in (V(K_{m,n})\backslash D)\cap V(\overline{K}_n)$ and let $f \in D$. Suppose that $f \in D\cap V(\overline{K}_n)$, then

$$\begin{aligned} N_{K_{m,n}}(f)\cap(V(K_{m,n})\backslash D) &= V(\overline{K}_m)\cap((V(K_{m,n})\backslash D) \\ &= \emptyset, \quad \text{since } |(V(K_{m,n})\backslash D)\cap V(\overline{K}_m)| = 0 \\ &\neq \{r\}. \end{aligned}$$

If $f \in D\cap V(\overline{K}_m)$, then

$$\begin{aligned} N_{K_{m,n}}(f)\cap(V(K_{m,n})\backslash D) &= V(\overline{K}_n)\cap((V(K_{m,n})\backslash D) \\ &\neq \{r\}, \quad \text{since } |(V(K_{m,n})\backslash D)\cap V(\overline{K}_m)| \geq 3. \end{aligned}$$

Thus D is not a super restrained dominating set of $K_{m,n}$. Hence, D is not a super restrained dominating set of $K_{m,n}$.

Thus, if $|V(K_{m,n})\backslash D| \geq 3$, then D is not a super restrained dominating set of $K_{m,n}$.

Therefore, from *Case* 1 and *Case* 2, if D is a super restrained dominating set of $K_{m,n}$ with $D \neq V(K_{m,n})$, then $D = V(K_{m,n})\backslash\{u_i, v_j\}$ where $u_i \in V(\overline{K}_m)$ and $v_j \in V(\overline{K}_n)$.

Suppose that $D = V(K_{m,n})$. Then D is a super restrained dominating set of $K_{m,n}$. Suppose that $D = V(K_{m,n})\backslash\{u_i, v_j\}$, where $u_i \in V(\overline{K}_m)$ and $v_j \in V(\overline{K}_n)$. Then $V(K_{m,n})\backslash D = \{u_i, v_j\}$, where $i \in \{1,2,3,\cdots,m\}$ and $j \in \{1,2,3,\cdots,n\}$. Since $n \geq 2$, then there exists $v_k \in V(\overline{K}_n)\cap D$ with $j \neq k$ such that

$$N_{K_{m,n}}(v_k)\cap(V(K_{m,n})\backslash D) = V(\overline{K}_m)\cap\{u_i, v_j\} = \{u_i\}$$

and since $m \geq 2$, then there exist $u_t \in V(\overline{K}_m)\cap D$ with $i \neq t$ such that

$$N_{K_{m,n}}(u_t)\cap(V(K_{m,n})\backslash D) = V(\overline{K}_n)\cap\{u_i, v_j\} = \{v_j\}.$$

Thus, for all $x \in V(K_{m,n})\backslash D$, there exist $y \in D$ such that

$$N_{K_{m,n}}(y)\cap(V(K_{m,n})\backslash D) = \{x\}.$$

Hence, D is a super dominating set of $K_{m,n}$. Moreover, since $u_i \in V(\overline{K}_m)$ and $v_j \in (\overline{K}_n)$, the subgraph of $K_{m,n}$ induced by $\{u_i, v_j\}$ is isomorphic to the path graph P_2. Hence, D is a super restrained dominating set of $K_{m,n}$. □

Corollary 2.14. *Let $K_{m,n}$ be a complete bipartite graph such that $m \geq 2$ and $n \geq 2$, then $\gamma_{spr}(K_{m,n}) = m + n - 2$.*

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] G. Chartrand and P. Zhang, *Chromatic Graph Theory*, 1st edition, Chapman and Hall/CRC, New York, 504 pages (2008), DOI: 10.1201/9781584888017.

[2] G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. C. Laskar and L. R. Markus, Restrained domination in graphs, *Discrete Mathematics* **203**(1-3) (1999), 61 – 69, DOI: 10.1016/S0012-365X(99)00016-3.

[3] E. L. Enriquez, Super restrained domination in the corona of graphs, *International Journal of Latest Engineering Research and Applications* **3**(5) (2018), 1 – 6.

[4] G. B. Monsanto and H. M. Rara, Resolving restrained domination in graphs, *European Journal of Pure and Applied Mathematics* **14**(3) (2021), 829 – 841, DOI: 10.29020/nybg.ejpam.v14i3.3985.

[5] J. A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, *SIAM Journal on Discrete Mathematics* **10**(4) (1997), 529 – 550, DOI: 10.1137/S0895480194275825.