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Research Article

# Perturbed MAP/PH Risk Model With Possible Delayed By-Claims and a Constant Dividend Barrier

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**Abstract.** In this study, we consider a perturbed risk model with MAP claim arrivals, PH claim sizes that incorporates possible by-claims and dividend barrier affected by a Brownian motion. The by-claims can occur along with the main claim, but their settlement is always delayed due to some necessary investigation. In order to analyze the model, we consider associated Markovian fluid models defined in the original timeline and an auxiliary timeline. We develop systems of second order *integro-differential equations* (IDE) for the *Gerber-Shiu functions* (GSF) of both the models without as well as with the barrier and solve them explicitly. Working on the same line we derive expressions for the Moment of the total dividends paid until ruin. Furthermore, a dividends-penalty identity is established. To showcase the effectiveness of the method, we numerically illustrate it using a two-phase model. Finally, we conduct a sensitive analysis by varying some of the parameters involved in the model.

**Keywords.** MAP claim arrivals, Phase type claims, By-claims, Risk reserve process, Brownian motion, Dividend barrier, Lundberg equation, Gerber-Shiu function (GSF)

**Mathematics Subject Classification (2020).** 62P05, 60J28, 91B30



## 1. Introduction

In this study, we focus on a Perturbed MAP/PH risk model that integrates both a barrier strategy and delayed settlement of by-claims. The MAP/PH risk model extends the classical Compound Poisson risk model by incorporating interdependence among claim arrivals through a *Markovian Arrival Process* (MAP). This interdependence enhances the model's ability to capture realistic claim occurrences, where the arrival of one claim can influence the likelihood of subsequent claims. Additionally, the model approximates claim sizes using *Phase-Type* (PH) distributions, which offer great flexibility in representing a wide range of positive random variables. The PH approximation enables the model to accurately capture various claim size distributions, accommodating both heavy-tailed and light-tailed distributions. By combining the MAP for claim arrivals and the PH approximation for claim sizes, the MAP/PH risk model provides an enhanced framework for assessing risk and estimating reserves in insurance settings.

Perturbation refers to the introduction of small changes or disturbances to a system, which allows for the examination of the system's response and sensitivity to such changes. In the context of insurance risk models, perturbation analysis provides valuable insights into the stability, robustness, and overall behavior of these models under different scenarios.

In 1957, DeFinetti [7] proposed a dividend strategy that revolves around the payment to shareholders. Dividend payments are a crucial aspect in evaluating a company's performance, reflecting the distribution of profits to shareholders. One commonly accepted approach to dividend payment involves distributing the surplus beyond a predetermined threshold immediately to shareholders. This strategy ensures that any excess surplus above the fixed barrier is promptly distributed among shareholders, aligning with the goal of maximizing shareholder value and enhancing overall company performance.

Risk models that incorporate delayed claims play a crucial role in effectively representing real-life scenarios. Various situations, such as in motor insurance, involve the necessity of immediate settlement for certain claims following an accident, while additional claims may require investigations and subsequent delays in payment. For instance, major auto accidents can give rise to diverse types of claims, including those related to automobile damage, injuries, and fatalities. To address such circumstances, risk models are designed to consider by-claims that are generated with specific probabilities alongside primary claims. By incorporating possible delayed by-claims, Markovian risk models achieve a more practical and realistic representation. This approach enables insurers to effectively manage the complexities associated with claim settlement, providing a comprehensive perspective for risk assessment and the establishment of appropriate dividend barriers.

Neuts [11], Latouche and Ramaswami [9] initiated the discussion on MAP (Markovian arrival process) in the literature. Ahn and Badescu [1] utilized Ramaswami [12] passage-time matrix analytic techniques to analyze a risk model incorporating MAP claim arrivals and PH claim sizes. Dibu and Jacob [4] conducted recent analyses on MAP/PH models, considering stochastic income and delayed capital injunction strategies, respectively, with MAP claim inter-arrival times and PH claim sizes. Lin *et al*. [10] investigated the *Gerber-Shiu Function* (GSF) for classical risk processes with a constant dividend barrier. In terms of claim settlement with delay, Waters and Papatriandafylou [15] introduced the concept in discrete-time risk models. Xie and Zou [16] analyzed the classical risk model with delayed claims. Xie *et al*. [17] discussed a compound binomial model with an interest rate dependent on the state's Markov chain. Dufresne and Gerber [5] consider the uncertainty in the classical risk process by adding a wiener process. Cheng and Landriault [3] handle MAP risk models with barrier and perturbation. Cheng and Wang [2] studied a similar model with a threshold dividend strategy. In the present work, we extend and generalize the research conducted in Sreeshamim *et al*. [14] by incorporating the concept of perturbation in the surplus process.

Subsequent sections of the paper are organized as follows. In Section 2, we introduce the model and provide necessary notations to facilitate understanding. Section 3 contains a comprehensive analysis of the *Gerber-Shiu Function* (GSF), both in the absence and presence of the barrier strategy. In Section 3.1, we derive an explicit expression for the GSF of the model without a barrier, while in Section 3.2, we present an intriguing result regarding the GSF of the model with a barrier, establishing its relationship with the GSF of the model without a barrier. In Section 4, we delve into the calculation of Moment of the cumulative dividend paid, offering insightful expressions. Section 5 unveils an important identity for the dividend penalty, while Section 6 provides a numerical illustration of the methodology by considering a two-phase model. By following this structured organization, we aim to present a clear and cohesive analysis of the model and its various components throughout the rest of the paper.

## 2. Model

We consider the MAP/PH Risk reserve process $U_w^d(t)$ with possible delayed by-claims and perturbation defined by

$$U_w^d(t) = x + ct - \sum_{i=1}^{N_t} X_i - B_d(t) + \sigma W(t), \quad t \geq 0. \tag{2.1}$$

Here $x$ represents the initial surplus or funds available to the insurance firm at time zero. The parameter $c$ is the rate at which premiums arrive or are collected by the firm. The term $ct$ accounts for the cumulative premiums received up to time $t$. $N_t$ is the number of claims that occur in the time interval $(0,t)$ with $N_0 = 0$ and $X_i$ denotes the $i$th claim amount. $B_d(t)$ is the total of all delayed by-claims that are settled till time $t \geq 0$. $W(t)$ is the Wiener process with mean 0 and volatility $\sigma$ which describes the perturbation in the surplus process.

The inter arrival time follows MAP (*Markovian Arrival Process*) with $m \geq 1$ transient phases and representation $\mathrm{MAP}_m(\boldsymbol{\alpha}, \mathbf{D}_0, \mathbf{D}_1)$. It is a two-dimensional *continuous-time Markov chain* (CTMC) $\{(N_t, J_t), t \geq 0\}$ with state space $\mathbb{N} \times S$, where $S = \{1, 2, \ldots, m\}$. $J_t$ is the state of the underlying CTMC of claim arrivals at time $t \geq 0$. The matrix $\mathbf{D}_0$ represents the rate of transition of states in $S$ without claim arrival and the matrix $\mathbf{D}_1$ represents the rate of transitions with a claim arrival. $(\boldsymbol{\alpha}, \boldsymbol{\alpha}_{m+1})$ is the initial probability vector of the CTMC where $\boldsymbol{\alpha}$ is $m$-dimensional.

The $\{X_i\}_{i\geq 1}$ are the main claim sizes and assumed to be iid positive random variables having $n_1$ ordered phase-type distribution $\mathrm{PH}_{n_1}(\boldsymbol{v}_{\mathbf{M}}, \mathbf{H}_{\mathbf{M}})$. $\boldsymbol{v}_{\mathbf{M}}$ gives the initial probabilities and $\mathbf{H}_{\mathbf{M}}$ gives the transition rates for transient states for the underlying CTMC.

The risk model considered is with possible by-claims for which the payments are delayed till the next claim arrival. Each main claim may induce a by-claim with a probability $\theta$, which is paid only at the next claim arrival instant. The occurrence of main claims and by-claims is assumed to be independent. We assume that the by-claims are iid positive random variables having a phase-type distribution with order $n_2$ with representations $\mathrm{PH}_{n_2}(\boldsymbol{v}_{\mathbf{B}},\mathbf{H}_{\mathbf{B}})$. In order to analyze the delayed feature in the model, we consider two timelines, a primary and an auxiliary (Delayed) timeline. This methodology is used in the literature since Xie and Zou [16]. In the primary timeline the first settlement can have only the main claim whereas, in the auxiliary timeline, the first settlement will have both the main claim and by-claim. In both timelines at the second claim epoch, the payment will depend on whether a by-claim occurred or not in the previous claim instant. The sum of main claim and by-claim is phase type with representation

$$\mathrm{PH}_{n_1+n_2}(\boldsymbol{v}_{\mathbf{M+B}},\mathbf{H}_{\mathbf{M+B}}),$$

where $\boldsymbol{v}_{\mathbf{M+B}}=(\boldsymbol{v}_{\mathbf{M}},(1-\sum_{i=1}^{n_1}v_{\mathbf{M}i})\boldsymbol{v}_{\mathbf{B}})$ and $\mathbf{H}_{\mathbf{M+B}}$ is the composite S-TRM

$$\mathbf{H}_{\mathbf{M+B}}=\begin{pmatrix}\mathbf{H}_{\mathbf{M}} & \boldsymbol{v}_{\mathbf{B}}\otimes\mathbf{h}_{\mathbf{M}}^{\top}\\ \mathbf{0} & \mathbf{H}_{\mathbf{B}}\end{pmatrix},$$

where $\mathbf{h}_{\mathbf{M}}^{\top}=-\mathbf{H}_{\mathbf{M}}\mathbf{e}_{n_1}^{\top}$.

Ren *et al*. [13] analyzed the perturbed risk reserve process as an embedded fluid flow process. This embedded process (say), $\{(L(t),C(t))\}_{t\ge 0}$, where $C(t)$ is a finite irreducible CTMC having the state space (say), $S\cup F$ and an infinitesimal (transient) sub-generator

$$\Lambda_M=\begin{pmatrix}\Lambda'_{11_{m\times m}} & \Lambda'_{12_{m\times mn_1}}\\ \Lambda'_{21_{mn_1\times m}} & \Lambda'_{22_{mn_1\times mn_1}}\end{pmatrix}=\begin{pmatrix}\mathbf{D}_0 & \boldsymbol{v}_{\mathrm{M}}\otimes\mathbf{D}_1\\ \mathbf{h}_{\mathbf{M}}^{\top}\otimes\mathbf{I}_m & \mathbf{H}_{\mathbf{M}}\otimes\mathbf{I}_m\end{pmatrix}. \tag{2.2}$$

$F=\{m+1,\dots,m+n\}$ with $n=mn_1$, having elements corresponding to the Cartesian product of $S$ and the set of phases of the PH claim distribution.

The $\Lambda_M$ given in equation (2.2) represents the transient generator of the CTMC effect in the main claims. The transient generator due to the sum of the main claims and by-claims is given by

$$\begin{aligned}\Lambda_{M+B}&=\begin{bmatrix}\Lambda''_{11_{m\times m}} & \Lambda''_{12_{m\times m(n_1+n_2)}}\\ \Lambda''_{21_{m(n_1+n_2)\times m}} & \Lambda''_{22_{m(n_1+n_2)\times m(n_1+n_2)}}\end{bmatrix}\\ &=\begin{bmatrix}\mathbf{D}_0 & \boldsymbol{v}_{\mathbf{M+B}}\otimes\mathbf{D}_1\\ \mathbf{h}_{\mathbf{M+B}}^{\top}\otimes\mathbf{I}_m & \mathbf{H}_{\mathbf{M+B}}\otimes\mathbf{I}_m\end{bmatrix}.\end{aligned} \tag{2.3}$$

It is to be noted that the condition satisfied by the density function of main claims and by-claims are

$$\Lambda_1=\int_{x=0}^{\infty}\Lambda'_{12}e^{\Lambda'_{22}x}\Lambda'_{21}\mathrm{d}x=\int_{x=0}^{\infty}\Lambda''_{12}e^{\Lambda''_{22}x}\Lambda''_{21}\mathrm{d}x. \tag{2.4}$$

The risk reserve process given in (2.1) is further restricted with a horizontal barrier $b\ge x$, satisfying the equation

$$\mathrm{d}U_{w,b}^{d}(t)=\begin{cases}c\ \mathrm{d}t-\mathrm{d}\sum_{i=1}^{N_\Lambda}X_i-\ \mathrm{d}B_d(t)+\sigma\ \mathrm{d}W(t), & U_{w,b}^{d}(t)<b,\\ -\mathrm{d}\sum_{i=1}^{N_\Lambda}X_i-\ \mathrm{d}B_d(t)+\sigma\ \mathrm{d}W(t), & U_{w,b}^{d}(t)=b.\end{cases} \tag{2.5}$$

In risk models with a barrier strategy, by paying the surplus above the barrier as dividends, the insurance firm will be able to satisfy the economic interest of shareholders.

Let $Q(t) = \sup\{U_w^d(y) \mid 0 \le y \le t\}$ be the running maximum of risk reserve process $U_w^d(t)$. Denoting $D_{b,w}(t) = \max\{Q(t) - b, 0\}$ as the aggregate dividends paid by the firm up to time $t$, the revised risk reserve process $U_{w,b}^d(t)$ is given by

$$U_{w,b}^d(t) = U_w^d(t) - D_{b,w}(t), \quad \text{for } t \ge 0. \tag{2.6}$$

It is assumed that the cumulative dividends paid till time $t \ge 0$ is zero if $U_{w,b}^d(t)$ remains less than $b$.

The ultimate time of ruin is defined by $T = \inf\{t \ge 0 : U_{b,w}^d(t) \le 0\}$,

$$R(u) = P(T < \infty \mid U_w^d(0) = u), \quad u \ge 0$$

and

$$R_b(u) = P(T < \infty \mid U_{w,b}^d(0) = u), \quad 0 \le u \le b$$

to be the ultimate ruin probabilities defines for perturbed risk model without and with barrier, respectively. Further,

$$R_{b,w}(u) = P(T < \infty, U_{w,b}^d(T) = 0 \mid U_{w,b}^d(0) = u), \quad 0 \le u \le b$$

and

$$R_{b,s}(u) = P(T < \infty, U_{w,b}^d(T) < 0 \mid U_{w,b}^d(0) = u), \quad 0 \le u \le b$$

are the ruin probabilities caused by wiener process and claim respectively for the perturbed risk process with barrier model. Similar way we can define the ruin probabilities caused by wiener process and claim for the perturbed risk process with out barrier model. We have that, $R_b(u) = R_{b,w}(u) + R_{b,s}(u)$, with $R_{b,w}(0) = 1$ and $R_{b,s}(0) = 0$.

For risk process without barrier, deficit at ruin is $|U_w^d(T)|$ and surplus immediately before ruin is $U_w^d(T^-)$. Similarly, $|U_{w,b}^d(T)|$ and $U_{w,b}^d(T^-)$ are defined for risk process with barrier.Define the $m$-dimensional vectors

$$\phi_d^\top(x) = (\phi_{d,1}(x), \phi_{d,2}(x), \phi_{d,3}(x), \dots, \phi_{d,m}(x))^\top$$

and

$$\phi_{d,b}^\top(x) = (\phi_{d,b,1}(x), \phi_{d,b,2}(x), \phi_{d,b,3}(x), \dots, \phi_{d,b,m}(x))^\top,$$

where for $i \in \mathfrak{E}$,

$$\phi_{d,i}(x) = \phi_{d,i}^w(x) + \phi_{d,i}^c(x),$$

$$\phi_{d,b,i}(x) = \phi_{d,b,i}^w(x) + \phi_{d,b,i}^c(x).$$

$\phi_{d,i}^w(x)$ and $\phi_{d,b,i}^w(x)$ are the Gerber-Shiu functions of the perturbed risk process without a barrier and with a barrier if the ruin caused by wiener process. Similarly, $\phi_{d,i}^c(x)$ and $\phi_{d,b,i}^c(x)$ are the the Gerber-Shiu function of risk process without a barrier and with a barrier respectively if the ruin due to claim arrival.

$$\phi_{d,i}^c(x) = \mathbb{E}[e^{-\delta T}\omega(U_w^d(T^-), |U_w^d(T)|)\mathbb{I}(T < \infty, U_w^d(T) < 0) \mid U_w^d(0) = x, J(0) = i],$$

$$\phi_{d,i}^w(x) = \mathbb{E}[e^{-\delta T}\mathbb{I}(T < \infty, \mid U_w^d(T) = 0) \mid U_w^d(0) = x, J(0) = i],$$

$$\phi_{d,b,i}^c(x) = \mathbb{E}[e^{-\delta T}\omega(U_{w,b}^d(T^-), |U_{w,b}^d(T)|)\mathbb{I}(T < \infty, \mid U_{w,b}^d(T) < 0) \mid U_{w,b}^d(0) = x, J(0) = i],$$

$$\phi^{w}_{d,b,i}(x) = \mathbb{E}[e^{-\delta T}\mathbb{I}(T < \infty, \mid U^{d}_{w,b}(T) = 0) \mid U^{d}_{w,b}(0) = x, J(0) = i].$$

Also consider $\phi^{w}_{d,i}(0) = 1$, $\phi^{w}_{d,b,i}(0) = 1$, $\phi^{c}_{d,i}(0) = 0$ and $\phi^{c}_{d,b,i}(0) = 0$. The GSFs $\phi_d(x)$, $\phi_{d,b}(x)$ in the primary timeline and the GSFs $\phi^{*}{}_{d}(x)$, $\phi^{*}{}_{d,b}(x)$ in auxiliary timeline are given by

$$\phi_d(x) = \sum_{i-1}^{m} \boldsymbol{\alpha}_i\ \phi_{d,i}(x), \qquad \text{for } x \geq 0, \tag{2.7}$$

$$\phi^{*}{}_{d}(x) = \sum_{i=1}^{m} \boldsymbol{\alpha}_i\ \phi^{*}{}_{d,i}(x), \qquad \text{for } x \geq 0, \tag{2.8}$$

$$\phi_{d,b}(x) = \sum_{i-1}^{m} \boldsymbol{\alpha}_i\ \phi_{d,b,i}(x), \quad \text{for } 0 \leq x \leq b, \tag{2.9}$$

and

$$\phi^{*}{}_{d,b}(x) = \sum_{i=1}^{m} \boldsymbol{\alpha}_i\ \phi^{*}{}_{d,b,i}(x), \quad \text{for } 0 \leq x \leq b. \tag{2.10}$$

The present value of the cumulative dividend payments up to the ruin time $T$ is

$$D_{\delta,b,w} = \int_0^T e^{-\delta t}\, \mathrm{d}D_{b,w}(t) \mid U^{d}_{b,w}(0) = x, \quad 0 \leq x \leq b.$$

In a similar way, we define the $m$ dimensional MGF of $D_{\delta,b}$ by

$$W^{\top}_{b,w}(x,z) = (W_{b,w,1}(x,z), W_{b,w,2}(x,z), W_{b,w,3}(x,z), \dots, W_{b,w,m}(x,z))^{\top},$$

where

$$W_{b,w,i}(x,z) = \mathbf{E}[e^{zD_{\delta,b,w}} \mid U^{d}_{b,w}(0) = x, J(0) = i],$$

provided $z$ is such that $W_{b,w,i}(x,z)$ exists.

The $k$th order of moment of $D_{\delta,b,w}$ is defined by,

$$g^{k^{\top}}_{w}(x,b) = (g^{k}_{1}(x,b), g^{k}_{2}(x,b), g^{k}_{3}(x,b), \dots, g^{k}_{3}(x,b))^{\top},$$

where

$$g^{k}_{i}(x,b) = \mathbf{E}[D^{k}_{\delta,b,w} \mid U^{d}_{b,w}(0) = x, j(0) = i], \quad k \in \mathbf{N}, \text{ and } {}^{0}_{i}(x,b) = 1.$$

The $W^{*^{\top}}_{b,w}(x,y)$ and $g^{*^{k^{\top}}}_{w}(x,b)$ are the $m$ dimensional vector of MGF and $k$th order moment of the total dividend paid in auxiliary timeline.

The *expected discounted dividend paid until ruin* (EDDR) $g_w(x,b)$ and the EDDR of auxiliary timeline $g^{*}_{w}(x,b)$ are given by

$$g_w(x,b) = \sum_{i-1}^{m} \boldsymbol{\alpha}_i\ g^{1}_{i}(x,b), \qquad \text{for } 0 \leq x \leq b \tag{2.11}$$

and

$$g^{*}_{w}(x,b) = \sum_{i=1}^{m} \boldsymbol{\alpha}_i\ g^{1^{*}}_{i}(x,b), \quad \text{for } 0 \leq x \leq b. \tag{2.12}$$

Finally, representing the stationary probability vector for continuous-time Markov chain $J_t$ as $\boldsymbol{\pi}$. In our model we are taking a constant AIF (*Average Inflow*). We can represent the average outflow (AOF) as,

$$\text{AOF} = -\boldsymbol{\pi}[(\boldsymbol{v}_{\mathbf{M}} \otimes \mathbf{D}_1)(\mathbf{H}_{\mathbf{M}} \otimes \mathbf{I}_m)^{-1}\mathbf{e}^{\top}_{mn_1} + \theta(\boldsymbol{v}_{\mathbf{B}} \otimes \mathbf{D}_1)(\mathbf{H}_{\mathbf{M}} \otimes \mathbf{I}_m)^{-1}\mathbf{e}^{\top}_{mn_2}],$$

For the model we are assuming that AIF > AOF and the security loading factor is

$$L = \frac{\text{AIF}}{\text{AOF}} - 1, \tag{2.13}$$

where $\mathbf{e}_{mn_1}^{\top}$ and $\mathbf{e}_{mn_2}^{\top}$ denotes column vectors having all entries one with dimensions $mn_1$ and $mn_2$.

## 3. Gerber-Shiu Function

In this section, we start by deriving the IDE for the GSF of risk reserve process (2.1) and then find the GSF of the risk reserve process (2.6) which is related to the GSF of risk reserve process (2.1).

### 3.1 Gerber-Shiu Function Without Barrier Strategy

We develop a system of IDE satisfied by $\phi_d^{\top}(x)$ and ${\phi^*}_d^{\top}(x)$ to find the GSF of risk reserve process (2.1). Applying Laplace transform and numerical inverse Laplace we solve the system of IDE.

#### 3.1.1 System of Integro-Differential Equations

In the next theorem, we provide a system of IDE for $\boldsymbol{\phi}_d^{\top}(x)$ and ${\boldsymbol{\phi}^*}^{\top}{}_d(x)$.

**Theorem 1.** *The GSFs $\phi_d^{\top}(x)$ and ${\phi^*}_d^{\top}(x)$ will satisfy the system of second order IDEs given below:*

*For $x \geq 0$,*

$$\begin{aligned} 0 = &\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi}_d^{\top}(x) + c\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}_d^{\top}(x) - [\delta\mathbf{I}_m - \Lambda'_{11}]\boldsymbol{\phi}_d^{\top}(x) \\ &- (1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}_d^{\top}(x-y)\mathrm{d}y + W_1(x)\right] \\ &- \theta\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}{\boldsymbol{\phi}_d^*}^{\top}(x-y)\mathrm{d}y + W_1(x)\right] \end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned} 0 = &\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}{\boldsymbol{\phi}_d^*}^{\top}(x) + c\frac{\mathrm{d}}{\mathrm{d}x}{\boldsymbol{\phi}_d^*}^{\top}(x)[\delta\mathbf{I}_m - \Lambda'_{11}]{\boldsymbol{\phi}_d^*}^{\top}(x) \\ &- (1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{\phi}_d^{\top}(x-y)\mathrm{d}y + W_2(x)\right] \\ &- \theta\Lambda'_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}{\boldsymbol{\phi}_d^*}^{\top}(x-y)\mathrm{d}y + W_2(x)\right], \end{aligned} \tag{3.2}$$

*where*

$$\Lambda_2'^{\top} = -\Lambda'_{22}\mathbf{e}_{mn_1}^{\top},$$

$$\Lambda_2''^{\top} = -\Lambda''_{22}\mathbf{e}_{m(n_1+n_2)}^{\top},$$

$$W_1(x) = \int_{x+ch+\sigma W(h)}^{\infty} \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda_2'^{\top}\omega(x+ch+\sigma W(h), y-x-ch-\sigma W(h))\mathrm{d}y$$

*and*

$$W_2(x) = \int_{x+ch+\sigma W(h)}^{\infty} \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda_2''^{\top}\omega(x+ch+\sigma W(h), y-x-ch-\sigma W(h))\mathrm{d}y.$$

*Proof.* For $0 \le x < \infty$, consider a very small time period $[0,h]$, there are three possibilities:

(i) no claim arrivals in $[0,h]$,

(ii) one main claim arrival in $[0,h]$ but it does not induce any by-claim which happens with $(1-\theta)$ probability (it may or may not cause the ruin),

(iii) one main claim occur in $[0,h]$ and it induces a by-claim that can happen with probability $\theta$ (here also it may or may not cause the ruin).

Conditioning on the above possible events in $[0,h]$, we obtain

$$\begin{aligned}\boldsymbol{\phi}_d^\top(x) &= e^{-\delta \mathbf{I_m} h} e^{\Lambda'_{11}h} \boldsymbol{\phi}_\delta^\top(x+ch+\sigma W(h)) + (1-\theta) e^{-\delta \mathbf{I}_m h}(\mathbf{I}_m - e^{\Lambda'_{11}h}) \\ &\cdot \left[\int_0^{x+ch+\sigma W(h)} \Lambda'_{12} e^{\Lambda'_{22}y} \Lambda'_{21} \boldsymbol{\phi}_d^\top(x+ch+\sigma W(h)-y) \mathrm{d}y + W_1(x)\right] + \theta e^{-\delta \mathbf{I}_m h}(\mathbf{I}_m - e^{\Lambda'_{11}h}) \\ &\cdot \left[\int_0^{x+ch+\sigma W(h)} \Lambda'_{12} e^{\Lambda'_{22}y} \Lambda'_{21} \boldsymbol{\phi}_d^{*\top}(x+ch+\sigma W(h)-y) \mathrm{d}y + W_1(x)\right] + o(h). \end{aligned} \tag{3.3}$$

Apply the Taylor series expansion in (3.3) and dividing the above equation by $h$ then taking limit $h \to 0$ we get the equation (3.1).

Similar argument yields the auxiliary risk reserve process

$$\begin{aligned}\boldsymbol{\phi}_d^{*\top}(x) &= e^{-\delta \mathbf{I_m} h} e^{\mathbf{T}'_{11}\mathbf{h}} \boldsymbol{\phi}_d^{*\top}(x+ch+\sigma W(h)) + (1-\theta) e^{-\delta \mathbf{I}_m h}(\mathbf{I}_m - e^{\Lambda'_{11}h}) \\ &\cdot \left[\int_0^{x+ch+\sigma W(h)} \Lambda''_{12} e^{\Lambda''_{22}y} \Lambda''_{21} \boldsymbol{\phi}_d^\top(x+ch+\sigma W(h)-y) \mathrm{d}y + W_2(x)\right] + \theta e^{-\delta \mathbf{I}_m h}(\mathbf{I}_m - e^{\Lambda''_{11}h}) \\ &\cdot \left[\int_0^{x+ch+\sigma W(h)} \Lambda''_{12} e^{\Lambda''_{22}y} \Lambda''_{21} \boldsymbol{\phi}_d^{*\top}(x+ch+\sigma W(h)-y) \mathrm{d}y + W_2(x)\right] + o(h). \end{aligned} \tag{3.4}$$

Apply the Taylor series expansion in (3.4) then dividing by $h$ and taking limit $h \to 0$ we get the equation (3.2). □

### 3.1.2 Analytic Solution

In this section, Laplace transforms is used on (3.1) and (3.2) to obtain expressions for $\boldsymbol{\phi}_d^\top(x)$ and $\boldsymbol{\phi}{*}_d^\top(x)$. Let $\mathcal{F}$ be a real-valued integrable function. So Laplace transform of $\mathcal{F}$ is defined as

$$\tilde{\mathcal{F}}(s) = \int_{y=0}^{\infty} e^{-sy} \mathcal{F}(y)\,\mathrm{d}y, \tag{3.5}$$

where $s \in \mathbb{C}$.

$\widetilde{W}_1(s)$ and $\widetilde{W}_2(s)$ are the Laplace transforms of $W_1(x)$ and $W_2(x)$. Applying the Laplace transform in eqs. (3.1) and (3.2) yield the proposition.

**Proposition 2.** *Let* $\mathbf{m}'(s) = \int_0^\infty e^{-sy} \Lambda'_{12} e^{\Lambda'_{22}y} \Lambda'_{21} \mathrm{d}y$ *and* $\mathbf{m}''(s) = \int_0^\infty e^{-sy} \Lambda''_{12} e^{\Lambda''_{22}y} \Lambda''_{21} \mathrm{d}y$. *Then, the Laplace transforms* $\widetilde{\boldsymbol{\phi}}_d^\top(s)$ *and* $\widetilde{\boldsymbol{\phi}}_d^{*\top}(s)$ *of measures* $\boldsymbol{\phi}_d^\top(x)$ *and* $\boldsymbol{\phi}_d^{*\top}(x)$ *can be expressed as*

$$\widetilde{\boldsymbol{\phi}}_d^\top(s) = \frac{[\operatorname{adj} \boldsymbol{A}^{\boldsymbol{w}}(s)]}{\det \boldsymbol{A}^{\boldsymbol{w}}(s)} [l_1^w(s)] \tag{3.6}$$

*and*

$$\widetilde{\boldsymbol{\phi}}_d^{*\top}(s) = \frac{[\operatorname{adj} \boldsymbol{A}^{\boldsymbol{w}}(s)]}{\det \boldsymbol{A}^{\boldsymbol{w}}(s)} [l_2^w(s)] \tag{3.7}$$

*where*

$$A^w(s) = \left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11}\right]\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11} - \Lambda'_{11}\left[(1-\theta)\mathbf{m}'(s) + \theta\mathbf{m}''(s)\right]\right],$$

$$l_1^w(s) = \left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11}\right] - \Lambda'_{11}\theta\mathbf{m}''(s)\right]\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^{\top}(\mathbf{0}) + \Lambda'_{11}\widetilde{W}_1(s) + \left[\frac{\sigma^2}{2}s + c\right]e^{\top}\right]$$
$$+ \Lambda'_{11}\theta\mathbf{m}(s)\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^{*\top}(0) + \Lambda'_{11}\widetilde{W}_2(s) + \left[\frac{\sigma^2}{2}s + c\right]e^{\top}\right]$$

*and*

$$l_2^w(s) = \left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \mathbf{T}'_{11}\right] - \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\right]\left[\frac{\sigma^2}{2}\boldsymbol{\phi}^{*'\top}(0) + \Lambda'_{11}\widetilde{W}_2(s) + \left[\frac{\sigma^2}{2}s + c\right]e^{\top}\right]$$
$$+ \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^{\top}(0) + \Lambda'_{11}\widetilde{W}_1(s) + \left[\frac{\sigma^2}{2}s + c\right]e^{\top}\right].$$

Thus, the matrix Lundberg equation $A^w(s)$ is obtained for the proposed risk model. We can use the method given in Sreeshamim *et al*. [14] to find $\phi_d'^{\top}(0)$ and $\phi_d'^{*\top}(0)$ and find explicit solution. Once we compute $\phi_d'^{\top}(0)$ and $\phi_d'^{*\top}(0)$, explicit expressions are obtained for the equations (3.6) and (3.7), respectively.

### 3.1.3 The Closed-Form Analytical Solutions

In this subsection, we develop explicit expressions for the GSFs both under original and modified timelines that are derived explicitly in closed forms. We shall take Laplace inverse of equations (3.6) and (3.7) to find $\phi_d^{\top}(x)$ and $\phi_d^{*\top}(x)$. The scheme given in Section 2.1(a) in Cheung and Landriault [3] is used for finding the Laplace transform inversion. In our work, we have used the matrix structure of the embedded fluid process associated with MAP/PH as given in Ahn and Badescu [1]. Instead of $T(s)$ in the system of equations (12) given in Cheung and Landriault [3], we are having $M(s) = (1-\theta)\mathbf{M}'(s) + \theta\mathbf{M}''(s)$ in equations (3.6) and (3.7). Thus, the same scheme can be followed with necessary changes.

**Theorem 3.** *$m(s)$ has the rational form $m(s) = \frac{p_{ij}(s)}{q_{ij}(s)}$, $i,j \in E$, where $p_{ij}(s)$ is a polynomial of degree less than $r_{ij}$ and $q_{ij}(s)$ is a polynomial of degree $r_{ij}$ with $p_{ij}(0)/q_{ij}(0) = 1$. If $\{\rho_i\}_{i=1}^{2m+r}$ be the distinct root of the equation $q(s)\det[A(s)] = 0$, where $q(s) = \prod_{i=1}^{m}\prod_{j=1}^{m} q_{ij}(s)$. Then, for an arbitrary $\kappa$ with $\kappa \neq \rho_i$ for $i = 1,\dots,2m+r$, then using the Lagrange's interpolating polynomial $\boldsymbol{\phi}_d^{\top}(x)$ and $\boldsymbol{\phi}_d^{*\top}(x)$ have the closed-form expressions*

$$\boldsymbol{\phi}_d^{\top}(x) = \frac{1}{h(\kappa)}\sum_{l=1}^{2m+r} h_1(\rho_l)\beta_l(\kappa)(\kappa-\rho_l)\mathrm{e}^{\rho_l x} \tag{3.8}$$

*and*

$$\boldsymbol{\phi}_d^{*\top}(x) = \frac{1}{h(\kappa)}\sum_{l=1}^{2m+r} h_2(\rho_l)\beta_l(\kappa)(\kappa-\rho_l)\mathrm{e}^{\rho_l x}, \tag{3.9}$$

*where*

$$\beta_l(s) = \prod_{k=1,k\neq l}^{2m+r}((\rho_k - s)/(\rho_k - \rho_l)),$$

$$h_1(s) = q(s)[\operatorname{adj} \boldsymbol{A}^w(s)][l_1(s)]$$

*and*

$$h_2(s) = q(s)[\operatorname{adj} \boldsymbol{A}^w(s)][l_2(s)]$$

*are polynomials of degrees less than* $2m+r$ *and* $h(s) = q(s)\det \boldsymbol{A}^w(s)$ *is polynomial with degree* $2m+r$.

*Proof.* $m(s)$ have rational form $m(s) = \frac{p_{ij}(s)}{q_{ij}(s)}$ (see Dufresne [6]). We can represent $\widetilde{\boldsymbol{\phi}}_\delta^\top(s)$ and $\widetilde{\boldsymbol{\phi}}_\delta^{*\top}(s)$ as

$$\widetilde{\boldsymbol{\phi}}_\delta^\top(s) = \frac{q(s)[\operatorname{adj} \boldsymbol{A}^w(s)]}{q(s)\det \boldsymbol{A}^w(s)}[l_1(s)] = \frac{h_1(s)}{h(s)} \tag{3.10}$$

and

$$\widetilde{\boldsymbol{\phi}}_\delta^{*\top}(s) = \frac{q(s)[\operatorname{adj} \boldsymbol{A}^w(s)]}{q(s)\det \boldsymbol{A}^w(s)}[l_2(s)] = \frac{h_2(s)}{h(s)} \tag{3.11}$$

For an arbitrary $\kappa$ with $\kappa \neq \rho_i$ for $i = 1,\dots,2m+r$, apply Lagrange's interpolating polynomial allows us to change the equations (3.6) and (3.7) as

$$\widetilde{\boldsymbol{\phi}}_\delta^\top(s) = \frac{1}{h(\kappa)}\sum_{l=1}^{2m+r} h_1(\rho_l)\beta_l(\kappa)\frac{\rho_l-\kappa}{\rho_l-s} \tag{3.12}$$

and

$$\widetilde{\boldsymbol{\phi}}_\delta^{*\top}(s) = \frac{1}{h(\kappa)}\sum_{l=1}^{2m+r} h_2(\rho_l)\beta_l(\kappa)\frac{\rho_l-\kappa}{\rho_l-s}. \tag{3.13}$$

Inverting (3.12) and (3.13) yields (3.8) and (3.9). □

## 3.2 Gerber-Shiu Function With Barrier Strategy

The expression for the GSF of the risk reserve process (2.6) is derived in this section. First, we derive the IDE system with certain boundary conditions for the GSF. Then, we show that its solution can be expressed as the linear combination of the solution to the GSF in the risk model without a barrier (risk reserve process (2.1)) and solutions to the associated homogeneous IDE.

### 3.2.1 System of Integro-Differential Equations

In the following theorem, our aim is to give a IDE system with boundary conditions for GSFs $\boldsymbol{\phi}_{d,b}^\top(x)$ and $\boldsymbol{\phi}_{d,b}^{*\top}(x)$.

**Theorem 4.** *For* $0 \le x \le b$, *the GSFs* $\phi_{d,b}^\top(x)$ *and* ${\phi^*}_{d,b}^\top(x)$ *satisfy the IDE with boundary conditions given below:*

$$\begin{aligned}0 &= \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi}_{d,b}^\top(x) + c\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}_{d,b}^\top(x) - [\delta\mathbf{I}_m - \Lambda'_{11}]\boldsymbol{\phi}_{d,b}^\top(x)\\ &\quad -(1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}_{d,b}^\top(x-y)\mathrm{d}y + W_1(x)\right]\\ &\quad -\theta\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}_{d,b}^{*\top}(x-y)\mathrm{d}y + W_1(x)\right]\end{aligned} \tag{3.14}$$

*and*

$$0 = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi}^{*\top}_{d,b}(x) + c\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{*\top}_{d,b}(x)[\delta \mathbf{I}_m - \Lambda'_{11}]\boldsymbol{\phi}^{*\top}_{d,b}(x)$$
$$-(1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{\phi}^{\top}_{d,b}(x-y)\mathrm{d}y + W_2(x)\right]$$
$$-\theta\Lambda'_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{\phi}^{*\top}_{d,b}(x-y)\mathrm{d}y + W_2(x)\right], \tag{3.15}$$

*with boundary conditions*

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{\top}_{d,b}(x)\Big|_{x=b} = 0$$

*and*

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{*\top}_{d,b}(x)\Big|_{x=b} = 0.$$

*Proof.* For $x = b$, conditioning on the possible events in $[0,h]$, we obtain

$$\boldsymbol{\phi}^{\top}_{d,b}(b) = e^{-(\delta I_m - \Lambda'_{11})h}\boldsymbol{\phi}^{\top}_{d,b}(b+\sigma W(h)) - (1-\theta)\Lambda'_{11}h$$
$$\cdot\left[\int_0^b \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}^{\top}_{d,b}(b+\sigma W(h)-y)\mathrm{d}y + W_1(b)\right]$$
$$-\theta\Lambda'_{11}h\left[\int_0^b \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}^{*\top}_{d,b}(b+\sigma W(h)-y)\mathrm{d}y + W_1(b)\right] + o(h), \tag{3.16}$$

divide by $h$ then taking limit $h \to 0$,

$$0 = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi}^{\top}_{d,b}(b) - [\delta \mathbf{I}_m - \Lambda'_{11}]\,\boldsymbol{\phi}^{\top}_{d,b}(b) - (1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}^{\top}_{d,b}(b-y)\mathrm{d}y + W_1(b)\right]$$
$$-\theta\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi}^{*\top}_{d,b}(b-y)\mathrm{d}y + W_1(b)\right]. \tag{3.17}$$

Arranging $x = b$ in (3.14) and comparing with (3.17) we can develop the condition

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{\top}_{d.b}(x)\Big|_{x=b} = 0.$$

With the auxiliary risk reserve process, using the same argument we can develop

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{*\top}_{d,b}(x)\Big|_{x=b} = 0.$$ □

### 3.2.2 Solution Analysis

We derive non-homogeneous second order IDE (3.14) and (3.15) for $\boldsymbol{\phi}^{\top}_{d,b}(x)$ and $\boldsymbol{\phi}^{*\top}_{d,b}(x)$. We begin by looking at the following homogeneous IDE of (3.18) and (3.19) to find the solution of the same IDEs.

For $x \geq 0$,

$$0 = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi^o}^{\top}_{d,b}(x) + c\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi^o}^{\top}_{d,b}(x) - [\delta \mathbf{I}_m - \Lambda'_{11}]\boldsymbol{\phi^o}^{\top}_{d,b}(x)$$
$$-(1-\theta)\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi^o}^{\top}_{d,b}(x-y)\mathrm{d}y\right]$$
$$-\theta\Lambda'_{11}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{\phi^o}^{*\top}_{d,b}(x-y)\mathrm{d}y\right] \tag{3.18}$$

and

$$\begin{aligned}0 &= \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\boldsymbol{\phi}^{\boldsymbol{o}^*}{}^{\top}_{d,b}(x) + c\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{\phi}^{\boldsymbol{o}^*}{}^{\top}_{d,b}(x) - [\delta \mathbf{I}_m - \Lambda''_{11}]\boldsymbol{\phi}^{\boldsymbol{o}^*}{}^{\top}_{d,b}(x)\\ &\quad -(1-\theta)\Lambda''_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{\phi}^{\boldsymbol{o}}{}^{\top}_{d,b}(x-y)\mathrm{d}y\right]\\ &\quad -\theta\Lambda''_{11}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{\phi}^{\boldsymbol{o}^*}{}^{\top}_{d,b}(x-y)\mathrm{d}y\right]. \end{aligned} \tag{3.19}$$

Solution of IDE system (3.18) and (3.19) derived using the general theory of DE (differential equation) has the form, for $x \ge 0$

$$\begin{bmatrix} \boldsymbol{\phi}^{o^\top}_{d,b}(x) \\ \boldsymbol{\phi}^{o^{*\top}}_{d,b}(x)\end{bmatrix}_{2m\times 1} = [V_w^A(x)]_{2m\times 2m}\begin{bmatrix}\mu_i^\top(b)\\ \mu_i^{*\top}(b)\end{bmatrix}_{i=1}^m + [V_w^B(x)]_{2m\times 2m}\begin{bmatrix}\eta_i^\top(b)\\ \eta_i^{*\top}(b)\end{bmatrix}_{i=1}^m. \tag{3.20}$$

Here the matrices $V_w^A(x)$ and $V_w^B(x)$ represents the solutions of system of the equation system (3.18) and (3.19) from Lakshmikantham and Rao [8], i.e., each columns of the matrices are the linearly independent solution of the system (3.18) and (3.19). $\eta_i^\top(b)$, $\eta_i^{*\top}(b)$, $\mu_i^\top(b)$ and $\mu_i^{*\top}(b)$ are the vectors of arbitrary constants.

Now we look for some explicit results. Consider the boundary conditions in Theorem 4 and using the same arguments similar to the paper those in Zou and Xie [18], we have the general solutions of the non-homogeneous IDE (3.18) and (3.19):

$$\begin{bmatrix} \boldsymbol{\phi}^{\top}_{d,b}(x) \\ \boldsymbol{\phi}^{*\top}_{d,b}(x)\end{bmatrix}_{2m\times 1} = \begin{bmatrix} \boldsymbol{\phi}^{\top}_{d}(x) \\ \boldsymbol{\phi}^{*\top}_{d}(x)\end{bmatrix}_{2m\times 1} + [V_w^B(x)]_{2m\times 2m}\begin{bmatrix}\boldsymbol{\phi}'^{\top}_{d,b}(0) - \boldsymbol{\phi}'^{\top}_{d}(0)\\ \boldsymbol{\phi}'^{*\top}_{d,b}(0) - \boldsymbol{\phi}'^{*\top}_{d}(0)\end{bmatrix}_{2m\times 1}. \tag{3.21}$$

Note that, $\eta_i{}^\top(b) = \boldsymbol{\phi}'^{\top}_{d,b}(0) - \boldsymbol{\phi}'^{\top}_{d}(0)$ and $\eta_i{}^{*\top}(b) = \boldsymbol{\phi}'^{*\top}_{d,b}(0) - \boldsymbol{\phi}'^{*\top}_{d}(0)$. Furthermore, $\eta_i{}^\top(b)$ and $\eta_i^{*\top}(b)$ can be obtained by the boundary conditions given in Theorem 4. So to determine $\boldsymbol{\phi}^{\top}_{d,b}(x)$ and $\boldsymbol{\phi}^{*\top}_{d,b}(x)$, we only need to find $[V_w^B(x)]_{2m\times 2m}$. Next, using the similar method in Section 3.1.3 we can find the explicit expression for $V_w^B(x)$.

**Proposition 5.** *Let $[V_w^B(x)]_{2m\times 2m}$ be the solution matrix of the system of homogeneous IDE* (3.18) *and* (3.19)*, $[V^1_{ij}(x)]$ and $[V^2_{ij}(x)]$ are two $m\times 2m$ dimensional matrices which are formed by considering the first $m$ rows and last $m$ rows of the matrix $[V_w^B(x)]_{2m\times 2m}$, respectively. Then, the Laplace transform $[\widetilde{V}^1_{ij}(s)]$ and $[\widetilde{V}^2_{ij}(s)]$ of matrices $[V^1_{ij}(x)]$ and $[V^2_{ij}(x)]$ can be represented as,*

*For $i = \{1,2,3\dots m\}$, $j = \{1,2,3\dots 2m\}$*

$$[\widetilde{V}^1_{ij}(s)]_{m\times 2m} = \frac{[\operatorname{adj}\boldsymbol{A}^w(s)]_{m\times m}}{\det \boldsymbol{A}^w(s)}[L_1(s)]_{m\times 2m} \tag{3.22}$$

*and for $i = \{m+1, m+2, m+3\dots 2m\}$, $j = \{1,2,3\dots 2m\}$*

$$[\widetilde{V}^2_{ij}(s)]_{m\times 2m} = \frac{[\operatorname{adj}\boldsymbol{A}^w(s)]_{m\times m}}{\det \boldsymbol{A}^w(s)}[L_2(s)]_{m\times 2m} \tag{3.23}$$

*where*

$$\begin{aligned} L_1^w(s) &= \left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11}\right] - \Lambda'_{11}\theta\mathbf{m}''(s)\right]\left[\frac{\sigma^2}{2}V_w^{B'}(0)_{ij} + \operatorname{diag}\left[\frac{\sigma^2}{2}s + c\right]\right]\\ &\quad + \Lambda'_{11}\theta\mathbf{m}(s)\left[\frac{\sigma^2}{2}V_w^{B'}(0)_{i-m,j} + \operatorname{diag}\left[\frac{\sigma^2}{2}s + c\right]\right] \end{aligned}$$

*and*

$$L_2^w(s) = \left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11}\right] - \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\right]\left[\frac{\sigma^2}{2}V_w^{B'}(0)_{ij} + \mathrm{diag}\left[\frac{\sigma^2}{2}s + c\right]\right] + \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\left[\frac{\sigma^2}{2}V_w^{B'}(0)_{i-m,j} + \mathrm{diag}\left[\frac{\sigma^2}{2}s + c\right]\right].$$

*Proof.* Taking the Laplace transform of homogeneous IDE (3.18) and (3.19). Then, making some simplifications, we get

$$\widetilde{\boldsymbol{\phi}}_{d,b}^{o^\top}(s) = \frac{\left(\begin{array}{l}\left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \Lambda'_{11}\right] - \Lambda'_{11}\theta\mathbf{m}''(s)\right] \\ \quad \times\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^\top(\mathbf{0}) + [\frac{\sigma^2}{2}s + c]e^\top\right] + \Lambda'_{11}\theta\mathbf{m}(s)\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^{*\top}(0) + [\frac{\sigma^2}{2}s + c]e^\top\right]\end{array}\right)}{\boldsymbol{A^w(s)}} \tag{3.24}$$

and

$$\widetilde{\boldsymbol{\phi}}_{d,b}^{0^{*\top}}(s) = \frac{\left(\begin{array}{l}\left[\left[\left(\frac{\sigma^2}{2}s^2 + cs - \delta\right)I_m + \mathbf{T}'_{11}\right] - \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\right] \\ \quad \times\left[\frac{\sigma^2}{2}\boldsymbol{\phi}^{*'\top}(0) + [\frac{\sigma^2}{2}s + c]e^\top\right] + \Lambda'_{11}(1-\theta)\mathbf{m}'(s)\left[\frac{\sigma^2}{2}\boldsymbol{\phi}'^\top(0) + [\frac{\sigma^2}{2}s + c]e^\top\right]\end{array}\right)}{\boldsymbol{A^w(s)}} \tag{3.25}$$

from equation (3.21) and the assumption $V_w^{B'}(0) = I$, we can derive the results. □

### 3.2.3 The Closed-Form Analytical Solutions

Here, we derive explicit expression for the matrices $[V_{ij}^1(x)$ and $[V_{ij}^2(x)]$ in closed forms. The matrices $[V_{ij}^1(x)]$ and $[V_{ij}^2(x)]$ together form the solution matrix $[Vij(x)]$ of the homogeneous IDE system (3.18) and (3.19). We shall take Laplace transform inversion to equations (3.22) and (4.1) to find the expressions for $[V_{ij}^1(x)]$ and $[V_{ij}^2(x)]$. The method given in Section 2.1(a) in Cheung and Landriault [3] is used for finding the Laplace transform inversion. In our work, we have used the matrix structure of the embedded fluid process associated with MAP/PH as given in Ahn and Badescu [1]. Instead of $T(s)$ in the system of equations (12) given in Cheung and Landriault [3], we are having $M(s) = (1-\theta)\mathbf{M}'(s) + \theta\mathbf{M}''(s)$ in equations (3.22) and (4.1). Thus, the same scheme can be followed with some necessary changes.

**Theorem 6.** *Let $M(s)$ has the rational form $M(s) = \frac{p_{ij}(s)}{q_{ij}(s)}$, $i,j \in E$, where $p_{ij}(s)$ is a polynomial of degree less than $r_{ij}$ and $q_{ij}(s)$ is a polynomial of degree $r_{ij}$ with $p_{ij}(0)/q_{ij}(0) = 1$. If $\{\rho_i\}_{i=1}^{2m+r}$ be the distinct root of the equation $q(s)\det[A(s)] = 0$, where $q(s) = \prod_{i=1}^m \prod_{j=1}^m q_{ij}(s)$. Then, for an arbitrary $\kappa$ with $\kappa \neq \rho_i$ for $i = 1,\dots,2m+r$, then using the Lagrange's interpolating polynomial $[V_{ij}^1(x)$ and $[V_{ij}^2(x)]$ have the closed-form expressions.*

*For $i = \{1,2,3,\dots,m\}$, $j = \{1,2,3,\dots,2m\}$,*

$$[V_{ij}^1(x)]_{m\times 2m} = \frac{1}{\gamma(\kappa)}\sum_{l=1}^{2m+r}\gamma_{1ij}(\rho_l)\beta_l(\kappa)(\kappa-\rho_l)\mathrm{e}^{\rho_l x}, \tag{3.26}$$

*and, for $i = \{m+1, m+2, m+3,\dots,2m\}$, $j = \{1,2,3,\dots,2m\}$,*

$$[V_{ij}^2(x)]_{m\times 2m} = \frac{1}{\gamma(\kappa)}\sum_{l=1}^{2m+r}\gamma_{2ij}(\rho_l)\beta_l(\kappa)(\kappa-\rho_l)\mathrm{e}^{\rho_l x}, \tag{3.27}$$

*where* $\gamma_{1_{ij}}(s) = q(s)[\operatorname{adj}\boldsymbol{A}(s)][L_1(s)]$, $\gamma_{2_{ij}}(s) = q(s)[\operatorname{adj}\boldsymbol{A}(s)][L_2(s)]$ *are polynomials of degrees less than* $2m+r$, $\gamma(s) = q(s)\det\boldsymbol{A}(s)$ *is polynomial with degree* $2m+r$ *and* $\beta_l(s) = \prod_{k=1,k\neq l}^{2m+r}((\rho_k - s)/(\rho_k-\rho_l))$.

*Proof.* $M(s)$ has rational form $M(s) = \frac{p_{ij}(s)}{q_{ij}(s)}$ (see Dufresne [6]). We can represent $[\widetilde{V}^1_{ij}(s)]_{m\times 2m}$ and $[\widetilde{V}^2_{ij}(s)]_{m\times 2m}$ as:

For $i = \{1,2,3,\dots,m\}$, $j = \{1,2,3,\dots,2m\}$,

$$[\widetilde{V}^1_{ij}(s)]_{m\times 2m} = \frac{q(s)[\operatorname{adj}\boldsymbol{A}(s)][L_1(s)]}{q(s)\det\boldsymbol{A}(s)} = \frac{\gamma_{1_{ij}}(s)}{\gamma(s)} \tag{3.28}$$

and for $i = \{m+1,m+2,m+3,\dots,2m\}$, $j = \{1,2,3,\dots,2m\}$,

$$[\widetilde{V}^2_{ij}(s)]_{m\times 2m} = \frac{q(s)[\operatorname{adj}\boldsymbol{A}(s)][L_2(s)]}{q(s)\det\boldsymbol{A}(s)} = \frac{\gamma_{2_{ij}}(s)}{\gamma(s)}. \tag{3.29}$$

For an arbitrary $\kappa$ with $\kappa \neq \rho_i$ for $i = 1,\dots,2m+r$, apply Lagrange's interpolating polynomial on equations (3.28) and (3.29) gives:

For $i = \{1,2,3,\dots,m\}$, $j = \{1,2,3,\dots,2m\}$,

$$[\widetilde{V}^1_{ij}(s)]_{m\times 2m} = \frac{1}{\gamma(\kappa)}\sum_{l=1}^{2m+r}\gamma_{1_{ij}}(\rho_l)\beta_l(\kappa)\frac{\rho_l-\kappa}{\rho_l-s}. \tag{3.30}$$

For $i = \{m+1,m+2,m+3\dots 2m\}$, $j = \{1,2,3,\dots,2m\}$,

$$[\widetilde{V}^2_{ij}(s)]_{m\times 2m} = \frac{1}{\gamma(\kappa)}\sum_{l=1}^{2m+r}\gamma_{2_{ij}}(\rho_l)\beta_l(\kappa)\frac{\rho_l-\kappa}{\rho_l-s}. \tag{3.31}$$

Inverting (3.30) and (3.31) yields (3.26) and (3.27). □

## 4. The Moment Generating Function and the Moment of Discounted Dividends Paid Until Ruin

Here we develop an IDE system satisfied by the MGF of the discounted dividend paid until ruin for the risk reserve process (2.6). Then, applying the Taylor series expansion gives an IDE system for the Moment of the discounted dividend payments. Further, with the help of the methodology used in Section 3 we solve this IDE system.

### 4.1 System of Integro-Differential Equation

In this section, IDE for the MGF of total dividend paid and Moment of total dividend paid are derived.

**Theorem 7.** *For* $0 \le x \le b$, *the MGFs of total dividend paid* $\mathbf{W}^{\top}_{b,w}(x,z)$ *and* $\mathbf{W}^{*\top}_{b,w}(x,z)$ *satisfy the following system of second order IDE with boundary conditions:*

$$0 = \left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2} + c\frac{d}{dx} - \delta y\frac{d}{dz}\right)\mathbf{I}_m + \Lambda'_{11}\right]\mathbf{W}^{\top}_{b,w}(x,z) - (1-\theta)\Lambda'_{\mathbf{11}} \cdot\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\mathbf{W}^{\top}_{b,w}(x-y,z)\mathrm{d}y + \int_x^\infty \Lambda'_{12}e^{\Lambda'_{22}y}\mathbf{t}'^{\top}_2 1\mathrm{d}y\right]$$

$$-\theta\Lambda'_{\mathbf{11}}\left[\int_0^x \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\mathbf{W}^{*\top}_{b,w}(x-y,z)\mathrm{d}y+\int_x^\infty \Lambda'_{12}e^{\Lambda'_{22}y}\mathbf{t}'^{\top}_2 1\mathrm{d}y\right] \tag{4.1}$$

*and*

$$0=\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta z\frac{d}{dz}\right)\mathbf{I}_m+\Lambda'_{11}\right]\mathbf{W}^{*\top}_{b,w}(x,z)-(1-\theta)\Lambda'_{\mathbf{11}}$$
$$\cdot\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\mathbf{W}^{\top}_{b,w}(x-y,z)\mathrm{d}y+\int_x^\infty \Lambda''_{12}e^{\Lambda''_{22}y}\mathbf{t}^{\top}_2 1\mathrm{d}y\right]$$
$$-\theta\Lambda'_{\mathbf{11}}\left[\int_0^x \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\mathbf{W}^{*\top}_{b}(x-y,z)\mathrm{d}y+\int_x^\infty \Lambda'_{12}e^{\Lambda''_{22}y}\mathbf{t}''^{\top}_2 1\mathrm{d}y\right], \tag{4.2}$$

*with boundary conditions*

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{W}^{\top}_{b,w}(x,z)\bigg|_{x=b}=z\mathbf{W}^{\top}_{b,w}(b,z),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{W}^{*\top}_{b,w}(x,z)\bigg|_{x=b}=z\mathbf{W}^{*\top}_{b,w}(b,z). \tag{4.3}$$

*Also,* $\lim_{b\to\infty}\mathbf{W}^{\top}_{b,w}(x,z)=e^{\top}$ *and* $\lim_{b\to\infty}\mathbf{W}^{*\top}_{b,w}(x,z)=e^{\top}$.

*Proof.* For $0\le x<b$, consider a very small time period $[0,h]$, there are three possibilities:

(i) no claim arrivals in $[0,h]$,

(ii) one main claim occurs in $[0,h]$ but it does not induce any by-claim which happens with $(1-\theta)$ probability (it may or may not cause the ruin),

(iii) one main claim occur in $[0,h]$ and it induces a by-claim that can happen with probability $\theta$ (here also it may or may not cause the ruin).

Conditioning on the above possible events in $[0,h]$, we obtain

$$\mathbf{W}^{\top}_{b,w}(x,z)=e^{\mathbf{T}'_{11}h}\mathbf{W}^{\top}_{b,w}(x+ch+\sigma W(h),ze^{-\delta h})+(1-\theta)(\mathbf{I}_m-e^{\Lambda'_{11}h})$$
$$\cdot\left[\int_0^{x+ch+\sigma W(h)}\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\mathbf{W}^{\top}_{b,w}(x+ch+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\right.$$
$$\left.+\int_{x+ch+\sigma W(h)}^\infty \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'^{\top}_1 1\mathrm{d}y\right]+\theta(\mathbf{I}_m-e^{\Lambda'_{11}h})$$
$$\cdot\left[\int_0^{x+ch+\sigma W(h)}\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\mathbf{W}^{*\top}_{b,w}(x+ch+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\right.$$
$$\left.+\int_{x+ch+\sigma W(h)}^\infty \Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'^{\top}_2 1\mathrm{d}y\right]+o(h). \tag{4.4}$$

Similarly,

$$\mathbf{W}^{*\top}_{b,w}(x,z)=e^{\mathbf{T}'_{11}h}\mathbf{W}^{*\top}_{b,w}(x+ch+\sigma W(h),ze^{-\delta h})+(1-\theta)(\mathbf{I}_m-e^{\Lambda'_{11}h})$$
$$\cdot\left[\int_0^{x+ch+\sigma W(h)}\Lambda'_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\mathbf{W}^{\top}_{b,w}(x+ch+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\right.$$
$$\left.+\int_{x+ch+\sigma W(h)}^\infty \Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''^{\top}_1 1\mathrm{d}y\right]+\theta(\mathbf{I}_m-e^{\Lambda'_{11}h})$$
$$\cdot\left[\int_0^{x+ch+\sigma W(h)}\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\mathbf{W}^{*\top}_{b,w}(x+ch+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\right.$$

$$+\int_{x+ch+\sigma W(h)}^{\infty}\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_2''^{\top}1\mathrm{d}y\Big]+o(h). \tag{4.5}$$

Expanding the equations (4.4) and (4.5) in Taylor series then dividing the equation by $h$ and taking limit $h\to 0$, we get the equation (4.1) and (4.2).

For $u=b$, use the same arguments

$$\begin{aligned}\mathbf{W}_{b,w}^{\top}(b,z)&=e^{\mathbf{T}_{11}'h}e^{zch}\mathbf{W}_{b,w}^{\top}(b+\sigma W(h),ze^{-\delta h})+(1-\theta)(\mathbf{I}_m-e^{\Lambda_{11}'h})\\&\cdot\Big[\int_0^b\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_{21}'\mathbf{W}_{b,w}^{\top}(b+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\\&+\int_b^{\infty}\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_1'^{\top}1\mathrm{d}y\Big]+\theta(\mathbf{I}_m-e^{\Lambda_{11}'h})\\&\cdot\Big[\int_0^b\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_{21}'\mathbf{W}_{b,w}^{*\top}(b+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\\&+\int_b^{\infty}\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_1'^{\top}1\mathrm{d}y\Big]+o(h)\end{aligned} \tag{4.6}$$

for the auxiliary risk reserve process

$$\begin{aligned}\mathbf{W}_{b,w}^{*\top}(b,z)&=e^{\mathbf{T}_{11}'h}e^{zch}\mathbf{W}_{b,w}^{*\top}(b+\sigma W(h),ze^{-\delta h})+(1-\theta)(\mathbf{I}_m-e^{\Lambda_{11}'h})\\&\cdot\Big[\int_0^b\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_{21}''\mathbf{W}_{b,w}^{\top}(b+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\\&+\int_b^{\infty}\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_2''^{\top}1\mathrm{d}y\Big]+\theta(\mathbf{I}_m-e^{\Lambda_{11}'h})\\&\cdot\Big[\int_0^b\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_{21}''\mathbf{W}_{b,w}^{*\top}(b+\sigma W(h)-y,ze^{-\delta h})\mathrm{d}y\\&+\int_b^{\infty}\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_2''^{\top}1\mathrm{d}y\Big]+o(h).\end{aligned} \tag{4.7}$$

Once again, substituting Taylor series expansion in the equations (4.6) and (4.7) then dividing it by $h$ and taking limit $h\to 0$

$$\begin{aligned}0&=-\delta z\frac{d}{dz}\mathbf{W}_{b,w}^{\top}(b+\sigma W(h),z)+(zc\mathbf{I}_m+\mathbf{T}_{\mathbf{11}}')\mathbf{W}_{b,w}^{\top}(b+\sigma W(h),z)-(1-\theta)\mathbf{T}_{\mathbf{11}}'\\&\cdot\Big[\int_0^b\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_{21}'\mathbf{W}_{b,w}^{\top}(b+\sigma W(h)-y,z)\mathrm{d}y+\int_b^{\infty}\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_1'^{\top}1\mathrm{d}y\Big]\\&-\theta\mathbf{T}_{\mathbf{11}}'\Big[\int_0^b\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_{21}'\mathbf{W}_b^{*\top}(b+\sigma W(h)-y,z)\mathrm{d}y\Big]+\Big[\int_b^{\infty}\Lambda_{12}'e^{\Lambda_{22}'y}\Lambda_1'^{\top}1\mathrm{d}y\Big]\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}0&=-\delta z\frac{d}{dz}\mathbf{W}_{b,w}^{*\top}(b+\sigma W(h),z)+(zc\mathbf{I}_m+\mathbf{T}_{\mathbf{11}}')\mathbf{W}_{b,w}^{*\top}(b+\sigma W(h),z)-(1-\theta)\mathbf{T}_{\mathbf{11}}'\\&\cdot\Big[\int_0^b\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_{21}''\mathbf{W}_{b,w}^{\top}(b+\sigma W(h)-y,z)\mathrm{d}y+\int_b^{\infty}\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_2''^{\top}1\mathrm{d}x\Big]\\&-\theta\mathbf{T}_{\mathbf{11}}'\Big[\int_0^b\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_{21}''\mathbf{W}_b^{*\top}(b+\sigma W(h)-y,z)\mathrm{d}y\Big]+\Big[\int_b^{\infty}\Lambda_{12}''e^{\Lambda_{22}''y}\Lambda_2''^{\top}1\mathrm{d}y\Big].\end{aligned} \tag{4.9}$$

Setting $x=b$ in equations (4.1) and (4.2), using the equation (4.8), (4.9) and $\boldsymbol{W}^{\top}_{b,w}(x,z)$ and $\boldsymbol{W}^{*\top}_{b,w}(x,z)$ are continuous at $x=b$, we find the boundary conditions (4.3). □

**Proposition 8.** *For $0 \le x \le b$ and $k \ge 1$, the kth Moment $\boldsymbol{g}^{k^{\top}}_{w}(x,b)$ and $\boldsymbol{g}^{*^{k^{\top}}}(x,b)$ satisfy the following system of IDE with boundary conditions:*

$$0=\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{k^{\top}}_{w}(x,b)-(1-\theta)\Lambda'_{\mathbf{11}}\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{k^{\top}}_{w}(x-y,b)\mathrm{d}y$$
$$-\theta\Lambda'_{\mathbf{11}}\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x-y,b)\mathrm{d}y \tag{4.10}$$

*and*

$$0=\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{*^{k^{\top}}}_{w}(x,b)-(1-\theta)\Lambda'_{\mathbf{11}}\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{k^{\top}}_{w}(x-y,b)\mathrm{d}y$$
$$-\theta\Lambda'_{\mathbf{11}}\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x-y,b)\mathrm{d}y \tag{4.11}$$

*with boundary conditions*

$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{g}^{k^{\top}}_{w}(x,b)\bigg|_{x=b}=k\boldsymbol{g}^{k-1^{\top}}_{w}(b,b),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x,b)\bigg|_{x=b}=k\boldsymbol{g}^{*^{k-1^{\top}}}_{w}(b,b). \tag{4.12}$$

*Also,* $\lim\limits_{b\to\infty}\boldsymbol{g}^{k^{\top}}_{w}(x,b)=\lim\limits_{b\to\infty}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x,b)=0.$

*Proof.* Using Taylor's series expansion

$$\boldsymbol{W}^{\top}_{b,w}(x,z)=e^{\top}+\sum_{k=1}^{\infty}\frac{z^k}{k!}\boldsymbol{g}^{k^{\top}}_{w}(x,b)\quad\text{and}\quad\boldsymbol{W}^{*\top}_{b,w}(x,z)=e^{\top}+\sum_{k=1}^{\infty}\frac{z^k}{k!}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x,b), \tag{4.13}$$

into (4.10) and (4.11) it gives

$$0=\sum_{k=1}^{\infty}\frac{z^k}{k!}\left[\frac{\sigma^2}{2}\frac{d^2}{dx^2}+\left(c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{k^{\top}}_{w}(x,b)-(1-\theta)\Lambda'_{\mathbf{11}}$$
$$\cdot\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{k^{\top}}_{w}(x-y,b)\mathrm{d}y$$
$$-\theta\Lambda'_{11}\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x-y,b)\mathrm{d}y \tag{4.14}$$

and

$$0=\sum_{k=1}^{\infty}\frac{z^k}{k!}\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{*^{k^{\top}}}_{w}(x,b)-(1-\theta)\Lambda'_{11}$$
$$\cdot\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{k^{\top}}_{w}(x-y,b)\mathrm{d}y$$
$$-\theta\Lambda'_{11}\sum_{k=1}^{\infty}\frac{z^k}{k!}\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{*^{k^{\top}}}_{w}(x-y,b)\mathrm{d}y. \tag{4.15}$$

Then comparing the coefficients of $y^k$ in (4.14) and (4.15) yields (4.10) and (4.11). Also, from (4.13) and (4.3), we have

$$\sum_{k=1}^{\infty}\frac{z^k}{k!}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{g}_w^{k^\top}(x,b)\bigg|_{x=b}=\sum_{k=1}^{\infty}k\frac{z^k}{k!}\boldsymbol{g}_w^{k-1^\top}(x,b)\bigg|_{x=b},$$
$$\sum_{k=1}^{\infty}\frac{z^k}{k!}\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{g}_w^{*^{k^\top}}(x,b)\bigg|_{x=b}=\sum_{k=1}^{\infty}k\frac{z^k}{k!}\boldsymbol{g}_w^{*^{k-1^\top}}(x,b)\bigg|_{x=b}. \tag{4.16}$$

To get the boundary condition (4.12) just compare the coefficients of $y^m$ in (4.16). We omit the proof of limit result which is trivial. □

### 4.2 Solution Analysis

The solution of IDE system (4.10) and (4.11) with boundary condition (4.12) heavily depends on the solution to the following associated homogeneous system in $\boldsymbol{g}^{k^\top}_{w,\bar{\delta},b}(x)$ and $\boldsymbol{g}^{*^{k^\top}}_{w,\bar{\delta},b}(x)$.

For $u\ge 0$,

$$\begin{aligned}0=&\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{k^\top}_{w,\bar{\delta},b}(x,b)-(1-\theta)\Lambda'_{\mathbf{11}}\\&\cdot\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{k^\top}_{w,\bar{\delta},b}(x-y,b)\mathrm{d}y\\&-\theta\Lambda'_{\mathbf{11}}\int_0^x\Lambda'_{12}e^{\Lambda'_{22}y}\Lambda'_{21}\boldsymbol{g}^{*^{k^\top}}_{w,\bar{\delta},b}(x-y,b)\mathrm{d}y\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}0=&\left[\left(\frac{\sigma^2}{2}\frac{d^2}{dx^2}+c\frac{d}{dx}-\delta k\right)\mathbf{I}_m+\Lambda'_{11}\right]\boldsymbol{g}^{*^{k^\top}}_{w,\bar{\delta},b}(x,b)-(1-\theta)\Lambda'_{\mathbf{11}}\\&\cdot\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{k^\top}_{w,\bar{\delta},b}(x-y,b)\mathrm{d}y\\&-\theta\Lambda'_{\mathbf{11}}\int_0^x\Lambda''_{12}e^{\Lambda''_{22}y}\Lambda''_{21}\boldsymbol{g}^{*^{k^\top}}_{w,\bar{\delta},b}(x-y,b)\mathrm{d}y,\end{aligned}\tag{4.18}$$

where $\bar{\delta}=\delta k$. We observe now that (4.17) and (4.18) form a homogeneous IDE system and hence similar to Section 3.2.2, it holds that for $0\le x\le b$,

$$\begin{bmatrix}\boldsymbol{g}^{k^\top}_{w,\bar{\delta},b}(x)\\ \boldsymbol{g}^{*^{k^\top}}_{w,\bar{\delta},b}(x)\end{bmatrix}_{2m\times 1}=[V^k(x)]_{2m\times 2m}\begin{bmatrix}\eta^\top_{k,i}(b)\\ \eta^{*^\top}_{k,i}(b)\end{bmatrix}_{i=1}^m. \tag{4.19}$$

Then using the same method in Section 3.2.2, we can solve the equation system.

## 5. The Dividend Penalty Identity

In this section, we develop an identity of dividends-penalty for the risk reserve process (2.6). Denote for $0\le x\le b$, $\vec{\boldsymbol{\phi}}_{d,b}(x)=(\boldsymbol{\phi}^\top_{d,b}(x),\boldsymbol{\phi}^{*^\top}_{d,b}(x))^\top$ and $0\le x\le\infty$, $\vec{\boldsymbol{\phi}}_d(x)=(\boldsymbol{\phi}^\top_d(x),\boldsymbol{\phi}^{*^\top}_d(x))^\top$,

$$\vec{\boldsymbol{\phi}}_d(x)=\begin{bmatrix}\boldsymbol{\phi}^\top_d(x)\\ \boldsymbol{\phi}^{*^\top}_d(x)\end{bmatrix}_{2m\times 1},\quad 0\le x\le\infty,$$

$$\vec{\phi}_{d,b}(x) = \begin{bmatrix} \phi^{\top}_{d,b}(x) \\ \phi^{*\top}_{d,b}(x) \end{bmatrix}_{2m\times 1}, \quad 0 \le x \le b,$$

$\vec{\eta}(b) = (\eta_i^{\top}(b), \eta_i^{*\top}(b))^{\top}_{2m\times 1}$ and $\vec{\eta}_k(b) = (\eta^{\top}_{k.i}(b), \eta^{*\top}_{k,i}(b))^{\top}_{2m\times 1}$. We also consider the representations $\vec{g}_w{}^k(x,b) = (g_w^{k\top}(x,b), g_w^{k^{*\top}}(x,b))^{\top}$. Assume that $V(x)$ and $V^k(x)$ are same matrices with $\bar{\delta} = \delta k$ by replacing $\delta$. We give the dividends-penalty identity for the risk model (2.6) in the following theorem.

**Theorem 9.** *The identity of dividend penalty for the Markovian risk model with possible by claims and dividend barrier is as follows.*

*For* $0 \le x \le b$,

$$[\vec{\phi}_{d,b}(x)]_{2m\times 1} = [\vec{\phi}_d(x)]_{2m\times 1} - [V_w^B(x)]_{2m\times 2m}[V_w^{B'}(b)]^{-1}_{2m\times 2m}[\vec{\phi}'_d(b)]_{2m\times 1}, \tag{5.1}$$

$$[\vec{g}_w{}^1(x,b)]_{2m\times 1} = [V_w^B(x)]_{2m\times 2m}[V_w^{B'}(b)]^{-1}_{2m\times 2m}\cdot\vec{1}, \tag{5.2}$$

*where* $\vec{1}$ *is the* $2m\times 1$ *vector of ones and* $(')$ *represent for first derivative.*

*Proof.* From (3.21) we can represent the solution of the homogeneous IDE system (3.18) and (3.19) as, for $0 \le x \le b$,

$$[\vec{\phi}_{d,b}(x)]_{2m\times 1} = [\vec{\phi}_d(x)]_{2m\times 1} - [V_w^B(x)]_{2m\times 2m}[\vec{\eta}_k(b)]_{2m\times 1}, \tag{5.3}$$

where the vector $\vec{\eta}(b) = \vec{\phi}'_{d,b}(0) - \vec{\phi}'_d(0)$ can be obtained by the boundary condition in Theorem 4,

$$\vec{0} = [\vec{\phi}'_d(b)]_{2m\times 1} + [V_w^B(b)]'_{2m\times 2m}[\vec{\eta}_k(b)]_{2m\times 1}, \tag{5.4}$$

where $\vec{0}$ is the $2m\times 1$ vector of zeros.
From (5.3) and (5.4) we get (5.1).
From (4.19) solution of (4.10) and (4.11) can be expressed as, for $0 \le x \le b$,

$$[\vec{g}_w{}^k(x,b)]_{2m\times 1} = [V^k(x)]_{2m\times 2m}[\vec{\eta}_k(b)]_{2m\times 1}, \tag{5.5}$$

where the vector $\vec{\eta}_k(b) = \vec{g}_w{}^k(0,b)$ can be obtained by the boundary condition in Proposition 8. In fact,

$$[\vec{\eta}_k(b)]_{2m\times 1} = k[V^k(b)]^{-1}_{2m\times 2m}[V^{k-1}(b)]_{2m\times 2m}\vec{\eta}_{k-1}(b), \quad k \ge 1. \tag{5.6}$$

Setting $\bar{\delta} = \delta$ and $m = 1$ in (5.5) and (5.6) yields

$$[\vec{g}_w{}^1(x,b)]_{2m\times 1} = [V_w^B(x)]_{2m\times 2m}[\vec{\eta}_1(b)]_{2m\times 1}, \tag{5.7}$$

$$[\vec{\eta}_1(b)]_{2m\times 1} = [V_w^{B'}(b)]^{-1}_{2m\times 2m}\vec{1}. \tag{5.8}$$

Eqs. (5.7) and (5.8) yields (5.2), hence Theorem 9 proved. □

## 6. Numerical Examples

Numerical illustration of our results is considered here for a two-phase model. The effect of $\theta$, the by-claim probability and $b$, the dividend barrier on the EDDR, $G(u,b)$ is shown here.

Assume that inter-arrival times follow a two-phase MAP with

$$\mathbf{D}_0 = \begin{pmatrix} -0.8 & 0.8 \\ 0 & -0.9 \end{pmatrix} \quad \text{and} \quad \mathbf{D}_1 = \begin{pmatrix} 1 & -1 \\ 0.9 & 0 \end{pmatrix}.$$

The main claims follow $\mathrm{PH}_{n_1}(\boldsymbol{v}_\mathbf{M},\mathbf{H}_\mathbf{M})$ and the by-claims follow $\mathrm{PH}_{n_2}(\boldsymbol{v}_\mathbf{B},\mathbf{H}_\mathbf{B})$ with $\boldsymbol{v}_\mathbf{B} = (0.4, 0.6)$, $\boldsymbol{v}_\mathbf{M} = (0.2, 0.8)$,

$$\mathbf{H}_M = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here $c$ is taken as 1 and $\delta = 0.6$.

In the below graphs x-axis represents $G(u,b)$ and y-axis represents $u$. Here we consider $u$ as the initial surplus.

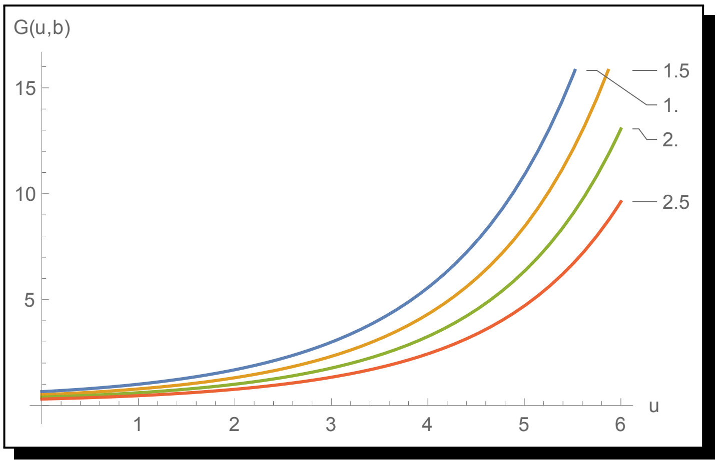


**Figure 1.** $G(u,b)$ with fixed $\theta = 0.2$ and $b = 1, 1.5, 2, 2.5$

In Figure 1 the $G(u,b)$ for $\theta = 0.2$, $u \in [0,4]$ and $b = 1, 1.2, 1.4, 1.6$. As expected, $G(u,b)$ is increasing as the initial surplus $u$ increases. Further, $G(u,b)$ decreases when $b$ increases.

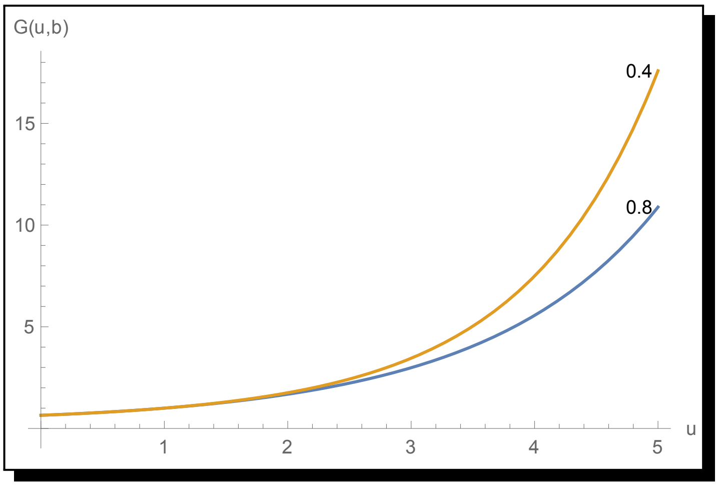


**Figure 2.** $G(u,b)$ with fixed $b = 1$ and $\theta = 0.4, 0.8$

Figure 2 shows the behavior of $G(u,b)$ for $b = 1$, $u \in [0,4]$ and $\theta = 0.4, 0.8$. Here, also $G(u,b)$ increases as the $u$ increases and it decreases as $\theta$ increases for the given $b$.

## 7. Concluding Remarks

MAP/PH risk models are highly valuable in capturing the multi-phase structure commonly found in real-world scenarios. In this study, we expanded upon the significance of these models by incorporating small fluctuations through the addition of a Brownian motion component. Specifically, we analyzed a perturbed MAP/PH risk model that incorporates a constant dividend barrier and two categories of claims: main claims and possible by-claims. Each main claim has the potential to generate a by-claim with a probability denoted as $\theta$. The payment of a by-claim is postponed until the arrival of the next main claim, which is particularly relevant when further investigation is required.

To analyze the model, we employed Markovian fluid queue processes, utilizing both the original timeline and an auxiliary timeline. We developed a system of *integro-differential equations* (IDE) for the *Gerber-Shiu function* (GSF) and the total dividend paid until ruin. Solving this system involves employing Laplace transforms and subsequently inverting them using the Lagrange interpolation formula. The resulting analysis yields explicit expressions for the GSF in both the models without a barrier and with a barrier. Additionally, we presented expressions for the Moment of the total dividends paid until ruin. Moreover, we established a dividend penalty identity that pertains to the investigated risk model.

To illustrate the effectiveness of our method, we provided a numerical demonstration using a two-phase model. Furthermore, we conduct sensitivity analysis by varying essential system parameters, allowing for a comprehensive understanding of the model's behavior and its robustness. The results obtained from this study lay the foundation for further research on a Markovian risk model with possible by-claims, a dividend barrier, and additional elements such as random incomes, capital injections, and taxes. Expanding the scope of investigation to include these factors would provide a more comprehensive understanding of risk dynamics in complex real-world scenarios.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

## References

[1] S. Ahn and A. L. Badescu, On the analysis of the Gerber-Shiu discounted penalty function for risk processes with Markovian arrivals, *Insurance: Mathematics and Economics* **41**(2) (2007), 234 – 249, DOI: 10.1016/j.insmatheco.2006.10.017.

[2] J. Cheng and D. Wang, On a perturbed MAP risk model under a threshold dividend strategy, *Journal of the Korean Statistical Society* **42**(4) (2013), 543 – 564, DOI: 10.1016/j.jkss.2013.03.004.

[3] E. C. K. Cheung and D. Landriault, Perturbed MAP risk models with dividend barrier strategies, *Journal of Applied Probability* **46**(2) (2009), 521 – 541, DOI: 10.1239/jap/1245676104.

[4] A. S. Dibu and M. J. Jacob, Analysis of a MAP risk model with stochastic incomes, inter-dependent phase-type claims and a constant barrier, in: *Applied Probability and Stochastic Processes*, V. Joshua, S. Varadhan and V. Vishnevsky (editors), pp. 235 – 262, Infosys Science Foundation Series, Springer, Singapore (2020), DOI: 10.1007/978-981-15-5951-8_15.

**[5]** F. Dufresne and H. U. Gerber, Risk theory for the compound Poisson process that is perturbed by diffusion, *Insurance: Mathematics and Economics* **10**(1) (1991), 51 – 59, DOI: 10.1016/0167-6687(91)90023-Q.

**[6]** D. Dufresne, *A General Class of Risk Models*, Centre for Actuarial Studies, Department of Economics, University of Melbourne, Australia, 1 – 24, (2002), URL: http://hdl.handle.net/11343/33696.

**[7]** B. De Finetti, Su un'impostazione alternativa della teoria collettiva del rischio, in: *Transactions of the XVth International Congress of Actuaries*, Vol. 2 (1) (1957), pp. 433 – 443.

**[8]** V. Lakshmikantham and M. R. M. Rao, *Theory of Integro-Differential Equations*, Gordon and Breach Science Publishers, Switzerland, x + 175 pages (1995).

**[9]** G. Latouche and V. Ramaswami, *Introduction to Matrix Analytic Methods in Stochastic Modelling*, ASA-SIAM Series on Statistics and Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA, xiv + 334 pages 1999.

**[10]** X. S. Lin, G. E. Willmot and S. Drekic, The classical risk model with a constant dividend barrier: Analysis of the Gerber-Shiu discounted penalty function, *Insurance: Mathematics and Economics* **33**(3) (2003), 551 – 566, DOI: 10.1016/j.insmatheco.2003.08.004

**[11]** M. F. Neuts, A versatile Markovian point process, *Journal of Applied Probability* **16**(4) (1979), 764 – 779, DOI: 10.2307/3213143.

**[12]** V. Ramaswami, Passage times in fluid models with application to risk processes, *Methodology and Computing in Applied Probability* **8**(4) (2006), 497 – 515, DOI: 10.1007/s11009-006-0426-9.

**[13]** J. Ren, L. Breuer, D. A. Stanford and K. Yu, Perturbed risk processes analyzed as fluid flows, *Stochastic Models* **25**(3) (2009), 522 – 544, DOI: 10.1080/15326340903088867.

**[14]** P. P. Sreeshamim, M. J. Jacob and A. S. Dibu, A Markovian risk model with possible by-claims and dividend barrier, *International Journal of Mathematics in Operational Research* **26**(4) (2022), 475 – 501, DOI: 10.1504/IJMOR.2022.10052301.

**[15]** H. R. Waters and A. Papatriandafylou, Ruin probabilities allowing for delay in claims settlement, *Insurance: Mathematics and Economics* **4**(2) (1985), 113 – 122, DOI: 10.1016/0167-6687(85)90005-8.

**[16]** J.-H. Xie and W. Zou, On the expected discounted penalty function for the compound Poisson risk model with delayed claims, *Journal of Computational and Applied Mathematics* **235**(8) (2011), 2392 – 2404, DOI: 10.1016/j.cam.2010.10.039.

**[17]** J.-H. Xie, J.-W. Gao and W. Zou, On a risk model with delayed claims under stochastic interest rates, *Communications in Statistics - Theory and Methods* **44**(4) (2015), 3022 – 3041, DOI: 10.1080/03610926.2013.799698.

**[18]** W. Zou and J. Xie, A risk process with delayed claims and constant dividend barrier, *Theory of Probability and Its Applications* **64**(1) (2019), 103 – 123, DOI: 10.1137/S0040585X97T989416.