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An Explicit Isomorphism in $\mathbb{R}/\mathbb{Z}$-K-Homology

Research Article

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Abstract. In this paper, we construct an explicit isomorphism between the flat part of differential K-homology and the Deeley $\mathbb{R}/\mathbb{Z}$-K-homology.

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1. Introduction

K-homology is the homology theory dual to topological K-theory.

A geometric model for K-homology was introduced by Baum-Douglas (see [1]), and proved to be an extremely important tool in index theory and physics (see [5]). Motivated by generalizing the pairings between K-theory and K-homology to the case of $\mathbb{R}/\mathbb{Z}$-coefficients, Deeley defined in [2] a model for geometric K-homology with $\mathbb{R}/\mathbb{Z}$-coefficients using approach of operators algebras.

Let X be a finite CW-complex and N be a II_1-factor. A cycle in the Deeley $\mathbb{R}/\mathbb{Z}$-K-homology (which we call $\mathbb{R}/\mathbb{Z}$-K-cycle) over X is a triple $(W,(H,\varepsilon,\alpha)^{(\nabla^H,\nabla^\varepsilon)},g)$ where W is a smooth compact $Spin^c$-manifold, H is a fiber bundle over W with fibers are finitely generated projective Hermitian Hilbert N-modules, with a Hermitian connection ∇^H, ε is a Hermitian vector bundle over ∂W with a Hermitian connection ∇^ε, α is an isomorphism from $H|_{\partial W}$ to $\varepsilon\otimes\mathsf{N}$, and $g:W\to X$ is a continuous map. The Deeley $\mathbb{R}/\mathbb{Z}$-K-homology group $K_*(X,\mathbb{R}/\mathbb{Z})$ is the quotient of the set of isomorphism classes of $\mathbb{R}/\mathbb{Z}$-K-cycles over X by the equivalence relation generated by bordism and vector bundle modification (Definition 3.6).

On the other hand, we defined in [3] the differential K-homology group $\check{K}_*(X)$ of a smooth compact manifold X. A cycle in $\check{K}_*(X)$ is called a differential K-cycle over X and consisting of a pair $((M,E^{\nabla^E},f),\phi)$ of a cycle of Baum-Douglas (M,E^{∇^E},f) over X and a class of currents $\phi \in \frac{\Omega_*(X)}{\mathrm{img}(\partial)}$. A flat differential K-cycle is a differential K-cycle (M,E^{∇^E},f,ϕ) such that $\partial\phi = \int_M Td(\nabla^M)ch(\nabla^E)f^*$.

The flat differential K-homology group $\check{K}^f_*(X)$ is the subgroup of $\check{K}_*(X)$ consisting of classes of flat differential K-cycles over X, and then fits into the exact sequence

$$0 \longrightarrow \check{K}^f_*(X) \hookrightarrow \check{K}_*(X) \longrightarrow \Omega^0_*(X) \longrightarrow 0 \ ,$$

where $\Omega^0_*(X)$ denotes the group of closed continuous currents whose de Rham homology class lies in the image of the geometric Chern character.

In this paper we show that the groups $K_*(X,\mathbb{R}/\mathbb{Z})$ and $\check{K}^f_{*-1}(X)$ are isomorphic.

2. The Functor $\check{K}^f$

For the benefit of the reader, we recall the construction of flat K-homology groups defined in [3]. Let E be a smooth Hermitian vector bundle over a smooth compact manifold M with a Hermitian connection ∇. The Chern character form of ∇ is given by

$$ch(\nabla) := \mathrm{Tr}\left(e^{\frac{-\nabla^2}{2i\pi}}\right).$$

It is a closed real-valued form on M, and then defines a class in the de Rham cohomology of M. Let $ch_k(\nabla) \in \Omega^{2k}(M,\mathbb{R})$ with $ch_k(\nabla) := \frac{1}{k!}\mathrm{Tr}\left(\left(\frac{-\nabla^2}{2i\pi}\right)^k\right)$. It is obvious that

$$ch(\nabla) = \sum_{k\geq 0} ch_k(\nabla).$$

If ∇_1 and ∇_2 are two Hermitian connections on E, there is a canonically-defined Chern-Simons class $CS(\nabla_1,\nabla_2) \in \frac{\Omega^{\mathrm{odd}}(M)}{\mathrm{img}(d)}$ (see [4]) such that

$$dCS(\nabla_1,\nabla_2) = ch(\nabla_1) - ch(\nabla_2).$$

If M is an n-dimensional smooth $Spin^c$-manifold and ∇^M is the Levi-Civita connection on M, the todd form of ∇^M is the closed form defined by

$$Td(\nabla^M) := \sqrt{\det\left(\frac{\frac{\nabla^{M2}}{2}}{\sinh\left(\frac{\nabla^{M2}}{2}\right)}\right)} \wedge e^{ch_1(\nabla^L)},$$

where L is the Hermitian line bundle associated with the $Spin^c$ structure on M and ∇^L is the induced Hermitian connection on L.

In all the following, we denote by X a smooth compact manifold.

Definition 2.1. A flat differential K-cycle over X is a quadruple $(M, E^{\nabla^E}, f, \phi)$ consisting of:

- A smooth closed $Spin^c$-manifold M;
- A smooth Hermitian vector bundle E over M with a Hermitian connection ∇^E;
- A smooth map $f : M \to X$;
- A de Rham homology class of continuous currents $\phi \in \frac{\Omega_*(X)}{\mathrm{img}(\partial)}$ with $\partial\phi = \int_M Td(\nabla^M) ch(\nabla^E) f^*$.

There are no connectedness requirements made upon M, and hence the bundle E can have different fibre dimensions on the different connected components of M. It follows that the disjoint union,

$$(M, E^{\nabla^E}, f, \phi) \sqcup (M', E'^{\nabla^{E'}}, f', \phi') := (M \sqcup M', E \sqcup E'^{\nabla^E \sqcup \nabla^{E'}}, f \sqcup f', \phi + \phi'),$$

is a well-defined operation on the set of flat differential K-cycles over X.

A flat differential K-cycle $(M, E^{\nabla^E}, f, \phi)$ is called even (resp. odd), if all connected components of M are of even (resp. odd) dimension and $\phi \in \frac{\Omega_{\mathrm{odd}}(X)}{\mathrm{img}(\partial)}$ (resp. $\phi \in \frac{\Omega_{\mathrm{even}}(X)}{\mathrm{img}(\partial)}$).

There are several kinds of relations involving flat differential K-cycles.

Definition 2.2 (Isomorphism)**.** Two flat differential K-cycles $(M, E^{\nabla^E}, f, \phi)$ and $(M', E'^{\nabla^{E'}}, f', \phi')$ over X are *isomorphic* if there exists a diffeomorphism $h : M \to M'$ such that

- h preserves the $Spin^c$-structures;
- $h^* E' \cong E$;
- the diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ X & & \end{array}$$

  commutes;
- $\phi - \phi' = \left[\int_{M \times [0,1]} Td(\nabla^{M \times [0,1]}) ch(B) (f \circ p)^* \right]$ where B is the connection on the pullback of E by the natural projection $p : M \times [0,1] \to M$ given by $B = (1-t)\nabla^E + t h^* \nabla^{E'} + dt \frac{d}{dt}$.

The semigroup for the disjoint union of isomorphism classes of flat differential K-cycles over X will be denoted by $C_*(X)$.

Definition 2.3 (Bordism)**.** Two flat differential K-cycles $(M, E^{\nabla^E}, f, \phi)$ and $(M', E'^{\nabla^{E'}}, f', \phi')$ over X are said to be *bordant* if there exist a smooth compact $Spin^c$-manifold W, a smooth Hermitian vector bundle ε over W, and a smooth map $g : W \to X$ such that

$$(M \sqcup M'^-, E \sqcup E'^{\nabla^E \sqcup \nabla^{E'}}, f \sqcup f') = (\partial W, \varepsilon|_{\partial W}^{\nabla^\varepsilon|_{\partial W}}, g|_{\partial W})$$

and

$$\phi - \phi' = \left[\int_W Td(\nabla^W) ch(\nabla^\varepsilon) g^* \right],$$

where M'^- denotes M' with its $Spin^c$-structure reversed (see [1]).

Let $(M, E^{\nabla^E}, f, \phi)$ be a flat differential K-cycle over X and V be a $Spin^c$-vector bundle of even rank over M with an Euclidean connection ∇^V. Let 1_M denote the trivial rank-one real vector bundle over M. The direct sum $V \oplus 1_M$ is a $Spin^c$-vector bundle, and moreover the total space of this bundle may be equipped with a $Spin^c$-structure in a canonical way. This is because its tangent bundle fits into an exact sequence

$$0 \to \pi^*[V \oplus 1_M] \to T(V \oplus 1_M) \to \pi^*[TM] \to 0$$

where π is the projection from $V \oplus 1_M$ onto M.
Let us now denote by $\hat{M}$ the unit sphere bundle of the bundle $V \oplus 1_M$. Since $\hat{M}$ is the boundary of the disk bundle, we may equip it with a natural $Spin^c$-structure by first restricting the given $Spin^c$-structure on the total space of $V \oplus 1_M$ to the disk bundle, and then taking the boundary of this $Spin^c$-structure to obtain a $Spin^c$-structure on the sphere bundle.
Denote by $S := S_- \oplus S_+$ the $\mathbb{Z}_2$-graded spinor bundle associated with the $Spin^c$-structure on the vertical tangent bundle of $\hat{M}$ carring with a Hermitian connection $\nabla^S := \nabla^{S_-} \oplus \nabla^{S_+}$ induced by ∇^V. Define $\hat{V}$ to be the dual of S_+ and $\nabla^{\hat{V}}$ to be the Hermitian connection on $\hat{V}$ induced by ∇^{S_+}. We obtain that the quadruple $(\hat{M}, \hat{V} \otimes \pi^* E^{\nabla^{\hat{V}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi)$ is a flat differential K-cycle over X.

Definition 2.4 (Vector bundle modification)**.** *The modification* of a flat differential K-cycle $(M, E^{\nabla^E}, f, \phi)$ associated to a $Spin^c$-vector bundle V of even rank over M carring with an Euclidean connection ∇^V is the flat differential K-cycle

$$(\hat{M}, \hat{V} \otimes \pi^* E^{\nabla^{\hat{V}} \otimes \pi^* \nabla^E}, f \circ \pi, \phi).$$

We are now ready to define the flat differential K-homology group $\check{K}^f_*(X)$.

Definition 2.5. *The flat differential K-homology group* $\check{K}^f_*(X)$ is the quotient of $C_*(X)$ by the equivalence relation ~ generated by

(i) *direct sum:* $(M, E^{\nabla^E}, f, \phi) \sqcup (M, E'^{\nabla^{E'}}, f, \phi') \sim (M, E \oplus E'^{\nabla^E \oplus \nabla^{E'}}, f, \phi + \phi')$;
(ii) *bordism*;
(iii) *vector bundle modification*.

The class of a flat differential K-cycle $(M, E^{\nabla^E}, f, \phi)$ in $\check{K}^f_*(X)$ will be denoted by $[M, E^{\nabla^E}, f, \phi]$. The neutral element of $\check{K}^f_*(X)$ is $[\emptyset, \emptyset, \emptyset, 0]$, and the inverse of a class $[M, E^{\nabla^E}, f, \phi]$ $(\in \check{K}^f_*(X))$ is $[M^-, E^{\nabla^E}, f, -\phi]$.
Since the equivalence relation ~ preserves the parity of flat differential K-cycles, this gives a $\mathbb{Z}_2$-gradation of $\check{K}^f_*(X)$:

$$\check{K}^f_*(X) = \check{K}^f_{\text{even}}(X) \oplus \check{K}^f_{\text{odd}}(X),$$

where $\check{K}^f_{\text{even}}(X)$ (resp. $\check{K}^f_{\text{odd}}(X)$) is the subgroup of $\check{K}^f_*(X)$ consisting of classes of even (resp. odd) flat differential K-cycles over X.

3. The Deeley Model for $\mathbb{R}/\mathbb{Z}$-K-Homology

In this section we recall the Deeley construction of a model for $\mathbb{R}/\mathbb{Z}$-K-homology (see [2]).
In all the following, we denote by N a II_1-factor and τ a faithful normal trace on N.

Definition 3.1. An $\mathbb{R}/\mathbb{Z}$-K-cycle over X is a triple $(W,(H,\varepsilon,\alpha)^{(\nabla^H,\nabla^\varepsilon)},g)$, where

- W is a smooth compact $Spin^c$-manifold;
- H is a fiber bundle over W with fibers are finitely generated projective Hermitian Hilbert N-modules, with a Hermitian connection ∇^H;
- ε is a Hermitian vector bundle over ∂W with a Hermitian connection ∇^ε;
- α is an isomorphism from $H|_{\partial W}$ to $\varepsilon\otimes\mathsf{N}$;
- $g: W\to X$ is a smooth map.

An $\mathbb{R}/\mathbb{Z}$-K-cycle $(W,(H,\varepsilon,\alpha)^{(\nabla^H,\nabla^\varepsilon)},g)$ is called even (resp. odd), if all connected components of W are of even (resp. odd) dimension.
The addition operation on the set of $\mathbb{R}/\mathbb{Z}$-K-cycles is defined using disjoint union operation.
Two $\mathbb{R}/\mathbb{Z}$-K-cycles over X are isomorphic if there are compatible isomorphisms of all of the above three components in the definition of $\mathbb{R}/\mathbb{Z}$-K-cycle.
The semigroup of isomorphism classes of $\mathbb{R}/\mathbb{Z}$-K-cycles over X will be denoted by $\Gamma_*(X)$.

Definition 3.2. A bordism of $\mathbb{R}/\mathbb{Z}$-K-cycles over X consists of the following data:

- Z is a smooth compact $Spin^c$-manifold;
- $W\subseteq\partial Z$ is a regular domain;
- V is a fiber bundle over Z with fibers are finitely generated projective Hermitian Hilbert N-modules, with a Hermitian connection ∇^V, and ϑ is a Hermitian vector bundle over $\partial Z - int(W)$ with a Hermitian connection ∇^ϑ such that $V|_{\partial Z-int(W)}\overset{\beta}{\cong}\vartheta\otimes\mathsf{N}$;
- $h: Z\to X$ is a smooth map.

Here, a regular domain W of ∂Z means a closed submanifold of ∂Z such that $int(W)\neq\emptyset$ and if $x\in\partial W$, then there exists a coordinate chart $\psi: U\to\mathbb{R}^n$ centred at x with $\psi(W\cap U)=\{(y_i)\in\psi(U)\mid y_n\geq 0\}$.
The boundary of a bordism $(Z,W,(V,\vartheta,\beta)^{(\nabla^V,\nabla^\vartheta)},h)$ is the $\mathbb{R}/\mathbb{Z}$-K-cycle

$$\partial(Z,W,(V,\vartheta,\beta)^{(\nabla^V,\nabla^\vartheta)},h):=(W,(V|_W,\vartheta|_{\partial W},\beta)^{\nabla^{V|_W},\nabla^{\vartheta|_{\partial W}}},h|_W).$$

Remark 3.3. If $(Z,W,(V,\vartheta,\beta)^{(\nabla^V,\nabla^\vartheta)},h)$ is a bordism, then $(\partial Z - int(W),\vartheta^{\nabla^\vartheta},h|_{\partial Z-int(W)})$ is a chain of Baum-Douglas with boundary $(\partial W,\vartheta|_{\partial W}^{\nabla^{\vartheta|_{\partial W}}},h|_{\partial W})$.

Definition 3.4. Two $\mathbb{R}/\mathbb{Z}$-K-cycles $(W_0,(H_0,\varepsilon_0,\alpha_0)^{(\nabla^{H_0},\nabla^{\varepsilon_0})},g_0)$ and $(W_1,(H_1,\varepsilon_1,\alpha_1)^{(\nabla^{H_1},\nabla^{\varepsilon_1})},g_1)$ are bordant if there exists a bordism ζ such that $(W_0,(H_0,\varepsilon_0,\alpha_0)^{(\nabla^{H_0},\nabla^{\varepsilon_0})},g_0)\sqcup$ $(W_1^-,(H_1,\varepsilon_1,\alpha_1)^{(\nabla^{H_1},\nabla^{\varepsilon_1})},g_1)$ is isomorphic to $\partial\zeta$.

Remark 3.5. If (M,E^{∇^E},f) is a cycle of Baum-Douglas over X, then its associated $\mathbb{R}/\mathbb{Z}$-K-cycle $(M,(E\otimes\mathsf{N},\emptyset,\emptyset)^{(\nabla^E,\emptyset)},f)$ is bordant to the trivial $\mathbb{R}/\mathbb{Z}$-K-cycle, where a bordism is given by $(M\times[0,1],M,(p_M^*E\otimes\mathsf{N},E)^{(p_M^*\nabla^E,\nabla^E)},f\circ p_M)$ with $p_M: M\times[0,1]\to M$ is the natural projection.

The vector bundle modification of an $\mathbb{R}/\mathbb{Z}$-K-cycle can be defined in the same way as the vector bundle modification of a flat differential K-cycle.

Definition 3.6. The Deeley $\mathbb{R}/\mathbb{Z}$-K-homology group $K_*(X,\mathbb{R}/\mathbb{Z})$ is the quotient of $\Gamma_*(X)$ by the equivalence relation generated by bordism and vector bundle modification.

$K_*(X,\mathbb{R}/\mathbb{Z})$ is $\mathbb{Z}_2$-graded by the parity of $\mathbb{R}/\mathbb{Z}$-K-cycles.
Note that if X is a smooth compact *Spin*-manifold, the group $K_*(X,\mathbb{R}/\mathbb{Z})$ is isomorphic to the Kasparov group $KK^{*-1}(C(X),\mathscr{C}_i)$ where $\mathscr{C}_i$ is the mapping cone of the inclusion $i:\mathbb{C}\hookrightarrow \mathsf{N}$ ([2, Theorem 3.10] together with [2, Theorem 5.2]).

4. The Isomorphism $K_*(X,\mathbb{R}/\mathbb{Z})\cong \check{K}^f_{*-1}(X)$

Recall that the geometric K-homology group of X is denoted by $K^{\text{geo}}_*(X)$.
Following the exact sequence in [3, p. 7] together with the fact that the geometric Chern character $Ch_*:K^{\text{geo}}_*(X)\to H^{dR}_*(X)$ is rationally injective, $\check{K}^f_*(X)$ fits into the exact sequence

$$0\to \frac{H^{dR}_{*+1}(X)}{\operatorname{img}(Ch_*)}\xrightarrow{a}\check{K}^f_*(X)\xrightarrow{i}\mathscr{T}(K^{\text{geo}}_*(X))\to 0$$

where $\mathscr{T}(K^{\text{geo}}_*(X))$ is the torsion subgroup of $K^{\text{geo}}_*(X)$, i is the forgetful map, and a is the map which associates to each $\phi\in H^{dR}_{*+1}(X)$ the class $[\emptyset,\emptyset,\emptyset,\phi]\in\check{K}^f_*(X)$.
Now, note that from [2] and [6], an element in the Kasparov's group $KK^*(C(X),\mathsf{N})$ can be described by a geometric cycle of the form (M,H^{∇^H},f) where M is a smooth closed $Spin^c$-manifold, H is a fiber bundle over M with fibers are finitely generated projective Hermitian Hilbert N-modules, with a Hermitian connection ∇^H, and $f:M\to X$ is a smooth map. $KK^*(C(X),\mathsf{N})$ is a model for the real K-homology of X; an isomorphism between $K^{\text{geo}}_*(X)\otimes\mathbb{R}$ and $KK^*(C(X),\mathsf{N})$ is given at level of cycles by

$$\nu((M,E^{\nabla^E},f),t):=[M,E\otimes p_t\mathsf{N}^{n\nabla^E},f],$$

where $p_t\in M_n(\mathsf{N})$ is a projection with $\tau(p_t)=t$.
Define a homomorphism $Ch_{\tau,*}:KK^*(C(X),\mathsf{N})\to H^{dR}_*(X,\mathbb{R})$ by setting

$$Ch_{\tau,*}[M,H^{\nabla^H},f]:=\left[\int_M Td(\nabla^M)ch_\tau(\nabla^H)f^*\right],$$

where $ch_\tau(\nabla^H):=(\tau\otimes\operatorname{Tr})(e^{\frac{-\nabla^{H^2}}{2i\pi}})\in\Omega^{2*}(X,\mathbb{R})$. It fits into the commutative diagram

$$\begin{array}{ccc} K^{\text{geo}}_*(X)\otimes\mathbb{R} & & \\ {\scriptstyle Ch^{\mathbb{R}}_*}\downarrow\cong & \overset{\nu}{\searrow} & \\ H^{dR}_*(X,\mathbb{R}) & \xleftarrow{Ch_{\tau,*}} & KK^*(C(X),\mathsf{N}) \end{array}$$

where $Ch^{\mathbb{R}}_*:K^{\text{geo}}_*(X)\otimes\mathbb{R}\to H^{dR}_*(X,\mathbb{R})$ is the Chern character.

Denote by $\delta' : KK^*(C(X),\mathsf{N}) \to K_*(X,\mathbb{R}/\mathbb{Z})$ the homomorphism given at the level of N-K-cycles by

$$\delta'(M,H^{\nabla^H},f) := [M,(H,\emptyset,\emptyset)^{(\nabla^H,\emptyset)},f],$$

and $\delta = \delta' \circ \nu : K_*^{\mathrm{geo}}(X) \otimes \mathbb{R} \to K_*(X,\mathbb{R}/\mathbb{Z})$. Let $\mu : K_*^{\mathrm{geo}}(X) \to K_*^{\mathrm{geo}}(X) \otimes \mathbb{R}$ be the homomorphism given by

$$\mu[M,E^{\nabla^E},f] := ([M,E^{\nabla^E},f],1).$$

By Remark 3.5, δ induces a well-defined homomorphism from $coker(\mu)$ to $K_*(X,\mathbb{R}/\mathbb{Z})$.

Theorem 4.1. The groups $K_*(X,\mathbb{R}/\mathbb{Z})$ and $\check{K}^f_{*-1}(X)$ are isomorphic.

To prove the theorem, we need the following lemma:

Lemma 4.2. The following sequence is exact:

$$0 \to coker(\mu) \xrightarrow{\delta} K_*(X,\mathbb{R}/\mathbb{Z}) \xrightarrow{\partial} \mathscr{T}(K^{\mathrm{geo}}_{*-1}(X)) \to 0,$$

where the map ∂ sends an $\mathbb{R}/\mathbb{Z}$-K-cycle $(W,(H,\varepsilon,\alpha)^{(\nabla^H,\nabla^\varepsilon)},g)$ to $(\partial W,\varepsilon^{\nabla^\varepsilon},g|_{\partial W})$.

Proof of Lemma 4.2. It is clear that ∂ is compatible with the relation of vector bundle modification. Compatibility with the relation of bordism follows from Remark 3.3.

Surjectivity of ∂. For $[M,E^{\nabla^E},f] \in \mathscr{T}(K_*^{\mathrm{geo}}(X))$, there exist a positive integer k and a chain of Baum-Douglas $(W,\vartheta^{\nabla^\vartheta},g)$ over X such that

$$(M,kE^{k\nabla^E},f) \overset{h}{\cong} (\partial W,\vartheta|_{\partial W}^{\nabla^\vartheta|_{\partial W}},g|_{\partial W}).$$

If we denote by $\alpha : \partial W \to M$ and $\beta : \vartheta|_{\partial W} \to k\alpha^*E$ the isomorphisms induced by h, then $(W,(\vartheta \otimes \mathsf{N},\alpha^*E,\beta \otimes 1)^{(\nabla^\vartheta,\alpha^*\nabla^E)},g)$ is an $\mathbb{R}/\mathbb{Z}$-K-cycle over X such that

$$\vartheta|_{\partial W} \otimes \mathsf{N} \overset{\beta\otimes 1}{\cong} \alpha^*E \otimes k\mathsf{N} \cong \alpha^*E \otimes \mathsf{N},$$

and satisfies

$$[\partial(W,(\vartheta \otimes \mathsf{N},\alpha^*E,\beta \otimes 1)^{(\nabla^\vartheta,\alpha^*\nabla^E)},g)] = 0 = [M,E^{\nabla^E},f].$$

Injectivity of δ. Let (M,E^{∇^E},f) be a cycle of Baum-Douglas over X and $t \in \mathbb{R}$ such that $\delta([M,E^{\nabla^E},f],t)$ is the trivial element. There exists a bordism $(Z,W,(V,\vartheta,\beta)^{(\nabla^V,\nabla^\vartheta)},h)$ over X such that:

$$\begin{aligned}\partial(Z,W,(V,\vartheta,\beta)^{(\nabla^V,\nabla^\vartheta)},h) &:= (W,(V|_W,\vartheta|_{\partial W},\beta)^{(\nabla^{V|_W},\nabla^{\vartheta|_{\partial W}})},h|_W)\\ &= (M,(E \otimes p_t\mathsf{N}^n,\emptyset,\emptyset)^{(\nabla^E,\emptyset)},f).\end{aligned}$$

Since

$$(\partial Z,V|_{\partial Z}{}^{\nabla^V|_{\partial Z}},h|_{\partial Z}) = (\partial Z - W,\vartheta \otimes \mathsf{N}^{\nabla^\vartheta},h|_{\partial Z-W}) \sqcup (W,V|_W{}^{\nabla^V|_W},h|_W),$$

(Z, V^{∇^V}, h) is a bordism in $KK^*(C(X), \mathsf{N})$ between the N-K-cycles $\nu((\partial Z - W, \vartheta^{\nabla^\vartheta}, g|_{\partial Z - W}), 1)$ and $\nu((M^-, E^{\nabla^E}, f), t)$. It follows that

$$(-[M, E^{\nabla^E}, f], t) = \mu([\partial Z - W, \vartheta^{\nabla^\vartheta}, h|_{\partial Z - W}]),$$

and then $([M, E^{\nabla^E}, f], t)$ determines the zero element in $coker(\mu)$.

In view of cycles of Baum-Douglas are without boundaries, the composition $\partial \circ \delta$ is zero.

It remains to show that $\mathrm{Ker}(\partial) \subseteq \mathrm{Img}(\delta)$. Let $(W, (H, \varepsilon, \alpha)^{(\nabla^H, \nabla^\varepsilon)}, g)$ be an $\mathbb{R}/\mathbb{Z}$-K-cycle over X with $(\partial W, \varepsilon^{\nabla^\varepsilon}, g|_{\partial W})$ is the boundary of a chain of Baum-Douglas (Z, F^{∇^F}, h). Form the closed smooth $Spin^c$ manifold $\widetilde{W} := W \cup_{\partial W \cong \partial Z} Z$. Denote that the fiber bundles and differentiable maps are compatible with the isomorphism $\partial W \cong \partial Z$. Hence, we can form the N-K-cycle $(\widetilde{W}, V^{\nabla^V}, j)$ with

$$V = H \cup_{\partial W \cong \partial Z} (F \otimes \mathsf{N}), \quad \nabla^V = \nabla^H \cup_{\partial W \cong \partial Z} \nabla^F$$

and

$$j = g \cup_{\partial W \cong \partial Z} h.$$

It determines a class in the KK-group $KK^*(C(X), \mathsf{N})$. We first show that there exists a bordism between $\delta'(\widetilde{W}, V^{\nabla^V}, j)$ and $(W, (H, \varepsilon, \alpha)^{(\nabla^H, \nabla^\varepsilon)}, g)$. This is given by the following quadruple

$$(\widetilde{W} \times [0,1], \widetilde{W} \sqcup W, (p^*V, F)^{(p^*\nabla^V, \nabla^F)}, j \circ p),$$

where $p : \widetilde{W} \times [0,1] \to \widetilde{W}$ is the natural projection.

Since $KK^*(C(X), \mathsf{N}) \cong K_*^{\mathrm{geo}}(X) \otimes \mathbb{R}$ and from the definition of δ, there exist $[M, E^{\nabla^E}, f] \in K_*^{\mathrm{geo}}(X)$ and $t \in \mathbb{R}$ such that

$$\begin{aligned}\delta([M, E^{\nabla^E}, f], t) &= \delta'[M, E \otimes p_t \mathsf{N}^{n\,\nabla^E}, f] \\ &= \delta'[\widetilde{W}, V^{\nabla^V}, j] \\ &= [W, (H, \varepsilon, \alpha)^{(\nabla^H, \nabla^\varepsilon)}, g].\end{aligned}$$

□

Proof of Theorem 4.1. Using Remark 3.3, the Atiyah-Singer index theorem on even spheres and the commutative diagram in page 9 relating $Ch_{\tau,*}$ and $Ch_*^{\mathbb{R}}$, we obtain that the map $\gamma : K_*(X, \mathbb{R}/\mathbb{Z}) \to \check{K}^f_{*-1}(X)$ given by

$$\gamma[W, (H, \varepsilon, \alpha)^{(\nabla^H, \nabla^\varepsilon)}, g] := \left[\partial W, \varepsilon^{\nabla^\varepsilon}, g|_{\partial W}, \left[\int_W Td(\nabla^W) ch_\tau(\nabla^H) g^*\right]\right]$$

is a well-defined homomorphism. The theorem results from the commutativity of the following diagram together with the five-lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & coker(\mu) & \xrightarrow{\delta} & K_*(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\partial} & \mathscr{T}(K^{\mathrm{geo}}_{*-1}(X)) & \longrightarrow & 0 \\ & & \chi \downarrow & \circlearrowleft & \downarrow \gamma & \circlearrowleft & \| & & \\ 0 & \longrightarrow & \dfrac{H^{dR}_*(X)}{\mathrm{img}(Ch_*)} & \xrightarrow{a} & \check{K}^f_{*-1}(X) & \xrightarrow{i} & \mathscr{T}(K^{\mathrm{geo}}_{*-1}(X)) & \longrightarrow & 0 \end{array}$$

where χ is the homomorphism induced by $Ch_*^{\mathbb{R}}$, which is obviously an isomorphism.
It is evident that $i \circ \gamma = \partial$. It remains to show that $\gamma \circ \delta = a \circ \chi$.
Let $[M, E^{\nabla^E}, f] \in K_*^{\text{geo}}(X)$ and $t \in \mathbb{R}$. We have

$$\begin{aligned}\gamma(\delta([M,E^{\nabla^E},f],t)) &= \gamma([M,(E \otimes p_t \mathsf{N}^n, \emptyset, \emptyset)^{(\nabla^E,\emptyset)}, f]) \\ &= \left[\emptyset,\emptyset,\emptyset, \left[\int_M Td(\nabla^M) ch_\tau(\nabla^{E\otimes p_t \mathsf{N}}) f^*\right]\right] \\ &= \left[\emptyset,\emptyset,\emptyset, \left[\tau(p_t)\int_M Td(\nabla^M) ch(\nabla^E) f^*\right]\right] \\ &= a(\chi([M,E^{\nabla^E},f],t)).\end{aligned}$$

This finishes the proof. □

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