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Research Article

The Regular Domination Number of Some Special Graphs

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Abstract. The purpose of this article is to illustrate the concept of regular domination on a variety of unique graph types, including complete graphs, path graphs, cycle graphs, lollipop graphs, barbell graphs, gear graphs, Petersen graphs, helm graphs, jellyfish graphs, jewel graphs, and complete bipartite graphs. We also determine the regular domination for specific operations, such as the join of two graphs and the corona product of two graphs.

Keywords. Domination, Regular domination, Regular domination number, Regular dominating set

Mathematics Subject Classification (2020). 05C69, 05C76

1. Introduction

Let G be an undirected, loop-free, and parallel edge-free graph. Let V_G and E_G stand for the collection of vertices and edges in a graph. The coefficient of external stability was the term Berge used in 1958 to refer to the domination number [2]. The term domination for undirected graphs was first used in 1962 by Ore [13], who also developed the notions of minimal and minimum dominating set of vertices in graphs.

The collection of vertices in G that are next to any vertex u in V_G is known as the open neighbourhood of u and mathematically, $N(u) = \{w \in V_G : uw \in E_G\}$. The closed neighbourhood of u is the set $N[u] = N(u) \cup \{u\}$.

In the case of a set $T \subseteq V_G$, the open neighbourhood of T is equal to the union of the open neighbourhoods of all the vertices that belong to T, i.e., $N(T) = \bigcup_{u \in T} N(u)$, and the closed neighbourhood of T is equal to $N[T] = N(T) \cup T$.

If $D \subseteq V_G$, then D is referred to as a dominant set of G since every vertex of $V - D$ is adjacent to at least one vertex of D. The number of vertices in a graph G's lowest minimal dominating set, D, is known as the graph G's domination number and is denoted by the symbol $\gamma(G)$ (Haynes *et al*. [7]).

If D is a dominating set of a graph G and each vertex in D has the same degree, then D is said to be a regular dominating set of G. Regular domination Number $\gamma_R(G)$ of graph G is defined as the minimum among all regular dominating sets.

In 2021, Prabakaran *et al*. [14] described *regular dominating set* (*RDS*) and regular dominating number $\gamma_R(G)$ in fuzzy graph and studied various properties and bounds of regular domination number in several fuzzy graphs. Inspiring by this idea, we assess the regular domination number of some simple, connected, and undirected graphs as well as the join and corona of two graphs.

2. Definitions and Preliminaries

Definition 2.1. Let G be a simple graph, a set $R \subseteq V_G$ is supposed to be a *regular dominating set* (*RDS*) of G if:

(i) each vertex $u \in V_G - R$ is adjacent to some vertex in R;

(ii) each vertex in $R \subseteq V_G$ has the same degree.

Example 2.2. Let G_1 be a graph as shown in Figure 1.

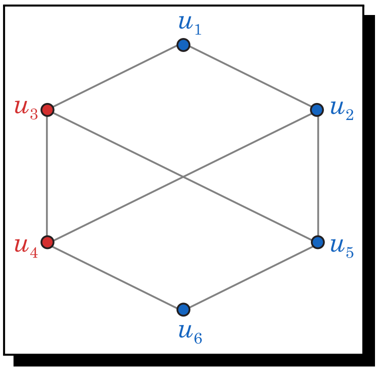

Figure 1. Graph G_1

If we consider R as $\{u_3, u_4\} \subseteq V_{G_1}$ and get $|N(u) \cap R| \geq 1$ for $u \in R$ and $\deg(u_3) = \deg(u_4)$, it is implied that R is a regular dominating set of graph G_1. Let $S = \{u_1, u_2, u_5\}$ be the subset of V_{G_1}. Due to the fact that S's vertices do not have the same degree, it is not a regular dominating set. Instead, S is a dominating set of G_1.

Definition 2.3. If for any vertex $u \in R$, $\langle R \setminus \{u\} \rangle$ is not a regular dominating set of G, then the regular dominating set R of G is minimal.

Definition 2.4. If R is the smallest minimal regular dominating set of graph G, then R is referred to as a γ_R-set of G.

Definition 2.5. The regular domination number of graph G is denoted by the symbol $\gamma_R(G)$ and refers to the number of vertices in a least minimal regular dominating set of graph G.

Definition 2.6 ([16])**.** The largest integer less than or equal to x is the floor function of a real number x, and it is represented by the symbol $\lfloor x \rfloor$. If n is an integer and $n \le x < n+1$, then $\lfloor x \rfloor = n$.

Definition 2.7 ([16])**.** The lowest integer greater than or equal to x is the ceiling function of a real number x, and it is represented by the symbol $\lceil x \rceil$. Assuming that n is an integer and $n-1 < x \le n$, then $\lceil x \rceil = n$.

Theorem 2.8 ([4])**.** *For $n \ge 3$, $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.*

3. Main Results

Theorem 3.1. *If G is a graph, then $|R| \ge 2$ where R is a γ_R-set of graph G.*

Proof. If $|R| \le 2$, it means that either $|R| = 2$ or $|R| = 1$. However, if $|R| = 0$, then $R = \phi$, which is not possible. Additionally, if $|R| = 1$, then R is a singleton set and each vertex must have the same degree in accordance with the definition of regular domination. As a result, $|R| \ge 2$. □

Theorem 3.2. *$\gamma_R(K_m) = 2$ for a complete graph K_m with $m \ge 2$ vertices.*

Proof. Any two vertices can form the lowest minimal regular dominating set in a complete graph K_m since all vertices have degree $m-1$. So, $\gamma_R(K_m) = 2$. □

Theorem 3.3. *For $n \ge 6$, $\gamma_R(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let the vertex set of P_n be $\{v_1, v_2, v_3, \dots, v_n\}$. Since we are aware that the path graph P_n comprises n vertices, $n-1$ edges, 2 pendant vertices, and $n-2$ vertices of degree 2. If R is a regular dominating set, there are two alternatives for R. If $v_1, v_n \in R$, then v_1, v_n cannot dominate $n-4$ vertices of P_n, which is in conflict with the concept of a regular dominating set. Now if $v_2, v_3, \dots, v_{n-1} \in R$ then this will be a regular dominating set but not smallest one. It is obvious that for a least minimal regular dominant set, v_2 and v_{n-1} must be members of R. Now we construct R as follows:

$$R = \left\{v_{2+3i} : 0 \le i \le \left\lceil \frac{m}{3} \right\rceil - 2\right\} \cup \{v_{n-1}\}.$$

Then $|R| = \lceil \frac{n}{3} \rceil$.
These $\lceil \frac{n}{3} \rceil$ vertices of R are of same degree and can dominate all remaining vertices of P_n, therefore R is regular dominating set. According to Theorem 2.8, making it the smallest minimal regular dominating set of P_n. Consequently, $\gamma_R(P_n) = \lceil \frac{n}{3} \rceil$. □

Theorem 3.4. *For $m \ge 4$, $\gamma_R(C_m) = \lceil \frac{m}{3} \rceil$.*

Proof. Let $\{v_1, v_2, \dots, v_m\}$ be the vertex set and $\deg(v_i) = 2 \ \forall \ i$. Now we construct a vertex set R as follows:

$$R = \left\{v_{1+3i} : 0 \le i \le \left\lceil \frac{m}{3} \right\rceil - 1\right\}.$$

Then $|R| = \lceil \frac{m}{3} \rceil$.
Now by using Theorem 2.8, R is the smallest minimal regular dominating set.
Thus $\gamma_R(C_m) = \lceil \frac{m}{3} \rceil$. □

Definition 3.5 ([5])**.** The lollipop graph is represented by the symbol $L_{n,m}$ and consists of a bridge between a complete graph K_n and a path graph P_m. The lollipop graph for $n=3$ and $m=5$ is as follows:

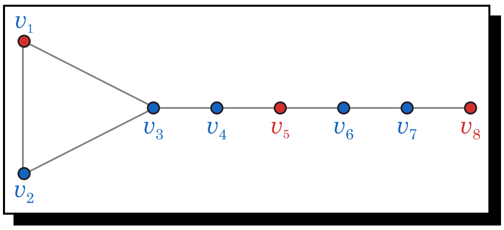

Figure 2. Lollipop graph $L_{3,5}$

Theorem 3.6. *For $m \neq 1$, the regular domination number of the lollipop graph $L_{3,m}$ is $\lceil \frac{m}{3} \rceil + 1$.*

Proof. Assume that $L_{3,m}$ is a lollipop graph with $m+3$ vertices and edges. The vertex set of $L_{3,m}$ is defined as $\{v_1, v_2, v_3, u_1, u_2, u_3, \dots, u_m\}$. Here, $\deg(v_2) = \deg(v_3) = 2$, $\deg(v_1) = 3$, $\deg(u_m) = 1$ and $\deg(u_i) = 2\ \forall\ 1 \le i \le m-1$. If $L_{3,m}$ has a regular dominating set, then R must include the vertices whose degrees are equal. This suggests that neither the degree three nor the degree one vertices can belong to R because they cannot dominate the other vertices of $L_{3,m}$. We now construct the following set using vertices of degree 2:

$$R_1 = \left\{v_2, u_{2+3i} : 0 \le i \le \left\lceil \frac{m}{3} \right\rceil - 2\right\} \cup \{v_{m-1}\}.$$

R_1 is a regular dominating set of $L_{3,m}$, in accordance with the definition of a regular dominating set. Additionally, the above set R_1 is the smallest minimal regular dominating set of $L_{3,m}$ since for any vertex $v \in R_1$, the set $R_1 - \{v\}$ does not dominate the vertices in $N(v)$. This means that $\gamma_R(L_{3,m}) = \lceil \frac{m}{3} \rceil + 1$. □

Corollary 3.7. *Lollipop graph $L_{3,m}$ has no γ_R-set for $m = 1$.*

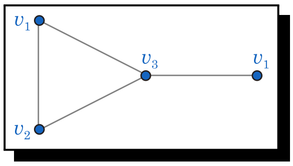

Figure 3. Lollipop graph $L_{3,1}$

Proof. In Figure 3, $\deg(v_1) = \deg(v_2) = 2$, $\deg(v_3) = 3$ and $\deg(u_1) = 1$. Given that the cardinality of the regular dominating set is greater than 2, if we consider a set $R = \{v_3\}$ to be the regular dominating set of $L_{3,1}$ then it leads to a contradiction because the regular dominating set's cardinality is larger than or equal to 2. If we consider the set $\{v_3, u_1\}$ as a regular dominant set, however, this would not be possible because both vertices have a different degree. Additionally, if we consider the set $\{v_1, v_2\}$ we see that it is not a regular dominant set since these vertices cannot dominate the vertex u_1. To construct a regular dominant set, all possible cases fail. As a result, it is implies that lollipop graph $L_{3,1}$ has no γ_R-set. □

Definition 3.8 ([8]). If we link an edge between two copies of complete graphs K_n, then the resulted graph is known as barbell graph and it is represented by B_n. For $n = 4$, the barbell graph B_4 is shown below:

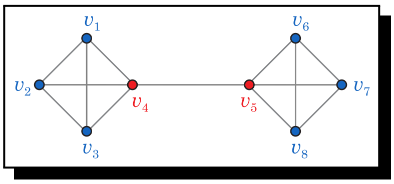

Figure 4. Barbell graph B_4

Theorem 3.9. *Barbell graph B_n has a regular dominating set with $\gamma_R(B_n) = 2$ for any n.*

Proof. Since B_n contains $2n$ vertices, $2(n-1)$ of them have degree $n-1$, while the remaining two have degree n. Let R represent a regular dominant set.

There are two choices for R here:
First, we need to choose one vertex of degree $n-1$ from each copy of the complete graph K_n if R has vertices of degree $n-1$. This is necessary for the regular dominant set. Also, this is the smallest regular dominating set that can exist because a regular dominating set must have at least two vertices of the same degree. Additionally, it is evident from Figure 4 that vertices of degree n dominate all other vertices. This one is also a minimal dominant set of cardinality 2. As a result, we can say that $\gamma_R(B_n) = 2$ for any n. □

Definition 3.10 ([3]). A wheel graph is represented by W_n. The resulting graph will be a Gear graph if we insert a new vertex between each pair of vertices in the outer cycle of W_n. By G_n, it is indicated. Following is the gear graph for $n = 5$:

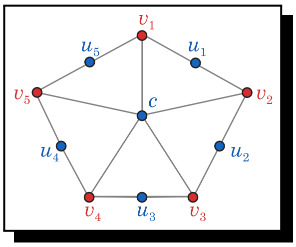

Figure 5. Gear graph G_5

Theorem 3.11. *For a gear graph G_n, $\gamma_R(G_n) = n$ for any n.*

Proof. As far as we know, the gear graph comprises a total of $2n+1$ vertices where n vertices $\{u_1, u_2, u_3, \dots, u_n\}$ are of degree two, n vertices $\{v_1, v_2, v_3, \dots, v_n\}$ are of degree three, and one central vertex, c, is of degree n. We must now choose at least two vertices of the same degree in order to construct a regular dominant set. Let R be a regular dominating set of G_n. No vertex of degree 2 can dominate the centre vertex c, hence if R contains all vertex of degree 2, it contradicts our assumption. Now, if we choose vertices of degree 3 in R, then R must be a regular dominating set as these vertices can dominate remaining vertices of the graph. Furthermore,

R is the smallest minimal regular dominating set since $R - \{v\}$ is not a dominating set for all values of $v \in R$. Consequently, $\gamma_R(G_n) = n$ for any n. □

Definition 3.12 ([9]). Assume C_5 is a cycle graph. If we find a pentagram inside C_5 and connect its vertices, the resulting graph is a Petersen graph. It has 10 vertices and 15 edges. The Petersen graph is depicted as Figure 6.

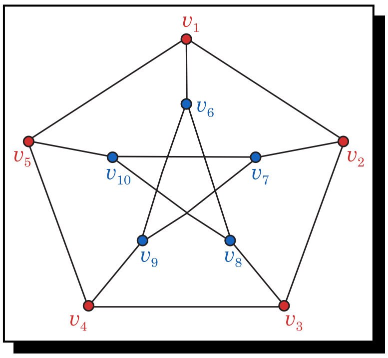

Figure 6. Petersen graph G

Theorem 3.13. *For a Petersen graph G, $\gamma_R(G) = 3$.*

Proof. Since we know that the Petersen graph has 10 vertices, each of which is of degree three. Let $\{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ represent G's vertex set. We know that the domination number of the Petersen graph is 3. Because all vertices are of the same degree, the regular domination number is also 3. The set $\{v_3, v_6, v_{10}\}$ in Figure 6 is the smallest minimal regular dominating set of G. As a result, $\gamma_R(G) = 3$. □

Theorem 3.14 ([10]). *Wheel graph W_n can be transformed into a Helm graph by adding a pendant or end vertex to each of its outer cycle vertex. It is represented by H_n. Helm's graph for $n = 6$ is depicted in Figure 7:*

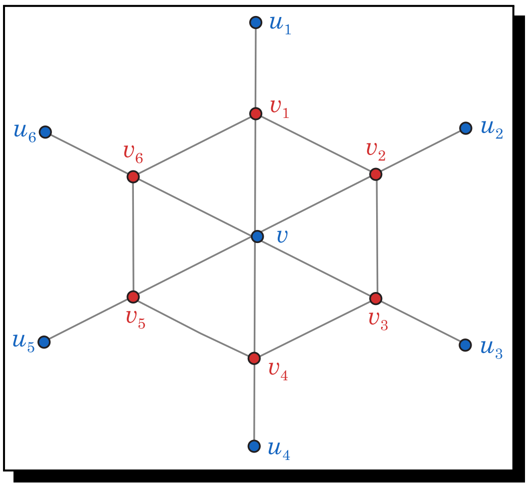

Figure 7. Helm graph H_6

Theorem 3.15. Helm graph H_n has a regular dominating set with $\gamma_R(H_n) = n$ for any n.

Proof. Helm graph H_n has a total of $2n+1$ vertices, with n vertices of degree 4, n pendant vertices, and one centre vertex of degree n. If R is a regular dominating set of H_n, then every vertex in R must be of the same degree. As a result, n vertices of degree 4 must belong to R since they dominate every vertex of H_n. Furthermore, R is the smallest minimal regular dominating set since, for every $v \in R$, $R - \{v\}$ cannot be a dominating set.
As a result, for each n, $\gamma_R(H_n) = n$. □

Definition 3.16 ([1])**.** The jellyfish graph $J_{m,n}$ is created from a 4-cycle with the vertices x, y, u and v by linking x and y with a prime edge and attaching m pendant edges to u and n pendant edges to v. The edge connecting the vertices x and y is referred to as the prime edge in jellyfish. It is shown in Figure 8.

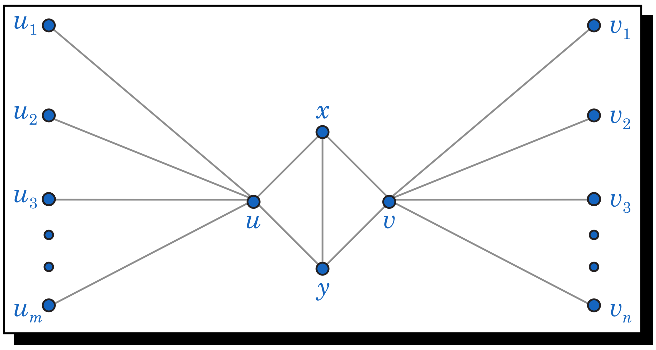

Figure 8. Jellyfish graph $J_{m,n}$

Theorem 3.17. *For a jellyfish graph $J_{m,n}$,*

$$\gamma_R(J_{m,n}) = \begin{cases} 2, & \text{if } m = n, \\ D.N.E, & \text{if } m \neq n. \end{cases}$$

Proof. A jellyfish graph has $m+n$ pendant vertices and $\deg(u) = m+2$, $\deg(v) = n+2$, $\deg(x) = \deg(y) = 3$. Here, we discuss two cases:

Case 1: When $m = n$
In this instance, $\deg(u) = \deg(v) = n+2$. Let R be a minimal regular dominant set. If $R = \{x, y\}$, then this will be in conflict with our definition because x and y cannot dominate pendant vertices that are connected to u and v. Now, if we select $R = \{u, v\}$, then this is the smallest minimal regular dominating set since both have the same degree and they dominate x, y and all $m+n$ pendent vertices. As a result, $\gamma_R(J_{m,n}) = 2$.

Case 2: When $m \neq n$
Let R_1 be the regular dominant set. Only vertices of the same degree belong to R_1 according to the concept of regular dominating set. If x and y belong to R_1, this is not possible since they cannot dominate $m+n$ pendent vertices. If $m+n$ pendant vertices belong to R_1, this is also not possible because these cannot dominate x and y. As a result, we can conclude that there is no regular dominating set of jellyfish graphs for $m \neq n$. □

Definition 3.18 ([1])**.** The jewel graph, J_n is derived from a four-cycle with the vertices x, y, u, v by linking x and y with a prime edge and also by adding the edges from u and v that meet at common vertices v_i, $1 \le i \le n$. The edge connecting the vertices x and y in a jewel graph is defined as the prime edge. It is depicted in Figure 9:

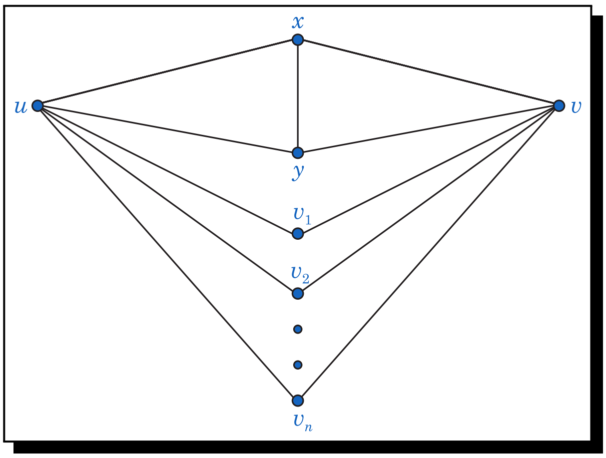

Figure 9. Jewel graph J_n

Theorem 3.19. *For a jewel graph J_n, $\gamma_R(J_n) = 2$.*

Proof. In the jewel graph J_n, $\deg(u) = \deg(v) = n+2$, $\deg(x) = \deg(y) = 3$ and $\deg(v_i) = 2$ for $1 \le i \le n$. We need at least two vertices of the same degree that can dominate all other vertices of the graph in order to have a regular dominating set. Here, we have two degree 3 vertices, x and y, but they are insufficient to construct a regular dominating set because they cannot dominate $v_1, v_2, \dots, v_n$. If we select all degree 2 vertices, then the degree 2 vertices that are selected cannot dominate x and y. We now choose the vertices u and v of degree $n+2$, and since they dominate all other vertices, they can form the lowest minimal regular dominating set. Hence $\gamma_R(J_n) = 2$. □

Definition 3.20 ([15])**.** A graph is said to be a complete bipartite graph in which the vertices can be divided into two subsets, say V_1 and V_2, so that no edge has both ends in the same subset and every vertex in V_1 set is connected to every vertex in V_2. It is represented by $K_{m,n}$. For $m = 5$ and $n = 6$ complete bipartite graph $K_{5,6}$ is shown below:

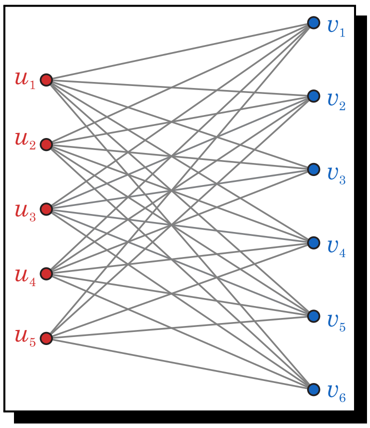

Figure 10. Complete bipartite graph $K_{5,6}$

Theorem 3.21. *For a complete bipartite graph $K_{m,n}$,*

$$\gamma_R(K_{m,n}) = \begin{cases} 2, & \text{if } m = n, \\ \min\{m,n\}, & \text{if } m \neq n. \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, u_3, \dots, u_m\}$ and $V_2 = \{v_1, v_2, v_3, \dots, v_n\}$ be two-partite sets of the complete bipartite graph $K_{m,n}$, which has m and n vertices, respectively. It is evident that $\deg(u_i) = \deg(u_m) = n+1$ and $\deg(u_i) = n+2$ for $2 \le i \le m-1$. Similarly, $\deg(v_i) = \deg(v_n) = m+1$ and $\deg(v_j) = m+2$ for $2 \le j \le n-1$. Here, we discuss two cases:

Case 1: If $m = n$
In order to construct a regular dominating set, we choose one vertex from set V_1 and another from set V_2. Additionally, this is a regular dominating set with minimal cardinality. Therefore, $\gamma_R(K_{m,n}) = 2$ for $m = n$.

Case 2: If $m \neq n$
Firstly, we consider $m > n$. Here, the set $V_2 = \{v_1, v_2, v_3, \dots, v_n\}$ constitute a regular dominating set of minimum cardinality. So, the regular domination number is n. Now, if we consider $m < n$, then the vertex set $V_1 = \{u_1, u_2, u_3, \dots, u_m\}$ constitutes a regular dominant set with a minimum cardinality. Thus, for $m \neq n$, $\gamma_R(K_{m,n}) = \min\{m,n\}$. □

Definition 3.22 ([6])**.** The union of the two graphs G_1 and G_2 with all of the edges joining V_{G_1} and V_{G_2} is the join $G_1 + G_2$ of the graphs with disjoint point sets V_{G_1} and V_{G_2} and edge sets E_{G_1} and E_{G_2}.
Mathematically, $V(G_1+G_2) = V_{G_1} \cup V_{G_2}$ and $E(G_1+G_2) = E_{G_1} \cup E_{G_2} \cup \{uv; u \in V_{G_1}, v \in V_{G_2}\}$.

Definition 3.23 ([12])**.** By combining two graphs K_1 and an empty graph $\overline{K_n}$, the star graph $S_{n,1}$ is formed. The star graph illustration for $n = 7$ is shown below:

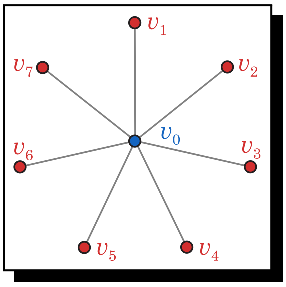

Figure 11. Star graph $S_{7,1}$

Theorem 3.24. *For star graph $S_{n,1}$ on $n+1$ vertices, $\gamma_R(S_{n,1}) = n$.*

Proof. There are n pendent vertices and one central vertex of degree n in a star graph. Two vertices of the same degree are required for a regular dominating set. As a result, the centre vertex v_0 cannot create a regular dominating set, so we must select all n pendant vertices to make a regular dominating set that is the least minimal regular dominating set.
As a result, $\gamma_R(S_{n,1}) = n$. □

Theorem 3.25. *Let P_n and P_m be two Path graphs on $n \geq 4$ and $m \geq 4$ vertices, respectively. Then*

$$\gamma_R(P_n + P_m) = \begin{cases} 2, & \text{if } n = m, \\ \lceil \frac{\min\{n,m\}}{3} \rceil, & \text{if } n \neq m. \end{cases}$$

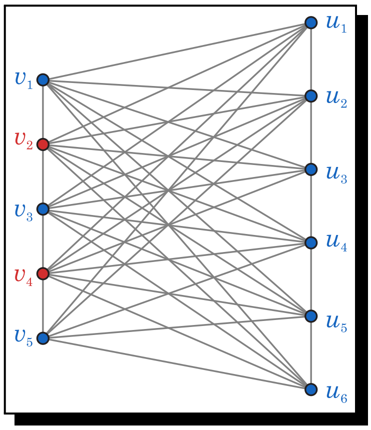

Figure 12. Join of two graphs: $P_5 + P_6$

Proof. Let $V(P_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and $V(P_m) = \{u_1, u_2, u_3, \dots, u_m\}$ denote the vertex sets of P_n and P_m, respectively. Now, according to the definition of join of graphs, every vertex of P_n is adjacent to every vertex of P_m; therefore, $\deg(v_1) = \deg(v_n) = m+1$ and $\deg(v_i) = m+2$, for all $2 \leq i \leq n-1$. Similarly, $\deg(u_1) = \deg(u_n) = n+1$ and $\deg(u_j) = n+2$, for all $2 \leq j \leq m-1$.

Here we consider three cases as follows:

Case 1: If $m = n$
To construct a regular dominating set, we require at least two vertices of the same degree that can dominate all other vertices of $V(P_n + P_m)$. In this case, any two vertices in the graph form a regular dominating set. Also, R must be minimum because any regular dominant set cannot be a singleton set. Thus, $\gamma_R(P_n + P_m) = 2$.

Case 2: If $n < m$
In this case, we create Table 1 to determine the regular domination number as follows:

Table 1. Regular domination number of $P_n + P_m$

S. No.	Values of n	Values of m	$\gamma_R(P_n+P_m)$	S. No.	Values of n	Values of m	$\gamma_R(P_n+P_m)$
1	$n=4$	$m=5,6,7,\dots$	2	7	$n=10$	$m=11,12,13,\dots$	4
2	$n=5$	$m=6,7,8,\dots$	2	8	$n=11$	$m=12,13,14,\dots$	4
3	$n=6$	$m=7,8,9,\dots$	2	9	$n=12$	$m=13,14,15,\dots$	4
4	$n=7$	$m=8,9,10,\dots$	3	10	$n=13$	$m=14,15,16,\dots$	5
5	$n=8$	$m=9,10,11,\dots$	3	11	$n=14$	$m=15,16,17,\dots$	5
6	$n=9$	$m=10,11,12,\dots$	3	12	$n=15$	$m=16,17,18,\dots$	5

Consequently, using the values from the above table as a generalisation, we have $\gamma_R(P_n+P_m)=\lceil\frac{n}{3}\rceil$.

Case 3: If $n>m$

The proof is the same as in *Case* 2.

From the above two cases, we conclude that $\gamma_R(P_n+P_m)=\lceil\frac{\min\{n,m\}}{3}\rceil$ for $n\neq m$. □

Corollary 3.26. *For $m=3$ or $n=3$ then $\gamma_R(P_n+P_m)=2$.*

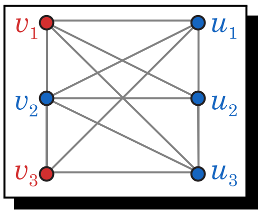

Figure 13. Join of two graphs: P_3+P_3

Proof. From Figure 13, it is obvious that $\deg(v_1)=\deg(v_3)=\deg(u_1)=\deg(u_3)=4$ and $\deg(v_2)=\deg(u_2)=5$. A regular dominant set of least cardinality can be formed here by any two vertices of the same degree. Hence $\gamma_R(P_3+P_3)=2$. □

Definition 3.27 ([12]). A cone graph, $C_{m,n}$, is produced when a cycle graph C_m on n vertices and an empty graph $\overline{K_n}$ on n vertices are joined. For $m=7$ and $n=2$, the cone graph is as follows:

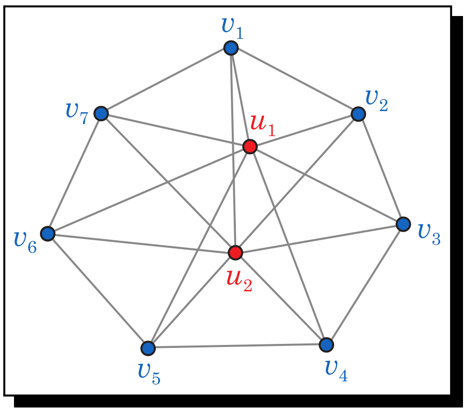

Figure 14. Cone graphs: $C_{7,2}$

Theorem 3.28. *For a cone graph $C_{m,n}$ with $m\geq 4$ and $n\geq 3$ vertices*

$$\gamma_R(C_{m,n})=\begin{cases}\lceil\frac{m}{3}\rceil, & \textit{whenever } \frac{m}{3}\leq n,\\ n, & \textit{whenever } \frac{m}{3}>n.\end{cases}$$

Proof. As we know, a cone graph $C_{m,n}$ is formed by joining a cycle graph C_m and an empty graph $\overline{K_n}$ on n vertices, i.e. $C_{m,n}\cong C_m+\overline{K_n}$. According to the notion of joining two graphs, every vertex of $\overline{K_n}$ is connected to every vertex of C_m in the cycle, which has vertex degrees of two. The degrees of each vertex in C_m and $\overline{K_n}$ will therefore be $n+2$ and m, respectively. We create the following table for $m\geq 4$ and $n=3$ to demonstrate our conclusion:

Table 2. Regular domination number of cone graph $\gamma_R(C_{m,3})$

S. No.	Values of m	Values of n	$\gamma_R(C_{m,3})$
1	$m=4$	$n=3$	2
2	$m=5$	$n=3$	2
3	$m=6$	$n=3$	2
4	$m=7$	$n=3$	2
5	$m=8$	$n=3$	2
6	$m=9$	$n=3$	2
7	$m=10$	$n=3$	2
8	$m=11$	$n=3$	2
9	$m=12$	$n=3$	2
10	$m=13$	$n=3$	2

The values in Table 2 show that the regular domination number is always n if the value of $\frac{m}{3} > n$, and that it is $\lceil \frac{m}{3} \rceil$ if the value of $\frac{m}{3} \le n$. □

Corollary 3.29. *For $m \ge 4$ and $n = 1$, the regular domination number of the cone graph $C_{m,n}$ is* $\lceil \frac{m}{3} \rceil$.

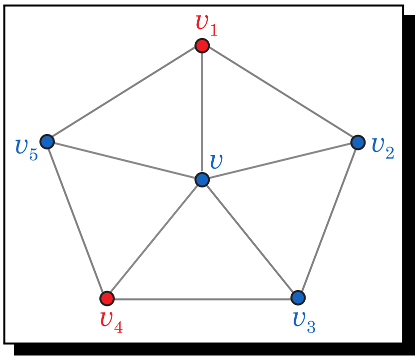

Figure 15. Cone graph: $C_{5,1}$

Proof. Let $C_{5,1}$ be a cone graph with $m = 5$ and $n = 1$ vertices, as illustrated in Figure 15. Each vertex of C_5 has a degree of 3 and the singleton vertex has a degree of 5. A regular dominating set must have at least two vertices, hence a singleton vertex v cannot form a regular dominating set. As a result, the vertices of C_5 form a regular dominating set, and Theorem 3.4 states that it is $\lceil \frac{5}{3} \rceil = 2$. □

Corollary 3.30. *For $m \ge 3$ and $n = 2$, the regular domination number of the cone graph $C_{m,n}$ is* 2.

Proof. The empty graph's vertices form a regular dominating set of cardinality 2 as can be seen in Figure 14. Additionally, vertices of cycle C_7 of cardinality 3 can be used to create a regular dominating set; however, this is not the smallest regular dominating set. Thus $\gamma_R(C_{7,2}) = 2$. As with all values of $m \ge 7$ with $n = 2$, the regular domination number will always be 2. □

Definition 3.31 ([12])**.** The generated graph is a fan graph $F_{m,n}$ if we link an empty graph $\overline{K_m}$ on m vertex and a path graph P_n on n vertex. Below is the fan graph for $m = 2$ and $n = 7$:

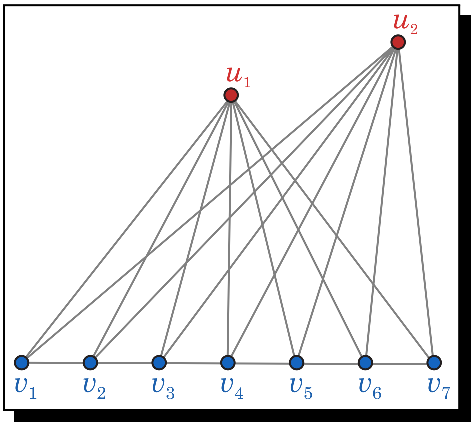

Figure 16. Fan graph: $F_{2,7}$

Theorem 3.32. *For any fan graph $F_{m,n}$ with $m \ge 2$ and $n \ge 4$ vertices,*

$$\gamma_R(F_{m,n}) = \begin{cases} m, & \text{if } \frac{n}{3} \ge m, \\ \lceil \frac{m}{3} \rceil, & \text{if } \frac{n}{3} < m. \end{cases}$$

Proof. Let the vertex set of the fan graph be $\{u_1, u_2, u_3, \dots, u_m, v_1, v_2, v_3, \dots, v_n\}$. As far as we know, an empty graph is a join of a path graph P_n on n vertices and an empty graph $\overline{K_m}$ on m vertices. Every vertex of the path graph is now connected to every vertex of the empty graph according to the joining of the two graphs, so $\deg(v_1) = \deg(v_n) = m+1$ and $\deg(v_i) = m+2$ for $2 \le i \le n-1$ and $\deg(u_j) = n \;\; \forall\, j$. Now, as indicated below, we create the tables below to calculate the regular domination number of $F_{m,n}$:

Table 3. Regular domination number of fan graph $\gamma_R(F_{2,n})$

S. No.	Values of m	Values of n	$\gamma_R(F_{2,n})$
1	$m = 2$	$n = 4$	2
2	$m = 2$	$n = 5$	2
3	$m = 2$	$n = 6$	2
4	$m = 2$	$n = 7$	2
5	$m = 2$	$n = 8$	2
6	$m = 2$	$n = 9$	2
7	$m = 2$	$n = 10$	2
8	$m = 2$	$n = 11$	2
9	$m = 2$	$n = 12$	2
10	$m = 2$	$n = 13$	2

Table 4. Regular domination number of fan graph $\gamma_R(F_{3,n})$

S. No.	Values of m	Values of n	$\gamma_R(F_{3,n})$
1	$m = 3$	$n = 4$	2
2	$m = 3$	$n = 5$	2
3	$m = 3$	$n = 6$	2
4	$m = 3$	$n = 7$	3
5	$m = 3$	$n = 8$	3
6	$m = 3$	$n = 9$	3
7	$m = 3$	$n = 10$	3
8	$m = 3$	$n = 11$	3
9	$m = 3$	$n = 12$	3
10	$m = 3$	$n = 13$	3

Tables 3 and 4 show that when $\frac{n}{3} \geq m$, the regular domination number is m, and if $\frac{n}{3} \leq m$, the regular domination number is $\lceil \frac{m}{3} \rceil$. □

Corollary 3.33. *For $m = 1$ and $n \geq 4$, the regular domination number of the fan graph $F_{m,n}$ is $\lceil \frac{m}{3} \rceil$.*

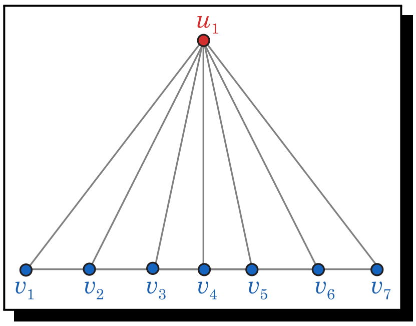

Figure 17. Fan graph: $F_{1,7}$

Proof. According to the definition, the singleton vertex u_1 does not form a regular dominating set. So, vertices of path P_7 form a regular dominating set. According to Theorem 3.3, $\gamma_R(P_n) = \lceil \frac{m}{3} \rceil$. Hence, $\gamma_R(F_{1,7}) = \lceil \frac{7}{3} \rceil = 3$. □

Corollary 3.34. *The regular domination number of fan graph $F_{m,n}$ for $m \geq 1$ and $n = 2$ or 3 is 2.*

Definition 3.35 ([11])**.** Windmill graph $W_{m,n}$ is an undirected graph constructed by joining m copies of complete graph K_n with a common vertex K_1. The figure of windmill graph for $m = 3$ and $n = 4$ is as below:

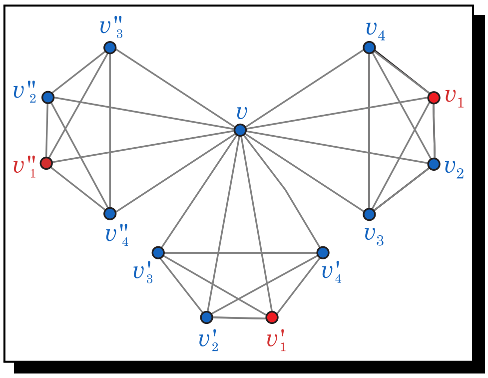

Figure 18. Windmill graph: $W_{3,4}$

Theorem 3.36. *For any windmill graph $W_{m,n}$ with $m \geq 2$ and $n \geq 3$, $\gamma_R(W_{m,n}) = m$.*

Proof. As we know, a windmill graph is created by combining m copies of the complete graph K_n at a common vertex, which is denoted by the notation $mK_n + K_1$. It is obvious that a regular dominating set cannot be formed by the common vertex v since a regular dominating set requires at least two vertices. Each vertex in the m copies of the complete graph has a degree of n. Now, we must choose at least one vertex from each copy of the complete graph in order to build a regular dominant set. Additionally, since $R - \{v\}$ cannot be a regular dominating set for every vertex $v \in R$, this is the least minimal regular dominating set. Therefore, $\gamma_R(W_{m,n}) = m$. □

Definition 3.37 ([6]). A pair of graphs, G and H, where $O(G) = m$ and $O(H) = n$, will be considered. A graph called the corona product $G \circ H$ of two graphs is created by taking one copy of G and $|V(G)| = m$ copies of H and connecting the i-th vertex of G to each vertex in the i-th copy of H.

Theorem 3.38. *Let P_n and P_m be two path graphs on n and m vertices respectively, then $\gamma_R(P_n \circ P_m) = n \times \lceil \frac{m}{3} \rceil$.*

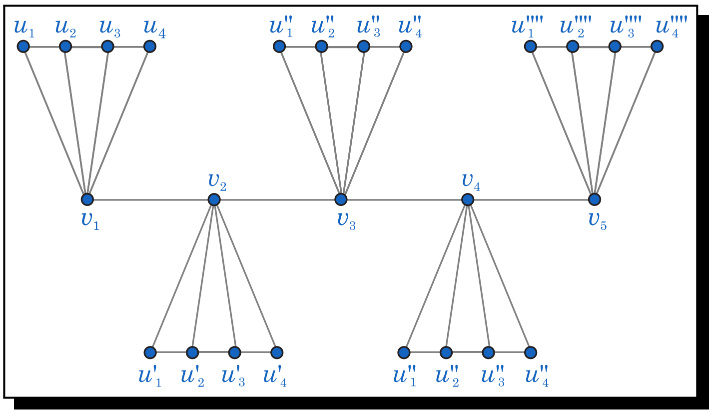

Figure 19. Corona of two graph: $P_5 \circ P_4$

Proof. As we know that P_n not a regular graph and $\deg(v_1) = \deg(v_n) = m+1$ and $\deg(v_i) = m+2$ for $2 \le i \le n-1$. It is clear that either vertices of degree $m+1$ or $m+2$ are not sufficient to form a regular dominating set of $P_n \circ P_m$. Now we have an another choice to form a regular dominating set with vertices of n copies of P_m. Let H^{v_i} be the ith copy of P_m. As we already prove that regular domination number of path graph is $\lceil \frac{m}{3} \rceil$ so that we have to select $\lceil \frac{m}{3} \rceil$ vertices of same degree from n copies of P_m. Thus, $\gamma_R(P_n \circ P_m) = n \times \lceil \frac{m}{3} \rceil$. □

Corollary 3.39.

(i) $\gamma_R(P_n \circ P_m) = 2$ *for* $n = 2$ *and* $m \in \mathbb{Z}^+$.

(ii) $\gamma_R(P_n \circ \overline{K_m}) = mn$.

(iii) $\gamma_R(C_n \circ \overline{K_m}) = n$.

Theorem 3.40. *Let G be a regular graph of order n and H be any graph of order m, then $\gamma_R(G \circ H) = n$.*

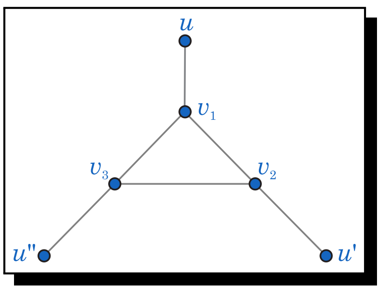

Figure 20. Corona of two graph: $C_3 \circ K_1$

Proof. Let $\{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of the regular graph G of order n, where each vertex has the same degree and let H be any graph of order m whose each vertex is attached to every vertex of G. Here, we have two choices for the minimum RDS. Since all the vertices of G have same degree, the vertex of G forms an RDS of cardinality n. Also, by choosing at least one vertex from n-copies of H, we construct another RDS of cardinality n. Both sets are regular dominating sets of minimum cardinaltiy. Therefore, $\gamma_R(G \circ H) = n$. □

Theorem 3.41. *Let G be a non-regular graph of order n and H be a complete graph K_m on m vertices, then $\gamma_R(G \circ H) = n$.*

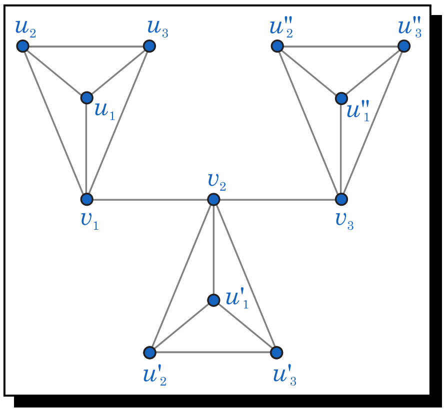

Figure 21. Corona of two graph: $P_3 \circ K_3$

Proof. Let G be a non-regular graph and $\{v_1, v_2, v_3, \dots, v_n\}$ be its vertex set. According to the definition of corona, there are n-copies of K_m attached to each vertex G. Since G is a non-regular graph, its vertex does not form an RDS. Now, to construct the RDS of $G \circ K_m$ of minimum cardinaltiy, select at least one vertex from n-copies of K_m. Therefore, $\gamma_R(G \circ H) = n$. □

4. Application of Regular Domination

The concept of domination has its application in identifying minimum number of security guards to guard a city. Also, total domination is about choosing minimum number of security guards where one guard is designated as backup to the other. Identifying the minimum number of security guards needed to protect a city while assigning each guard an equal amount of responsibilities (by allocating each guard an equal number of positions) is the application of regular domination.

5. Conclusion

Motivated by the concept of regular domination in fuzzy graph described by Prabakaran *et al.* [14] we introduced concept of regular domination for simple graphs. Here, we determined the regular domination number of several graphs like complete graphs, path graphs, cycle graphs, lollipop graphs, barbell graphs, gear graphs, Petersen graphs, helm graphs, jellyfish graphs, jewel graphs, and complete bipartite graphs. Further, regular domination number can also be determined for specific graph operations such as join and corona of two graphs.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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