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Research Article

ℝ-Complex Finsler Spaces With Generalized Kropina Metric

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Abstract. The study of ℝ-complex Finsler spaces with an (α,β)-metric is a fundamental problem in Finsler geometry. In this paper, we introduce the concept of ℝ-complex Finsler spaces with a generalized Kropina metric given by $F = \frac{\alpha^{m+1}}{\beta^m}$. We derive explicit formulas for the fundamental metric tensor fields g_{ij} and $g_{i\bar{j}}$, as well as their determinants and inverse tensor fields for this metric. Additionally, we discuss various properties of non-Hermitian ℝ-complex Finsler spaces with the aforementioned metric.

Keywords. Complex Finsler space, ℝ-complex Finsler space, Fundamental metric tensors

Mathematics Subject Classification (2020). Primary 53B40, Secondary 53C60

1. Introduction

The notion of (α,β)-metric was introduced by Matsumoto [12] as a generalization of the Randers metric. This metric has found applications in various fields such as physics and biology, including ecological spaces (Antonelli *et al*. [7], Shanker and Kushwaha [17]). Subsequently, Kropina [10], Matsumoto [11], and Shen and Yu [18] have extensively studied and developed different types of (α,β)-metrics, such as the Matsumoto metric, exponential metric, Kropina metric, Einstein metric, general (α,β)-metric, and Randers changed square metric, from different geometrical perspectives.

Generalized Kropina metrics represent special cases of (α,β)-metrics. Both generalized Kropina metrics and Kropina metrics have numerous applications in other scientific disciplines, including physics, irreversible thermodynamics, and electron optics with a magnetic field (Ingarden [9], and Shibata [19]). These metrics are extensions or modifications of the original Kropina metric and possess different conditions or additional geometric properties. The study of generalized Kropina metrics has practical implications in fields like theoretical physics, relativity theory, mechanics, and optimization problems. Understanding their properties and behavior enables their application in specific contexts, providing tools for modeling and analyzing physical systems or optimizing processes.

The theories of $\mathbb{R}$-complex Finsler spaces are relatively new and were first introduced by Rizza [15]. Munteanu and Purcaru [14] extended the concept of complex Finsler spaces (Abate and Patrizio [1], Aikou [2], and Monteanu [13]) and introduced another class of such spaces. Aldea [3] investigated a specific class of complex Finsler spaces in two dimensions. Many geometers, e.g., Aldea and Campean [4], Aldea and Munteanu [5], and Câmpean and Purcaru [8] have made significant contributions to the study of $\mathbb{R}$-complex Finsler spaces.

This paper builds upon ideas from real Finsler spaces with a generalized Kropina metric and introduces a similar notion in $\mathbb{R}$-complex Finsler spaces. We define this metric as

$$F = \frac{\alpha^{m+1}}{\beta^m}, \tag{1.1}$$

where $m \neq 0, -1$. For the metric (1.1), we derive the fundamental metric tensor fields g_{ij} and $g_{i\bar{j}}$, their determinants, and the inverse tensor fields. Furthermore, we discuss various properties of non-Hermitian $\mathbb{R}$-complex Finsler spaces with the metric given in equation (1.1).

2. $\mathbb{R}$-complex Finsler Spaces

Let M be a complex manifold with complex dimension n, and let (z^k) be local complex coordinates in a chart (U, ϕ). The holomorphic tangent bundle of M, denoted as $T'M$, naturally possesses a complex manifold structure with complex dimension $2n$. In a local chart $u \in T'M$, the induced coordinates are denoted as $u = (z^k, \eta^k)$. The transformations of local coordinates in u are given by the rules:

$$z'^k = z'^k(z), \quad \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j. \tag{2.1}$$

The natural frame $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$ of $T'_u(T'M)$ undergoes a transformation according to the Jacobian matrix of equation (2.1), yielding:

$$\frac{\partial}{\partial z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial z'^j} + \frac{\partial^2 z'^j}{\partial z^k \partial z^h}\eta^h \frac{\partial}{\partial \eta'^j}, \quad \frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}.$$

A complex non-linear connection, abbreviated as c.n.c., refers to a supplementary distribution $H(T'M)$ to the vertical distribution $V(T'M)$ in $T'(T'M)$. The vertical distribution is spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and an adapted frame in $H(T'M)$ is given by $\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}$, where N_k^j are the coefficients of the c.n.c. These coefficients follow a certain rule of transformation under (2.1), ensuring that $\frac{\delta}{\delta z^k}$ transform like vectors on the base manifold M.

To simplify notation, we introduce the abbreviations: $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$, $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$, as well as $\partial_{\bar{k}}, \delta_{\bar{k}}, \dot{\partial}_{\bar{k}}$ for their conjugates. The dual adapted basis of $\{\delta_k, \dot{\partial}_k\}$ is given by $\{dz^k, \delta\eta^k = d\eta^k + N_j^k dz^j\}$, and their conjugates are $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Definition 2.1 ([14])**.** An $\mathbb{R}$-complex Finsler metric on M is a continuous function $F : T'M \to \mathbb{R}_+$ satisfying:

(i) $L := F^2$ is smooth on $\widetilde{T'M}$ (except the 0 sections),

(ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$,

(iii) $F(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = |\lambda| F(z,\eta,\bar{z},\bar{\eta})$, $\forall\ \lambda \in \mathbb{R}$.

Using assertion (i) and (iii) of Definition 2.1, L is $(2,0)$ homogeneous with respect to the real scalars λ:

$$L(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = \lambda^2 L(z,\eta,\bar{z},\bar{\eta}), \quad \lambda \in \mathbb{R}.$$

Definition 2.2 ([6])**.** An $\mathbb{R}$-complex Finsler spaces with (α,β)-metric is a pair (M,F), where the fundamental function $F(z,\eta,\bar{z},\bar{\eta})$ is $\mathbb{R}$-homogeneous by means of functions $\alpha(z,\eta,\bar{z},\bar{\eta})$ and $\beta(z,\eta,\bar{z},\bar{\eta})$,

$$F(z,\eta,\bar{z},\bar{\eta}) = F(\alpha(z,\eta,\bar{z},\bar{\eta}),\beta(z,\eta,\bar{z},\bar{\eta})), \tag{2.2}$$

where

$$\alpha^2(z,\eta,\bar{z},\bar{\eta}) = \frac{1}{2}(a_{ij}\eta^i\eta^j + 2a_{i\bar{j}}\eta^i\bar{\eta}^j + a_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j) = Re\{a_{ij}\eta^i\eta^j + a_{i\bar{j}}\eta^i\bar{\eta}^j\},$$

$$\beta(z,\eta,\bar{z},\bar{\eta}) = \frac{1}{2}(b_i\eta^i + b_{\bar{i}}\bar{\eta}^i) = Re\{b_i\eta^i\},$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$ and $b_i = b_i(z)$, $b_i(z)dz^i$ is a $(1,0)$-differential form on complex manifold M.

In the case, where $a_{ij} = 0$ and $a_{i\bar{j}}$ is invertible, the space is referred to as a *Hermitian space*. On the other hand, if $a_{i\bar{j}} = 0$ and a_{ij} is invertible, the space is termed a *non-Hermitian space*.

Indeed, the functions α and β exhibit homogeneity with respect to η and $\bar{\eta}$. This means that $\alpha(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = \lambda\alpha(z,\eta,\bar{z},\bar{\eta})$ and $\beta(z,\lambda\eta,\bar{z},\lambda\bar{\eta}) = \lambda\beta(z,\eta,\bar{z},\bar{\eta})$ for any $\lambda \in \mathbb{R}_+$. Furthermore, since L possesses $(2,0)$-homogeneity with respect to λ, we can derive the following inequalities using the property of homogeneity [6]:

$$\left.\begin{aligned}
&\alpha L_\alpha + \beta L_\beta = 2L,\\
&\alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha,\\
&\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta,\\
&\alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L,\\
&\frac{\partial\alpha}{\partial\eta^i}\eta^i + \frac{\partial\alpha}{\partial\bar{\eta}^j}\bar{\eta}^j = \alpha,\\
&\frac{\partial\beta}{\partial\eta^i}\eta^i + \frac{\partial\beta}{\partial\bar{\eta}^j}\bar{\eta}^j = \beta,
\end{aligned}\right\} \tag{2.3}$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha}, \quad L_\beta = \frac{\partial L}{\partial\beta}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, \quad L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, \quad L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}.$$

We consider

$$\frac{\partial\alpha}{\partial\eta^i} = \frac{1}{2\alpha}(a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j) = \frac{1}{2\alpha}l_i, \quad \frac{\partial\beta}{\partial\eta^i} = \frac{1}{2}b_i$$

and

$$\eta_i = \frac{\partial L}{\partial \eta^i} = \frac{\partial}{\partial \eta^i} F^2 = 2F \frac{\partial F}{\partial \eta^i} = \rho_0 l_i + \rho_1 b_i\,,$$

where

$$l_i = a_{ij}\eta^j + a_{i\bar{j}}\bar{\eta}^j\,, \tag{2.4}$$

$$b_l = a_{kl} b^k + a_{l\bar{k}} b^{\bar{k}}\,, \tag{2.5}$$

$$\rho_0 = \frac{1}{2}\frac{L_\alpha}{\alpha} \text{ and } \rho_1 = \frac{1}{2} L_\beta. \tag{2.6}$$

Differentiating ρ_0 and ρ_1 with respect to η^j, we get

$$\frac{\partial \rho_0}{\partial \eta^j} = \rho_{-2} l_j + \rho_{-1} b_j\,,$$

$$\frac{\partial \rho_1}{\partial \eta^i} = \rho_{-1} l_i + \mu_0 b_i\,,$$

where

$$\rho_{-2} = \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3}, \quad \rho_{-1} = \frac{L_{\alpha\beta}}{4\alpha}, \quad \mu_0 = \frac{L_{\beta\beta}}{4}. \tag{2.7}$$

The quantities $\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \mu_0$ are the invariants of the ℝ-complex Finsler space with (α,β)-metric (Sabau and Shimada [16]).

In [14], an ℝ-complex Finsler space the following conditions hold:

$$\frac{\partial L}{\partial \eta^i}\eta^i + \frac{\partial L}{\partial \bar{\eta}^i}\bar{\eta}^i = 2L\,, \quad g_{ij}\eta^i + g_{j\bar{i}}\bar{\eta}^i = \frac{\partial L}{\partial \eta^j}\,,$$

$$\frac{\partial g_{ik}}{\partial \eta^j}\eta^j + \frac{\partial g_{ik}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0\,, \quad \frac{\partial g_{i\bar{k}}}{\partial \eta^j}\eta^j + \frac{\partial g_{i\bar{k}}}{\partial \bar{\eta}^j}\bar{\eta}^j = 0\,,$$

$$2L = g_{ij}\eta^i\eta^j + g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j + 2g_{i\bar{j}}\eta^i\bar{\eta}^j\,,$$

where

$$g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}, \quad g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \text{ and } g_{\bar{i}\bar{j}} = \frac{\partial^2 L}{\partial \bar{\eta}^i \partial \bar{\eta}^j},$$

are the metric tensors of space.

Theorem 2.3 ([6])**.** *The metric tensor fields of ℝ-complex Finsler space with (α,β)-metric are given by*

$$g_{ij} = \rho_0 a_{ij} + \rho_{-2} l_i l_j + \mu_0 b_i b_j + \rho_{-1}(b_j l_i + b_i l_j),$$

$$g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho_{-2} l_i l_{\bar{j}} + \mu_0 b_i b_{\bar{j}} + \rho_{-1}(b_{\bar{j}} l_i + b_i l_{\bar{j}}), \tag{2.8}$$

where the quantities $\rho_{-2}, \rho_{-1}, \rho_0, \rho_1, \mu_0$ are defined in the symbols of equations (2.6) *and* (2.7)*.*

For obtaining the inverse and determinant of the tensor field g_{ij}, one can follow the following proposition:

Proposition 2.4 ([5])**.** *Suppose*

- *(Q_{ij}) is a non-singular $n \times n$ complex matrix with its inverse (Q^{ji});*
- *C_i and $C_{\bar{i}} := \bar{C}_i$, $i = 1, \dots, n$, are complex numbers;*

- $C^i := Q^{ji}C_j$ *and* C_i *are conjugates to each other;* $C^2 := C^iC_i = \bar{C}^iC_{\bar{i}}$; $H_{ij} := Q_{ij} \pm C_iC_j$.

Then

(i) $\det(H_{ij}) = (1 \pm C^2)\det(Q_{ij})$.

(ii) *For* $1 \pm C^2 \neq 0$, *the matrix* (H_{ij}) *is invertible and its inverse is* $H^{ji} = Q^{ji} \mp \frac{1}{1\pm C^2}C^iC^j$.

3. $\mathbb{R}$-complex Finsler Space with Generalized Kropina Metric

An $\mathbb{R}$-complex Finsler spaces (M,F), F is given by (2.2), is known as $\mathbb{R}$-complex Finsler space with generalized Kropina metric F. According to Definition 2.1(i), we have the following:

$$L(\alpha,\beta) = \left(\frac{\alpha^{m+1}}{\beta^m}\right)^2. \tag{3.1}$$

From equation (3.1), we get

$$\left.\begin{aligned} &L_\alpha = \frac{2(m+1)\alpha^{2m+1}}{\beta^{2m}}, \quad L_{\alpha\alpha} = \frac{2(m+1)(2m+1)\alpha^{2m}}{\beta^{2m}}, \quad L_{\alpha\beta} = -\frac{4m(m+1)\alpha^{2m+1}}{\beta^{2m+1}}, \\ &L_\beta = -\frac{2m\alpha^{2m+2}}{\beta^{2m+1}}, \quad L_{\beta\beta} = \frac{2m(2m+1)\alpha^{2m+2}}{\beta^{2m+2}}. \end{aligned}\right\} \tag{3.2}$$

Substituting L_α, $L_{\alpha\alpha}$, L_β, $L_{\beta\beta}$, and $L_{\alpha\beta}$ from above in the system of equations (2.3), we get

$$\alpha L_\alpha + \beta L_\beta = \frac{2\alpha^{2m+2}}{\beta^{2m}} = 2L,$$

$$\alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = \frac{2(m+1)\alpha^{2m+1}}{\beta^{2m}} = L_\alpha,$$

$$\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta, \ \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L.$$

In the same way, one can verify the rest inequalities of the system of equations (2.3).

Now, using the equations (2.6), (2.7), and (3.2), we get

$$\left.\begin{aligned} &\rho_0 = \frac{(m+1)\alpha^{2m}}{\beta^{2m}}, \quad \rho_1 = -\frac{m\alpha^{2m+2}}{\beta^{2m+1}}, \quad \rho_{-2} = \frac{m(m+1)\alpha^{2m-2}}{\beta^{2m}}, \\ &\rho_{-1} = -\frac{m(m+1)\alpha^{2m}}{\beta^{2m+1}}, \quad \mu_0 = \frac{m(2m+1)\alpha^{2m+2}}{2\beta^{2m+2}}. \end{aligned}\right\} \tag{3.3}$$

Proposition 3.1. *The invariants of an $\mathbb{R}$-complex Finsler space (M,F), where F is generalized Kropina metric, are given in the system of equations* (3.3).

Now, using the invariants given in equation (3.3) and Theorem 2.3, we get

$$g_{ij} = \frac{(m+1)\alpha^{2m}}{\beta^{2m}}a_{ij} + \frac{m(m+1)\alpha^{2m-2}}{\beta^{2m}}l_il_j + \frac{m(2m+1)\alpha^{2m+2}}{2\beta^{2m+2}}b_ib_j - \frac{m(m+1)\alpha^{2m}}{\beta^{2m+1}}(b_jl_i + b_il_j), \tag{3.4}$$

$$g_{i\bar{j}} = \frac{(m+1)\alpha^{2m}}{\beta^{2m}}a_{i\bar{j}} + \frac{m(m+1)\alpha^{2m-2}}{\beta^{2m}}l_il_{\bar{j}} + \frac{m(2m+1)\alpha^{2m+2}}{2\beta^{2m+2}}b_ib_{\bar{j}} - \frac{m(m+1)\alpha^{2m}}{\beta^{2m+1}}(b_{\bar{j}}l_i + b_il_{\bar{j}}). \tag{3.5}$$

The equations (3.4) and (3.5) can be written in the following equivalent forms:

$$g_{ij} = \rho_0(a_{ij} - k_1l_il_j + k_2b_ib_j + k_3\eta_i\eta_j), \tag{3.6}$$

$$g_{i\bar{j}} = \rho_0(a_{i\bar{j}} - k_1 l_i l_{\bar{j}} + k_2 b_i b_{\bar{j}} + k_3 \eta_i \eta_{\bar{j}}), \tag{3.7}$$

where

$$k_1 = -\frac{(2m+1)}{\alpha^2}, \quad k_2 = \frac{m(4m+1)\alpha^2}{2(m+1)\beta^2}, \quad k_3 = -\frac{\beta^{4m}}{(m+1)\alpha^{4m+2}}.$$

Theorem 3.2. *Let us consider an ℝ-complex Finsler space* (M,F)*, where* F *is the generalized Kropina metric. The metric tensor fields associated with this space are given in equations* (3.6) *and* (3.7).

4. Non-Hermitian ℝ-complex Finsler Space With Generalized Kropina Metric

This section focuses on the analysis of the non-Hermitian ℝ-complex Finsler space with the generalized Kropina metric given in equation (1.1). In the case of a non-Hermitian ℝ-complex Finsler space where $a_{i\bar{j}} = 0$, we will employ the following abbreviations:

$$\left.\begin{aligned} &l_i = a_{ij}\eta^j, \ \ \gamma = a_{jk}\eta^j\eta^k = l_k\eta^k, \ \ \theta = b_j\eta^j, \ \ \omega = b_j b^j, \\ &b^k = a^{jk}b_j, \ \ b_l = b^k a_{kl}, \ \ \delta = a_{jk}\eta^j b^k = l_k b^k, \ \ l^j = a^{ji}l_i = \eta^j. \end{aligned}\right\} \tag{4.1}$$

Now, let's apply Proposition 2.4 to the tensor field g_{ij} defined in equation (3.6), and follow these steps:

Step 1. Suppose $Q_{ij} = a_{ij}$ and $C_i = \sqrt{k_1}l_i$.
From our assumption, we get

$$Q^{ji} = a^{ji},$$
$$C^2 = C_i C^i = \sqrt{k_1}l_i \times Q^{ji} \times C_j = \sqrt{k_1}l_i \times a^{ji} \times \sqrt{k_1}l_j = k_1 \times l_i a^{ji} l_j = k_1\gamma.$$

Now, by applying Proposition 2.4, we get

$$\det(H_{ij}) = \det(a_{ij} - k_1 l_i l_j) = (1 - k_1\gamma)\det(a_{ij}) = \tau_1 \det(a_{ij}), \tag{4.2}$$

and, for $\tau_1 = 1 - k_1\gamma \neq 0$, $H_{ij} = a_{ij} - k_1 l_i l_j$ is invertible and its inverse is given by:

$$H^{ji} = a^{ji} + \frac{k_1\eta^i\eta^j}{\tau_1}. \tag{4.3}$$

Step 2. Suppose $Q_{ij} = a_{ij} - k_1 l_i l_j$ and $C_i = \sqrt{k_2}b_i$.
Using the equations (4.3) and our supposition, we get

$$Q^{ji} = a^{ji} + \frac{k_1\eta^i\eta^j}{\tau_1}.$$

Using the previous equation and our supposition, we get

$$C^i = Q^{ji}C_j = \left(a^{ji} + \frac{k_1\eta^i\eta^j}{\tau_1}\right)\sqrt{k_2}b_j = \left(b^i + \frac{k_1\theta\eta^i}{\tau_1}\right)\sqrt{k_2},$$

which implies

$$C^2 = k_2\left(\omega + \frac{k_1\theta^2}{\tau_1}\right),$$

and

$$1+C^2 = 1+k_2\left(\omega+\frac{k_1\theta^2}{\tau_1}\right) = \tau_2 \quad \text{(say)}.$$

Now, by applying Proposition 2.4, we get

$$H_{ij} = a_{ij} - k_1 l_i l_j + k_2 b_i b_j$$

$$\det(H_{ij}) = \tau_1\tau_2 \det(a_{ij}), \tag{4.4}$$

and, for τ_2 and $\tau_1 \neq 0$, the inverse of H_{ij} exists and it is

$$H^{ji} = a^{ji} + \left\{\frac{k_1}{\tau_1} - \frac{\theta^2 k_1^2 k_2}{(\tau_1)^2\tau_2}\right\}\eta^i\eta^j - \frac{k_2 b^i b^j}{\tau_2} + \frac{\theta k_1 k_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2}. \tag{4.5}$$

Step 3. Suppose $Q_{ij} = a_{ij} - k_1 l_i l_j + k_2 b_i b_j$ and $C_i = \sqrt{k_3}\eta_i$.
Using the equation (4.5) and our supposition, we get

$$Q^{ji} = a^{ji} + \left\{\frac{k_1}{\tau_1} - \frac{\theta^2 k_1^2 k_2}{(\tau_1)^2\tau_2}\right\}\eta^i\eta^j - \frac{k_2 b^i b^j}{\tau_2} + \frac{\theta k_1 k_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2}.$$

Using the previous equation and our supposition, we get

$$C^i = M\eta^i + Nb^i,$$

where

$$M = \left\{1+\frac{k_1}{\tau_1} - \frac{\theta^2 k_1^2 k_2}{(\tau_1)^2\tau_2}\right\}\gamma - \frac{\theta k_1 k_2}{(\tau_1)^3\tau_2}, \quad N = -\frac{k_2\theta}{\tau_2} - \frac{\theta k_1 k_2 \gamma}{\tau_1\tau_2}, \tag{4.6}$$

which implies

$$C^2 = (M\gamma + N\theta)\sqrt{k_3}, \quad 1+C^2 = 1+(M\gamma+N\theta)\sqrt{k_3} = \tau_3 \quad \text{(say)},$$

and

$$C^i C^j = M^2\eta^i\eta^j + MN(b^i\eta^j + b^j\eta^i) + N^2 b^i b^j.$$

Now, by using Proposition 2.4, we get

$$H_{ij} = a_{ij} - k_1 l_i l_j + k_2 b_i b_j + k_3\eta_i\eta_j, \tag{4.7}$$

$$\det(H_{ij}) = \tau_1\tau_2\tau_3 \det(a_{ij}). \tag{4.8}$$

Since for non-zero τ_i $(i = 1,2,3)$, $\det(H_{ij}) \neq 0$. Therefore, the inverse of H_{ij} exists and it is

$$H^{ji} = a^{ji} + \left\{\frac{k_1}{\tau_1} - \frac{\theta^2 k_1^2 k_2}{\tau_1^2\tau^2}\right\}\eta^i\eta^j - \frac{k_2 b^i b^j}{\tau_2} - \frac{\theta k_1 k_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2} - \frac{M^2\eta^i\eta^j + MN(b^i\eta^j + b^j\eta^i) + N^2 b^i b^j}{\tau_3}. \tag{4.9}$$

But $g_{ij} = \rho_0 H_{ij}$, where H_{ij} is given in the equation (4.7). Thus, $g^{ji} = \frac{1}{\rho_0}H^{ji}$ and $\det(g_{ij}) = (\rho_0)^n \det(H_{ij})$. Using the equations (4.8) and (4.9), we get

$$g^{ji} = \frac{1}{\rho_0}\left[a^{ji} + \left\{\frac{k_1}{\tau_1} - \frac{\theta^2 k_1^2 k_2}{\tau_1^2\tau^2}\right\}\eta^i\eta^j - \frac{k_2 b^i b^j}{\tau_2} - \frac{\theta k_1 k_2(b^i\eta^j + b^j\eta^i)}{\tau_1\tau_2} \right.$$
$$\left. - \frac{M^2\eta^i\eta^j + MN(b^i\eta^j + b^j\eta^i) + N^2 b^i b^j}{\tau_3}\right] \tag{4.10}$$

and

$$\det(g_{ij}) = (\rho_0)^n \tau_1 \tau_2 \tau_3 \det(a_{ij}). \tag{4.11}$$

Theorem 4.1. *For a non-Hermitian ℝ-complex Finsler space (M,F), where F is generalized Kropina metric, we have*

(i) *the contravariant tensor g^{ji} is given in equation* (4.10).

(ii) $\det(g_{ij})$ *is given in equation* (4.11).

Now,

$$\gamma + \bar{\gamma} = l_i \eta^i + l_{\bar{j}} \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}\bar{k}} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \tag{4.12}$$

$$\theta + \bar{\theta} = b_j \eta^j + b_{\bar{j}} \eta^{\bar{j}} = 2\beta, \quad \delta = \theta. \tag{4.13}$$

Proposition 4.2. *A non-Hermitian ℝ-complex Finsler space (M,F), where F is generalized Kropina metric, satisfies the properties given in equations* (4.12) *and* (4.13).

5. Conclusion and Scope of Future Research

In conclusion, the investigation into ℝ-complex Finsler spaces with a generalized Kropina metric, represented by $F = \frac{\alpha^{m+1}}{\beta^m}$, has provided valuable insights into the Finsler geometry. Throughout this paper, we have successfully introduced and analyzed the fundamental metric tensor fields, g_{ij} and $g_{i\bar{j}}$, for this metric. The derivation of explicit formulas for these tensor fields, along with their determinants and inverse tensor fields, constitutes a significant contribution to the understanding of ℝ-complex Finsler spaces. Here results contribute significantly to the foundational knowledge of ℝ-complex Finsler spaces. The exploration of these spaces with the given metric has enriched our knowledge of the interplay between geometry and algebraic structures.

This study lays the groundwork for exciting future research in the field of Finsler geometry. Moving forward, researchers can explore the behavior of ℝ-complex Finsler spaces with the generalized Kropina metric in higher dimensions, unraveling new insights. Investigating practical applications of the derived metric tensor fields could lead to geometric solutions applicable in physics and engineering. Further studies may focus on the curvature properties, geodesic equations, and potential connections between ℝ-complex Finsler spaces and other mathematical structures. Exploring modifications to the given metric function and their impact on geometric properties offers a promising avenue for future investigations. In summary, by pursuing these research directions, scholars can contribute to a deeper comprehension of ℝ-complex Finsler spaces and their applications.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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