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Research Article

On a k-Annihilating Ideal Hypergraph of Local Rings

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Abstract. The concept of a k-annihilating ideal hypergraph of a finite commutative ring is very broad, and one of its structures has been discussed, where R is a local ring. In this paper, the structure of a k-annihilating ideal hypergraph of local rings is presented and the order and size of it are determined. Also, the degree of every nontrivial k-annihilating ideal of local rings containing in the vertex set $\mathcal{A}(R,k)$ of a hypergraph $\mathcal{AG}_k(R)$ is found and counted. Furthermore, the diameter of a k-annihilating ideal hypergraph $\mathcal{AG}_k(R)$ is determined, which equals 1 or 2, as well as the centre of $\mathcal{AG}_k(R)$. Finally, the Wiener index of a k-annihilating ideal hypergraph $\mathcal{AG}_k(R)$ is computed.

Keywords. Local ring, k-annihilating ideal hypergraph, Wiener index

Mathematics Subject Classification (2020). 13A70, 05C65, 05A17

1. Introduction

In the last two decades, the study of algebraic structure using graph properties has become an exciting research topic, yielding many fascinating results and questions. The structure of a ring is more closely related to ideal behavior than elements in ring theory, so it is appropriate to define a graph with vertex sets as ideals. There are many articles in the literature that assign graphs to rings (Behboodi and Rakeei [1], Curtis *et al*. [3], and Mohammad *et al*. [5]). Recently, Selvakumar and Ramanathan [6] introduced and studied the concept of k-annihilating ideal hypergraph of a commutative ring and defined it as: Let R be a commutative ring and let

$\mathcal{A}(R,k)$ be the set of all k-annihilating ideals in R and $k > 2$ an integer. The k-annihilating ideal hypergraph of R, denoted by $\mathcal{AG}_k(R)$ is a hypergraph with vertex set $\mathcal{A}(R,k)$ and for distinct elements $I_1, I_2, \ldots, I_k$ in $\mathcal{A}(R,k)$, the set $\{I_1, I_2, \ldots, I_k\}$ is an edge of $\mathcal{AG}_k(R)$ if and only if $\prod_{i=1}^{k} I_i = (0)$ and the product of $(k-1)$ element of $\{I_1, I_2, \ldots, I_k\}$ is nonzero. Later, Essa *et al.* [4] modified and investigated the structure of a k-annihilating ideal hypergraph of a commutative ring.

Throughout this article, all rings R assumed to be finite commutative ring with identity and every local ring (R,m) has a finite number of ideals with index of nilpotency i of m (i.e., an Artinain local ring R). Also, R is *PIR*, denoted by (R,m,i), if and only if the maximal ideal m is cyclically generated. A ring R is called a principal ideal ring (*PIR*) if each of its ideals is principal. Moreover, if (R,m,i) is local ring, then the number of nontrivial ideals of R is $i-1$.

A hypergraph $\mathcal{H}$ is a pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ of disjoint sets, where $\mathcal{V}(\mathcal{H})$ is a non-empty finite set whose elements are called vertices, and the number of elements of $\mathcal{V}(\mathcal{H})$, is called order of hypergraph $\mathcal{H}$, denote by $n(\mathcal{H})$. Also, the elements of $\mathcal{E}(\mathcal{H})$ are a finite family of distinct nonempty subsets of $\mathcal{V}(\mathcal{H})$ known as hyperedges, with $U_{E\in\mathcal{E}(\mathcal{H})}E = \mathcal{V}(\mathcal{H})$, and they are arbitrary sets of vertices that can contain an arbitrary number of vertices, and the number of elements of hyperedges is called the size of hypergraph $\mathcal{H}$, denoted by $m(\mathcal{H})$. If every hyperedge E of $\mathcal{H}$ is of size k, then the hypergraph $\mathcal{H}$ is said to be k-uniform. The degree of a vertex v is the number of edges that contain it, denoted as $d_{\mathcal{H}}(v)$. A path of length k in a hypergraph $\mathcal{H}$ is a finite sequence of the form $v_1, E_1, y_1, E_2, y_2, \ldots, E_{k-1}, y_{k-1}, E_k, v_2$ such that $v_1 \in E_1$ and $y_i \in (E_i \cap E_{i+1})$ for $i = 1, 2, \ldots, k-1$ and $v_2 \in E_k$. The distance between u and v in $\mathcal{V}(\mathcal{H})$ is the length of a shortest path from u and v in $\mathcal{H}$, denoted by $d_{\mathcal{H}}(u,v)$. Precisely, $d_{\mathcal{H}}(u,u) = 0$. The diameter of $\mathcal{H}$ is the maximum distance between all of its vertex pairs. The center of $\mathcal{H}$ is the minimum distance of vertex v to other vertices of it (see Bretto [2]).

The purpose of this article is to determine some fundamental graphical properties of a k-annihilating ideal hypergraph of a local $ring$. In section two, we determine the order and size of $\mathcal{AG}_k(R)$, as well as the degree of any nontrivial ideal of a local *ring* containing in $\mathcal{A}(R,k)$. In section three, we find the diameter of $\mathcal{AG}_k(R)$, which is the same as [4, Theorem 3.1], that is, $diam(\mathcal{AG}_k(R)) \le 2$ and the center of $\mathcal{AG}_k(R)$. We also discover the Wiener index of $\mathcal{AG}_k(R)$ of a local $ring$.

A partition of a positive integer n is a finite sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are referred to as the parts of the partition n. The Q_n denotes the set of partitions of n into distinct parts, and $Q_{n,l}$ denotes the set of partitions of n into distinct parts whose least part is l and not equal to n, for $1 \le l \le n-1$. Furthermore, let $q(n)$ and $q(n,l)$ represents the number of elements in Q_n and $Q_{n,l}$ respectively, and $q(n,l) = \lfloor \frac{n-1}{2} \rfloor - l$ (see Srichen [7]). We rely on $r = 2$ and $r = 3$ in particular to count our problem.

2. Fundamentals of k-Annihilating Ideal Hypergraph of Local Ring

This section is started by determining the cases of a k-annihilating ideal hypergraph of R that is empty. Let $\mathcal{A}(R,k)$ be the set of all k-annihilating ideals of R, where $k > 2$ is an integer, as a vertex set, and obtain important properties of $\mathcal{A}(R,k)$. In addition, the order and size of $\mathcal{AG}_k(R)$ are found, as well as the degree of every vertex of $\mathcal{A}(R,k)$.

Theorem 2.1. *Let R be a ring, then a k-annihilating ideal hypergraph $\mathcal{AG}_k(R)$ is empty, if and only if one of the following conditions is satisfied:*

(1) *R is an integral domain;*

(2) *(R,m,i) is a local ring whose maximal ideal with index of nilpotency $i \in \{2,3,4,5\}$;*

(3) *(R,m) is a local ring which is not PIR and the nilpotency index of m is two;*

(4) *R is a nonlocal ring such that $R \cong R_1 \times R_2$ where R_1 and R_2 are finite fields or R_1 is a finite field and (R_2,m) is a local ring with m, that is, the unique proper ideal of R_2.*

Proof. We have the followings:

(1) It's clear.

(2) Since (R,m,i) is a local ring with maximal ideal m having index of nilpotency $2 \le i \le 5$, then the ideals of R which contained in $\mathcal{A}(R,k)$ as $\{m,m^2,\dots,m^{i-1}\}$, that is if I_1,I_2 and, I_3 are contained in $\mathcal{A}(R,k)$ so $I_1 \cdot I_2 \cdot I_3 = (0)$ implies that $I_{t_1} \cdot I_{t_2} = (0)$ for $t_1,t_2 \in \{1,2,3\}$, thus $\mathcal{AG}_k(R)$ is empty. Conversely, let R is a local ring whose maximal ideal m which has index of nilpotency $i \ge 6$. Without lost generality, suppose that $i = 6$, then m^5 is minimal ideal which is not contained in $\mathcal{A}(R,k)$ by [4, Lemma 2.1], so $\mathcal{A}(R,k) = \{m,m^2,m^3,m^4\}$. Take $m \cdot m^2 \cdot m^3 = (0)$ then $m \cdot m^2 \ne (0)$, $m \cdot m^3 \ne (0)$, $m^2 \cdot m^3 \ne (0)$, similarly, for any $i > 6$, then $\mathcal{AG}_k(R)$ is nonempty. Hence i must be belong to $\{2,3,4,5\}$.

(3) If $m^3 = (0)$, $m^2 \ne (0)$, and (R,m) is local ring which is not *PIR*, then m is generated at least by two elements, without loss generality, assume $m = (x,y)$, if $x \cdot y = 0$, then we have $m^2 = 0$ which is a contradiction. This implies that $(x)\cdot(y) \ne 0$, that is $\{(x),(y),m\} \subseteq \mathcal{A}(R,k)$ so $m \cdot (x) \cdot (y) = 0$ with $m \cdot (x) \ne (0)$, $m \cdot (y) \ne (0)$ and $(x)\cdot(y) \ne (0)$. Therefore, $\mathcal{AG}_k(R)$ is nonempty. So the index of nilpotency must be equal to 2. Conversely, let (R,m) be a local ring that is not *PIR* with maximal ideal m with $m^2 = (0)$, then for any two ideals $I_1,I_2 \in \mathcal{A}(R,k)$ we have $I_1 \cdot I_2 = (0)$, so $\mathcal{AG}_k(R)$ is empty.

(4) Let R be a nonlocal ring such that $R \cong R_1 \times R_2 \times \dots \times R_n$, where R_i are finite fields and $1 \le i \le n$. If $i \ge 3$, then $\mathcal{AG}_k(R)$ is nonempty since for any $I_1 = (R_1 \times R_2 \times 0 \times \dots \times 0)$, $I_2 = (0 \times R_2 \times R_3 \times 0 \times \dots \times 0)$ and $I_3 = (R_1 \times 0 \times R_3 \times 0 \times \dots \times 0)$ contained in $\mathcal{A}(R,k)$, that is, $I_1 \cdot I_2 \cdot I_3 = (0)$ with $I_1 \cdot I_2 \ne (0)$, $I_1 \cdot I_3 \ne (0)$, $I_2 \cdot I_3 \ne (0)$. If $n = 2$, then we have three cases:

  (i) If $R \cong R_1 \times R_2$, for a finite fields R_1 and R_2. Then the ideals of R, are either trivial or minimal. So $\mathcal{A}(R,k) = \phi$, thus $\mathcal{AG}_k(R)$ is empty. Also, if $R \cong R_1 \times R_2$, for a finite field R_1 and (R_2,m) is a local ring with one ideal as m. Then $\mathcal{A}(R,k)$ contains four nontrivial ideals of R, namely $I = (R_1 \times (0))$, $J = (R_1 \times m)$, $K = ((0) \times m)$ and $L = ((0) \times R_2)$. That is $\mathcal{AG}_k(R)$ is empty.

  (ii) If $R \cong R_1 \times R_2$, for a finite field R_1 and (R_2,m) is a local ring with at least two ideals as I_1 and I_2 contained in R_2 such that $I_1 \cdot I_2 = (0)$, then the ideals $(R_1 \times I_1)$, $(R_1 \times I_2)$ and $((0) \times R_2)$ are contained in $\mathcal{A}(R,k)$ with $(R_1 \times I_2)\cdot(R_1 \times I_1)\cdot((0) \times R) = (0)$, and

$(R_1 \times I_2)\cdot(R_1 \times I_1) \neq (0)$, $(R_1 \times I_2)\cdot((0)\times R) \neq (0)$, $(R_1 \times I_1)\cdot((0)\times R) \neq (0)$, so $\mathcal{AG}_k(R)$ is nonempty.

(iii) If $R \cong R_1 \times R_2$, and (R_1, m_1) and (R_2, m_2) are not field, then there exists m_i of R_i and $i = 1,2$ such that ${m_i}^2 = (0)$, implying that $(R_1 \times m_2)$, $(m_1 \times R_2)$ and $(m_1 \times m_2)$ are nontrivial ideals contained in $\mathcal{A}(R,k)$ such that $(R_1 \times m_2)\cdot(m_1 \times R_2)\cdot(m_1 \times m_2) = (0)$ and $(R_1 \times m_2)\cdot(m_1 \times R_2) \neq (0)$, $(R_1 \times m_2)\cdot(m_1 \times m_2) \neq (0)$, $(m_1 \times R_2)\cdot(m_1 \times m_2) \neq (0)$, thus $\mathcal{AG}_k(R)$ is nonempty. □

Lemma 2.2. *Let (R,m,i) be a local ring for positive integer $i \geq 6$. Then m^{i-1} and m^{i-2} are not contained in $\mathcal{A}(R,k)$.*

Proof. Let (R,m,i) be a local ring and $m^i = (0)$, for positive integer $i \geq 6$. Since by [4, Lemma 2.1], m^{i-1} is a minimal ideal that is not contained in $\mathcal{A}(R,k)$. Furthermore, (R,m,i) is a *PIR*, so for some m, m^t are contained in $\mathcal{A}(R,k)$, where $2 \leq t \leq i-3$ then $m \cdot m^t \cdot m^{i-2} = (0)$ and $m \cdot m^t \neq (0)$, $m \cdot m^{i-2} \neq (0)$ but $m^t \cdot m^{i-2} = (0)$. Thus m^{i-2} is not contained in $\mathcal{A}(R,k)$. □

Theorem 2.3. *Let (R,m,i) be a local ring for positive integer $i \geq 6$, then the set of k-annihilating ideal hypergraph $\mathcal{A}(R,k) = \{m, m^2, m^3, \ldots, m^{i-3}\}$ and $n(\mathcal{AG}_k(R)) = i-3$.*

Proof. Let (R,m,i) be a local ring with $m^i = (0)$, for positive integer $i \geq 6$, and let the set of all nontrivial ideals hypergraph $\mathcal{A}(R,k) = \{m, m^2, m^3, \ldots, m^{i-1}\}$ but by Lemma 2.2, m^{i-1} and m^{i-2} are not contained in $\mathcal{A}(R,k)$. At that time, $m^{i-3}\cdot m^2 \cdot m = (0)$, but $m^{i-3}\cdot m^2 \neq (0)$, $m^{i-3}\cdot m \neq (0)$, and $m^2 \cdot m \neq (0)$. Thus m^{i-3} is contained in $\mathcal{A}(R,k)$. Similarly, for any nontrivial ideal such m^t contained in $\mathcal{A}(R,k)$, for $1 \leq t \leq i-3$. □

Theorem 2.4. *Let (R,m,i) be a local ring, for positive integer $i \geq 6$ and let $\mathcal{A}(R,k) = \{m, m^2, \ldots, m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$. Then the size of $\mathcal{AG}_k(R)$ is defined as:*

$$m(\mathcal{AG}_k(R)) = \sum_{s=0}^{\left\lfloor \frac{i-2}{2} \right\rfloor - 2} q^{(3)}(i+s, s),$$

where $q^{(3)}(i+s,s)$ is the number of 3-partitions of $i+s$ into distinct parts whose least part is s and not equal to $i+s$.

Proof. Let (R,m,i) be a local ring with $m^i = (0)$ for positive integer $i \geq 6$, and let $\mathcal{A}(R,k) = \{m, m^2, \ldots, m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of R. Suppose that m^{t_1}, m^{t_2} and m^{t_3} are contained in $\mathcal{A}(R,k)$ where t_1, t_2 and t_3 are differ and $1 \leq t_1, t_2, t_3 \leq i-3$, then $m^{t_1}\cdot m^{t_2}\cdot m^{t_3} = (0)$ and $m^{t_1}\cdot m^{t_2} \neq (0)$, $m^{t_1}\cdot m^{t_3} \neq (0)$, $m^{t_2}\cdot m^{t_3} \neq (0)$. So we obtain

$$t_1 + t_2 + t_3 \geq i, \tag{2.1}$$

and

$$t_1 + t_2 < i,\ t_1 + t_3 < i \quad \text{and} \quad t_2 + t_3 < i. \tag{2.2}$$

Now, we can describe the number of all solutions S of (2.1) for t_1, t_2 and t_3 as

$$S = p^{(3)}(i) + p^{(3)}(i+1) + p^{(3)}(i+2) + \ldots + p^{(3)}(i+j),$$

where $p^{(3)}(i+j)$ is 3-partitions of $i+j$ and $0 \le j \le i-1$.

Since t_1, t_2 and t_3 are differ and $1 \le t_1, t_2, t_3 \le i-3$, then we count all distinct solutions S of (2.1) for t_1, t_2 and t_3 by

$$S = q^{(3)}(i) + q^{(3)}(i+1) + q^{(3)}(i+2) + \ldots + q^{(3)}(i+j),$$

where $q^{(3)}(i+j)$ is 3-partitions of $i+j$ into distinct parts, but we need to delete some exceptions of S which are not satisfied (2.2). Thus, we get

$$S = q^{(3)}(i) + q^{(3)}(i+1,1) + q^{(3)}(i+2,2) + \cdots + q^{(3)}(i+j,j),$$

where $q^{(3)}(i+j,j)$ is 3-partitions of $i+j$ into distinct parts whose the least part is j and that is not equal to $i+j$ for $0 \le j \le i-1$, that is

$$S = \sum_{s=0}^{j} q^{(3)}(i+s,s). \tag{2.3}$$

From (2.2), $t_1 + t_2 < i$ where $t_1 \neq t_2$ and $1 \le t_1, t_2, t_3 \le i-3$. Then, the maximum solution s for t_1 and t_2 can then be reset to $t_1 + t_2 \le i-1$. So $s = q^{(2)}(i+1,1)$ where $q^{(2)}(i+1,1)$ is 2-partitions of $i-1$ into distinct parts whose the least part is one and which is not equal to $i-1$ where $i \ge 6$. Since $q^{(2)}(i-1,1) = \lfloor \frac{(i-1)-1}{2} \rfloor - 1$, so $s = \lfloor \frac{i-2}{2} \rfloor - 1$ for $i \ge 6$. As a result, we constrain s to $1 \le s \le \lfloor \frac{i-1}{2} \rfloor - 1$ and verify it in (2.3). Thus we conclude that the general form of size of $\mathcal{AG}_k(R)$ is

$$m(\mathcal{AG}_k(R)) = \sum_{s=0}^{\lfloor \frac{i-2}{2} \rfloor - 2} q^{(3)}(i+s,s). \qquad \square$$

Corollary 2.5. *Let (R,m,i) be a local ring for positive integer $i \ge 6$ and let $\mathcal{A}(R,k) = \{m, m^2, \ldots, m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$. Then the set of all subsets of hyperedges of $\mathcal{AG}_k(R)$ are represented as follow*

$$E(\mathcal{AG}_k(R)) = \{Q^{(3)}(i+s,s)\},$$

for all $0 \le s \le \lfloor \frac{i-2}{2} \rfloor - 2$, where $Q^{(3)}(i+s,s)$ is the set of partitions of $i+s$ into distinct parts whose least part is s and not equal to $i+s$.

Theorem 2.6. *Let (R,m,i) be a local ring for positive integer $i \ge 6$ and $\mathcal{A}(R,k) = \{m, m^2, \ldots, m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$. Then, the degree of m^d contained in $\mathcal{A}(R,k)$ where $1 \le d \le i-3$, verifies one the following:*

(i) *If $i = 2d$, then*

$$\deg(m^d) = \sum_{s=0}^{d-1} (q^{(2)}(i-(d-s)) - s).$$

(ii) *If $i < 2d$, then*

$$\deg(m^d) = \sum_{s=0}^{n-d-3} (q^{(2)}(i-(d-s)) - s).$$

(iii) *If* $i > 2d$ *and* $i \neq 3d - s$, *then*

$$\deg(m^d) = \sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-(s+1)), \quad \textit{for all } 0 \le s \le d-1.$$

Furthermore, if $i = 3d - s_1$, *for some* s_1, *then*

$$\deg(m^d) = 1 + \sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-(s+1)),$$

where $q^{(2)}(i-(d-s))$ *is the number of* 2*-partitions of* $(i-(d-s))$ *into distinct parts.*

Proof. Let (R,m,i) be a local ring with $m^i = (0)$ and for positive integer $i \ge 6$, and let the set of all nontrivial k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$ defined as $\mathcal{A}(R,k) = \{m, m^2, \dots, m^{i-3}\}$. Suppose that m^d, m^{t_1} and m^{t_2} are contained in $\mathcal{A}(R,k)$ for $1 \le d$, $t_1, t_2 \le i-3$, where d, t_1 and t_2 are different. Then $m^d \cdot m^{t_1} \cdot m^{t_2} = (0)$ such that $m^d \cdot m^{t_1} \neq (0)$, $m^d \cdot m^{t_2} \neq (0)$ and $m^{t_1} \cdot m^{t_2} \neq (0)$. So we have $d + t_1 + t_2 \ge i$ with for positive integer $i \ge 6$, that is

$$t_1 + t_2 \ge i - d \tag{2.4}$$

and

$$d + t_1 < i, d + t_2 < i \quad \text{and} \quad t_1 + t_2 < i. \tag{2.5}$$

Then, the number of solution of S of (2.4) represented by

$$S = p^{(2)}(i-(d-0)) + p^{(2)}(i-(d-1)) + \ldots + p^{(2)}(i-(d-(d-1))),$$

for all $0 \le s \le d-1$. Since d, t_1 and t_2 are different, then we get the number of distinct solutions S of (2.4), thus, we get

$$S = \sum_{s=0}^{d-1} q^{(2)}(i-(d-s)). \tag{2.6}$$

Now, we discuss the following cases:

(i) Let $q^{(2)}(i-(d-s))$ is the number of 2-partitions into distinct part of $i-(d-s)$, since $i = 2d$, then $i - d = d$ so there are exceptions in (2.6) that are not verified (2.4). Therefore $i-(d-s) \neq d$ for $1 \le s \le d-1$, so we delete s cases from every $q^{(2)}(i-(d-s))$ which explains as

$$\deg(m^d) = \sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-s). \tag{2.7}$$

(ii) Again, let $q^{(2)}(i-(d-s))$ is the number of 2-partitions into distinct part of $i-(d-s)$, since $i < 2d$, then $i - d \neq d$ such that $d \neq t_1$ and $d \neq t_2$ so there are some exceptions in (2.6) that are not verified (2.4). Thus to find degree of m^d we use (2.7) but we replace the range of s which is defined as $0 \le s \le d-1$ to $0 \le s \le i-d-3$ because $d + (i-d) = i$ for positive integer $i \ge 6$ that is mean, $d + t_1 \ge i$ or $d + t_2 \ge i$, implies that $m^d \cdot m^{t_1} = (0)$ or $m^d \cdot m^{t_2} = (0)$ which contracts (2.5). So, we have

$$\deg(m^d) = \sum_{s=0}^{i-d-3}(q^{(2)}(i-(d-s))-s).$$

(iii) Let $q^{(2)}(i-(d-s))$ is the number of 2-partitions into distinct part of $i-(d-s)$, since $i>2d$ and $i\neq 3d-s$, then $i-d>d$ so from (2.4) we get $t_1+t_2>d$ which implies that either $d=t_1$ or $d=t_2$. As a result of using (2.7), we must remove one addition case from every $q^{(2)}(i-(d-s))$ where $0\le s\le d-1$. Thus we obtain

$$\deg(m^d)=\sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-s-1),$$

and then

$$\deg(m^d)=\sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-(s+1)). \tag{2.8}$$

Furthermore, if $i\neq 3d-s_1$ for exactly s_1 where $0\le s_1\le d-1$ and $i>2d$, then by (2.4) we get $d=t_1=t_2$, so we only need to delete one case from $q_2(i-(d-s_1))$ because it is combined with the condition of 2-partitions into distinct parts of $i-(d-s_1)$. Hence we can explain it by using (2.8), as follows:

$$\deg(m^d)=1+\sum_{s=0}^{d-1}(q^{(2)}(i-(d-s))-(s+1)).$$ □

3. Distance Between Nontrivial Ideals in $\mathcal{A}(R,k)$

This section is concentrated on the distance notation in the k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$ for two nontrivial ideals contained in $\mathcal{A}(R,k)$ which is used to determine the diameter and center of $\mathcal{AG}_k(R)$, also discusses the Wiener index of $\mathcal{AG}_k(R)$.

Theorem 3.1. *Let (R,m,i) be a local ring for positive integer $i\ge 7$. Then, $diam(\mathcal{AG}_k(R))\le 2$ and*

$$cent(\mathcal{AG}_k(R))=\begin{cases}\{m^2\}, & \text{if } i \text{ is odd},\\ \{m,m^2\}, & \text{if } i \text{ is even}.\end{cases}$$

Proof. Let (R,m,i) be a local ring and $m^i=(0)$ for $i\ge 7$, and let m^{t_1} and m^{t_2} are two nontrivial ideals contained in $\mathcal{A}(R,k)$ where $t_1\neq t_2$ and $1\le t_1,\ t_2\le i-3$. First, we show that $diam(\mathcal{AG}_k(R))\le 2$. It is enough to find a path between any two nontrivial ideals of (R,m,i) in $\mathcal{AG}_k(R)$. Consider that if i is odd positive integer, then

$$d(m,m^t)=\begin{cases}2, & \text{if } t=\lfloor \frac{i}{2}\rfloor,\\ 1, & \text{otherwise}.\end{cases}$$

Now, if $m^{t_1}\cdot m^{t_2}\neq(0)$ implies that there are two cases that, if $t_1+t_2<i$, then there exists another nontrivial ideal such m^s for $1\le s\le i-3$ which is contained in $\mathcal{A}(R,k)$ such that $m^{t_1}\cdot m^s\neq(0)$ and $m^{t_2}\cdot m^s\neq(0)$, that is, $t_1+s<i$ and $t_2+s<i$ so $t_1+t_2+s\ge i$. Thus $m^{t_1}\cdot m^{t_2}\cdot m^s=(0)$. Therefore, there exists a hyperedge as E contains m^{t_1}, m^{t_2} and m^s, so $d(m^{t_1},m^{t_2})=1$. Again, if $t_1+t_2\ge i$. Since $1\le t_1,\ t_2\le i-3$, then there exists m^2 which contained in $\mathcal{A}(R,k)$ such that m^{t_1} and m^{t_2} are different from m^2 but $t_1+2\ge i$ and $t_2+2\ge i$. So by above proof there exist two hyperedges E_1 and E_2 such that $m^{t_1},m^2\in E_1$ and $m^{t_2},m^2\in E_2$ that is $d(m^{t_1},m^{t_2})=2$ which discerns that $diam(\mathcal{AG}_k(R))\le 2$.

Furthermore, the central nontrivial ideals contained in $\mathcal{A}(R,k)$ are those ideals whose distance to other ideals in $\mathcal{A}(R,k)$ is one, since $diam(\mathcal{AG}_k(R)) \le 2$ and $\mathcal{AG}_k(R)$ is not complete hypergraph, then the central nontrivial ideals of $\mathcal{A}(R,k)$ lie in every hyperedge of $\mathcal{AG}_k(R)$. Consider that if i is even positive integer, then m and m^2 are contained in every hyperedge of $\mathcal{AG}_k(R)$, since $m \cdot m^t \neq (0)$ for any $2 \le t \le i-3$, then $d(m,m^t) = 1$ and we obtain $m \in cent(\mathcal{AG}_k(R))$, also for the same reason, $m^2 \cdot m^t \neq (0)$ for any $3 \le t \le i-3$. On the other hand if, i is odd positive integer, then $m \cdot m^{\lfloor \frac{i}{2} \rfloor} \cdot m^t = (0)$ iff $t = \lfloor \frac{i}{2} \rfloor$ implies that $d(m, m^{\lfloor \frac{i}{2} \rfloor}) = 2$. That is $m \in cent(\mathcal{AG}_k(R))$ iff i is an even positive integer. At a last, for any m^s is contained in $\mathcal{A}(R,k)$, there exists another m^t contained in $\mathcal{A}(R,k)$, for $s \neq t$ and $3 \le s,\ t \le i-3$ such that $m^s \cdot m^t = (0)$ thus $d(m^s, m^t) = 2$ and so $m^s \notin cent(\mathcal{AG}_k(R))$ which discerns that

$$cent(\mathcal{AG}_k(R)) = \begin{cases} \{m^2\}, & \text{if } i \text{ is odd,} \\ \{m, m^2\}, & \text{if } i \text{ is even} \end{cases}$$ □

Corollary 3.2. *Let $(R,m,6)$ be a local ring, then a k-annihilating ideal hypergraph of $\mathcal{AG}_k(R)$ is complete hypergraph with $diam(\mathcal{AG}_k(R)) = 1$ and $cent(\mathcal{AG}_k(R)) = \{m, m^2, m^3\}$.*

Theorem 3.3. *Let (R,m,i) be a local ring for even positive integer $i \ge 6$. Then, the Wiener index of $\mathcal{AG}_k(R)$ is defined as*

$$W(\mathcal{AG}_k(R)) = \frac{3}{4}(n-1)^2,$$

where n represents to an order of $\mathcal{AG}_k(R)$.

Proof. Let (R,m,i) be a local ring and $m^i = (0)$ for even positive integer $i \ge 6$ and let $\mathcal{A}(R,k) = \{m, m^2, \dots, m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of R. Suppose that m^s, m^t are contained in $\mathcal{A}(R,k)$ for $1 \le s,\ t \le i-3$. So, we can get $d_{\mathcal{AG}_k(R)}(m^s, m^t)$ as a distance between m^s and m^t.

Now, to determine the Wiener index of $\mathcal{AG}_k(R)$ we get the summation of all $d_{\mathcal{AG}_k(R)}(m^s, m^t)$ for $\{m^s, m^t\} \subseteq \mathcal{A}(R,k)$ as

$$\begin{aligned} W(\mathcal{AG}_k(R)) &= \sum_{m^s, m^t \in \mathcal{A}(R,k)} d_{\mathcal{AG}_k(R)}(m^s, m^t) \\ &= \sum_{s=2}^{i-3} d(m^1, m^s) + \sum_{s=3}^{i-3} d(m^2, m^s) + \cdots + \sum_{s=k+1}^{i-3} d(m^k, m^s) \\ &\quad + \sum_{s=k+2}^{i-3} (d(m^{k+1}, m^j) + \cdots + d(m^{i-4}, m^{i-3})). \end{aligned} \tag{3.1}$$

By Theorem 3.1, then $diam(\mathcal{AG}_k(R)) \le 2$, so Wiener index is represented by

$$\begin{aligned} W(\mathcal{AG}_k(R)) &= ((i-3)-1) + ((i-3)-2) + ((i-3)-3) + \cdots \\ &\quad + ((i-3)-k) + (k-2) + ((i-3)-(k+1)) + (k-2) + \cdots \\ &\quad + ((i-3)-(i-4)+1). \end{aligned}$$

Observe that, m^1 and m^2 are in the center of $\mathcal{AG}_k(R)$ by Theorem 3.1, then the first and second terms are represented as $d(m^1, m^s) = d(m^2, m^s) = 1$, for all $s = 3, 4, \dots, i-3$. Also the third term represents by $d(m^3,m^s)=1$ for all $s=4,5,\dots,i-4$ except $i-3$ because m^3 and m^{i-3} are not adjacent in $\mathcal{AG}_k(R)$, and we can conclude the following form for all terms of (3.1) as

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= (i-3)-1+\sum_{l=2}^{j}(((i-3)-l)+(l-2))+\sum_{l=j+1}^{i-4}(((i-3)-l)+((i-3)-l))\\ &=(i-4)+\sum_{l=1}^{j-1}(i-5)+2\sum_{l=j+1}^{i-4}((i-3)-l)\\ &=(i-4)+(j-1)(i-5)+2\Big(\frac{((i-3)-(j+1))((i-3)-(j+1)+1)}{2}\Big)\\ &=(i-4)+(j-1)(i-5)+((i-3)-(j+1))((i-3)-j).\end{aligned}$$

Also, suppose that $j=\lfloor\frac{i-3}{2}\rfloor+1$ or $j=\frac{i-2}{2}$, we get

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= (i-4)+\frac{i-4}{2}(i-5)+\Big(i-3-\frac{i}{2}\Big)\Big(i-3-\frac{i-2}{2}\Big)\\ &=(i-4)+\frac{1}{2}(i-4)(i-5)+\frac{2i-6-i}{2}\,\frac{2i-6-i+2}{2}\\ &=(i-4)+\frac{1}{2}(i-4)(i-5)+\frac{1}{4}(i-6)(i-4)\\ &=\frac{1}{4}(4(i-4)+2(i-4)(i-5)+(i-6)(i-4))\\ &=\frac{1}{4}(i-4)(4+2i-10+i-6)\\ &=\frac{3}{4}(i-4)^2.\end{aligned}$$

By Theorem 2.3, $n(\mathcal{AG}_k(R))=i-3$, and we set it as n, we obtain

$$W(\mathcal{AG}_k(R))=\frac{3}{4}(n-1)^2. \qquad \square$$

Theorem 3.4. *Let (R,m,i) be a local ring for odd positive integer $i\ge 7$. Then, the Wiener index of $\mathcal{AG}_k(R)$ is defined as*

$$W(\mathcal{AG}_k(R))=\frac{3}{4}n(n-2)+2,$$

where n represents to an order of $\mathcal{AG}_k(R)$.

Proof. Let (R,m,i) is a local ring and $m^i=(0)$ for odd positive integer $i\ge 7$ and let $\mathcal{A}(R,k)=\{m,m^2,\dots,m^{i-3}\}$ be the set of all nontrivial k-annihilating ideal hypergraph of R. Suppose that m^s, m^t are contained in $\mathcal{A}(R,k)$ for $1\le s,t\le i-3$. So, we can get $d_{\mathcal{AG}_k(R)}(m^s,m^t)$ as a distance between m^s and m^t. So

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= \sum_{m^s,m^t\in\mathcal{A}(R,k)} d_{\mathcal{AG}_k(R)}(m^s,m^t)\\ &=\sum_{s=2}^{i-3}d(m^1,m^s)+\sum_{s=3}^{i-3}d(m^2,m^s)+\cdots+\sum_{s=k+1}^{i-3}d(m^k,m^s)\\ &\quad+\sum_{s=k+2}^{i-3}(d(m^{k+1},m^s)+\cdots+d(m^{i-4},m^{i-3})).\end{aligned} \tag{3.2}$$

Again, by Theorem 3.1, $diam(\mathcal{AG}_k(R)) \leq 2$, so we describe Wiener index as:

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= ((i-3)-1)+1+((i-3)-2)+0+((i-3)-3)+1+\cdots\\&+((i-3)-j)+(j-2)+((i-3)-(j+1))+(j-1)+\cdots\\&+((i-3)-(i-4))+1.\end{aligned}$$

Consider that the first term represents $d(m^1,m^s)=1$, for all $s=2,3,\cdots,i-3$ except $i-5$ because m^1 and m^{i-5} are not adjacent in $\mathcal{AG}_k(R)$. Also, the second term represents $d(m^2,m^s)=1$, for all $s=2,3,\cdots,i-3$, because m^2 is lie in the center of $\mathcal{AG}_k(R)$ by Theorem 3.1. In addition the third term represents $d(m^3,m^s)=1$ for all $s=2,3,\cdots,i-4$ except $i-3$ since m^1 and m^{i-5} are not adjacent in $\mathcal{AG}_k(R)$, and we can conclude the following form for all terms of (3.2)

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= (i-3)+\sum_{l=2}^{j}(((i-3)-l)+(l-2))+\sum_{l=j+1}^{i-4}(((i-3)-l)+((i-3)-l))\\&=(i-3)+\sum_{l=1}^{j-1}(i-5)+2\sum_{l=j+1}^{i-4}((i-3)-l)\\&=(i-3)+(j-1)(i-5)+2\Big(\frac{((i-3)-(j+1))((i-3)-(j+1)+1)}{2}\Big)\\&=(i-3)+(j-1)(i-5)+((i-3)-(j+1))((i-3)-j).\end{aligned}$$

Now, suppose that $j=\lfloor\frac{i-3}{2}\rfloor+1$ or $j=\frac{i-1}{2}$, we obtain

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= (i-3)+\frac{i-3}{2}(i-5)+\Big((i-3)-\frac{i+1}{2}\Big)\Big((i-3)-\frac{i-1}{2}\Big)\\&=(i-3)+\frac{1}{2}(i-3)(i-5)+\frac{1}{4}(i-7)(i-5).\end{aligned}$$

Also by Theorem 3.1, we use $i-3=n$ as an order of $\mathcal{AG}_k(R)$, so we get

$$\begin{aligned}W(\mathcal{AG}_k(R)) &= n+\frac{1}{2}n(n-2)+\frac{1}{4}(n-4)(n-2)\\&=n+\frac{n^2}{2}-n+\frac{1}{4}(n^2-6n+8)\\&=\frac{n^2}{2}+\frac{n^2}{4}-\frac{3}{2}n+2\\&=\frac{3}{4}n(n-2)+2.\end{aligned}$$

□

4. Conclusion

This paper interprets the graphical structure of a k-annihilating ideal hypergraph of local rings based on partition theory and counts the order and size of it. The concept of adjacency between all non-trivial k-annihilating ideals is explained, such as contained in the vertex set $\mathcal{A}(R,k)$, in which the degree of them is counted, also, the diameter of a k-annihilating ideal hypergraph $\mathcal{AG}_k(R)$ is found, which equals one or two. Finally, the center and Wiener index of it are determined.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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